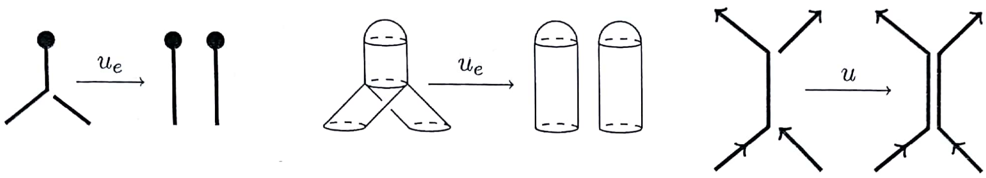


FIGURE 7. Disc unzip on the left and middle, strand unzip on the right.

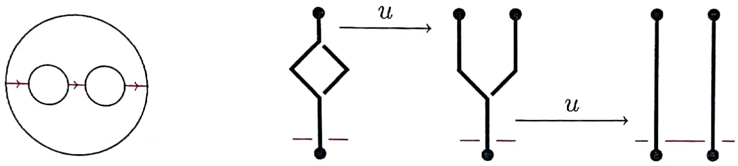


FIGURE 8. On the left, the circuit connection diagram for the connect sum operation, or concatenating along the red strings. On the right, two applications of disc unzips to achieve the doubled sphere.

The *disc unzip operation* u_e is defined for a capped strand labeled by e. Using the blackboard framing, u_e doubles the capped strand e and then attaches the ends of the doubled strand to the connecting ones, as shown Figure 7. Topologically, the blackboard framing of the diagram induces a framing of the corresponding tubes and discs in $\mathbb{R}^4$ via Satoh's tubing map [BD1, Section 3.1.1] and [Sa]. Unzip is the operation "pushing the disc off of itself slightly in the framing direction". See [BD2, Section 4.1.3] for details on framings and unzips.

A related operation not strictly necessary for this proof, *strand unzip*, is defined for strands which end in two vertices of opposite signs, as shown in the right of Figure 7. For the interested reader a detailed definition of crossing and vertex signs is in [BD2, Sections 3.4 and 4.1]. Strand unzip doubles the strand in the direction of the blackboard framing, and connects the ends of the doubled strands to the corresponding edge strands. Topologically, strand unzip pushes the tube off in the direction of the blackboard framing, as before. Note that unzips preserve the ribbon property.

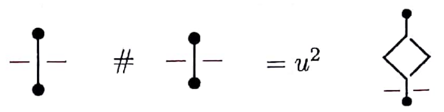
2.3. Interpreting "$1+1=2$" in w-foams. The threaded sphere of Section 2.1 can be described in $\widetilde{wTF}$ by the diagram , since a doubly capped tube is in fact a sphere. Recall that the "4D abacus" interpretation of "$1+1=2$" is $\mathbb{1}\#\mathbb{1} = \Delta_{\mathfrak{l}}\mathbb{1}$, where $\mathbb{1}$ is the threaded sphere, and $\Delta_{\mathfrak{l}}$ is the doubling of the sphere along a framing.

The connected sum $\#$ operation for the threaded sphere is the circuit algebra composition shown in Figure 8. The doubling $\Delta_{\mathfrak{l}}$ can be realized in $\widetilde{wTF}$ using the unzip operation. However, since unzipping a sphere is not one of the operations in $\widetilde{wTF}$, we need to start with a slightly more complex w-foam as shown in Figure 8: we perform two disc unzip operations on a w-foam with two vertices to achieve . To summarize, the topological statement "$1+1=2$" expressed in $\widetilde{wTF}$ is shown in Figure 9.

3. UNDERSTANDING THE DIAGRAMATIC STATEMENT

3.1. The associated graded structure $\mathcal{A}^{sw}$. As in [BD3], the space $\widetilde{wTF}$ is filtered by powers of the augmentation ideal and its associated graded space, denoted $\mathcal{A}^{sw}$, is a space of

Topological Statement



No!

FIGURE 9. The topological statement in w-foams.

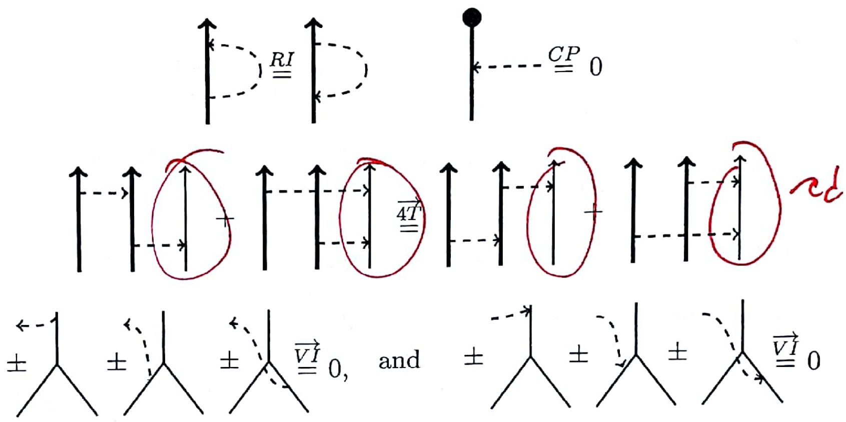
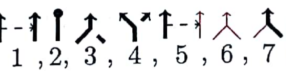


FIGURE 10. The relations RI, CP, $\overrightarrow{4T}$ and $\overrightarrow{VI}$.

arrow diagrams on foam skeleta. The *arrows* are drawn as black dashed oriented lines, and the skeleton w-foam elements are drawn with the usual thick black lines and thin red lines. Just like $\widetilde{wTF}$, $\mathcal{A}^{sw}$ is a circuit algebra presented in terms of generators and relations as follows:

$$\mathcal{A}^{sw} = CA\left\langle \text{1, 2, 3, 4, 5, 6, 7} \;\middle|\; RI,\, CP,\, \overrightarrow{4T},\, \overrightarrow{VI},\, TC \;\middle|\; u_e \right\rangle.$$



Generators 1 and 5 are *single arrows.* A single arrow is a degree 1 element of the associated graded space and represents the difference of a crossing and a non-crossing. The arrow head lies on the under (fly-through) strand of the crossing, and the tail lies on the over strand. All of the other generators of $\mathcal{A}^{sw}$ are *skeleton features* of degree zero. Note that circuit algebra products of the generators will never have an arrow tail on a red string, as red strings never cross over a strand. (Alternatively, we could introduce a relation asserting that arrow tails on red strings equal zero. This is necessary in [BD3] because a 'puncture' operation can give rise to a diagram with a tail on a red string, but not necessary here.) The relations of $\mathcal{A}^{sw}$ are briefly described below, see [BD2, Section 4.2.1] for more detail.

The RI (Rotation Invariance) relation is a consequence of $R1^s$, and CP is the diagrammatic analogue of the CP relation for $\widetilde{wTF}$. Both are shown in Figure 10. The $\overrightarrow{4T}$ and $\overrightarrow{VI}$ relations are also shown in Figure 10. In both cases the ambiguous strands can be either thick black or thin red, but must be consistent through the relation. The $\overrightarrow{VI}$ relation is the diagrammatic analogue of $R4$, and $4T$ has a slightly more complicated topological explanation. The TC (Tails Commute) relation is a consequence of OC and shown in Figure 11.

This is a mixup of an A^{sw} argument with a wTF argument

If this should remain, it should be demoted to a footnote.

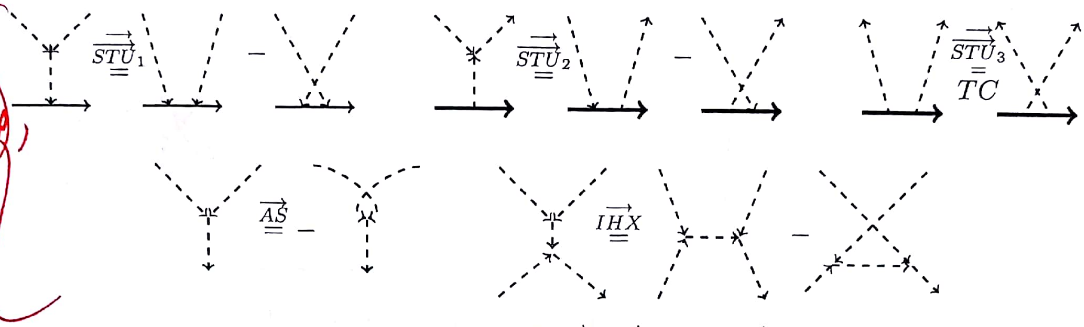


FIGURE 11. The relations $\overrightarrow{STU}$, $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ on $\mathcal{A}^{swt}$.

This presentation of $\mathcal{A}^{sw}$ is intuitive as a mirror of the circuit algebra presentation of $\widetilde{wTF}$. However, for relating $\mathcal{A}^{sw}$ to Lie algebras, it is more useful to use the following isomorphic formulation in terms of *w-Jacobi diagrams.*

A *w-Jacobi diagram* consist of a $\widetilde{wTF}$ skeleton and 'arrow graphs' between the components of the skeleton. An *arrow graph* is an oriented uni-trivalent graph, with the following three properties:

(1) Univalent vertices are attached to the skeleton.
(2) Trivalent vertices are equipped with a cyclic orientation.
(3) The edges are oriented so that every trivalent vertex has two 'in' arrows and one 'out' arrow. This is referred to as the *2-in-1-out* property.

Let $\mathcal{A}^{swt}$ denote the circuit algebra of linear combinations of w-Jacobi diagrams modulo the relations $\overrightarrow{STU}$, CP, RI and $\overrightarrow{VI}$ relations. The $\overrightarrow{STU}$ relation (really three relations) is shown in Figure 11, with TC (Tails Commute) being a degenerate case. The $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations, also shown in Figure 11, are consequences of $\overrightarrow{STU}$, and so is $\overrightarrow{4T}$. Once again, ambiguous strands can be either thick black or thin red, consistently throughout the relation. The following theorem is the arrow diagram analogue of a well-known fact about classical "chord diagrams":

Theorem 3.1. *[BD2, Theorem 3.8] The obvious inclusion $\mathcal{A}^{sw} \to \mathcal{A}^{swt}$ is a circuit algebra isomorphism.*

In light of this we drop the t superscript and write $\mathcal{A}^{sw}$ to denote the space of Jacobi diagrams. The advantage of Jacobi diagrams is that trivalent vertices satisfy the same properties as a Lie bracket—this will be made precise using the tensor interpretation map in Section 4.2.

We introduce the following notation: For a w-foam $F \in \widetilde{wTF}$, the circuit algebra $\mathcal{A}^{sw}(S(F))$ is the space of Jacobi diagrams with skeleton $S(F)$, where $S(F)$ is the skeleton of F as defined in Section 2.2. Often we will write $\mathcal{A}^{sw}(F)$ to mean $\mathcal{A}^{sw}(S(F))$.

The associated graded operation of the circuit algebra composition in $\widetilde{wTF}$ is the circuit algebra composition in $\mathcal{A}^{sw}$. As for the (disk) unzip u_e, given a w-foam F with a given strand e, the associated graded unzip operation $u_e : \mathcal{A}^{sw}(F) \to \mathcal{A}^{sw}(u_e(F))$ maps each arrow ending on e to a sum of two arrows, one ending on each of the two new strands which replace e. For example, an arrow diagram with k arrows ending on e – either heads or tails – is mapped to a sum of 2^k arrow diagrams. This sum is suppressed notationally as shown in Figure 12.

3.2. **The Homomorphic Expansion.** As proved by the first two authors in [BD2, BD3], there exists a (group-like) *homomorphic expansion* $Z : \widetilde{wTF} \to \mathcal{A}^{sw}$. An expansion is a

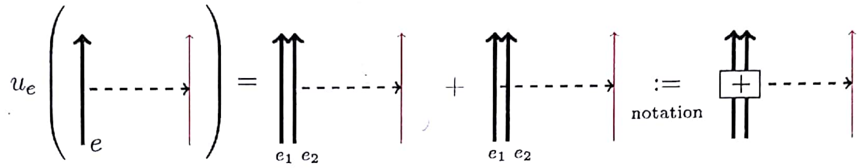


FIGURE 12. Unzipping the strand labeled e, where e_1 and e_2 are the two new strands replacing e.

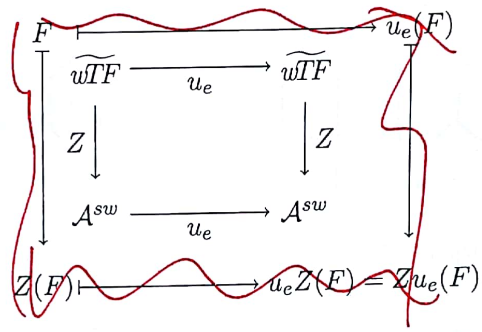


FIGURE 13. The commutativity of Z with unzips.

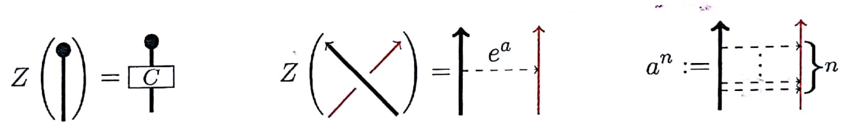


FIGURE 14. Values of Z on generating w-foams.

filtered linear map with the property that the associated graded map $\operatorname{gr} Z : \mathcal{A}^{sw} \to \mathcal{A}^{sw}$ is the identity map on $\mathcal{A}^{sw}$. A homomorphic expansion is an expansion that is a circuit algebra homomorphism and also intertwines each auxiliary operation of $\widetilde{wTF}$ with its arrow diagrammatic counterpart, meaning that the appropriate squares commute. For this paper, the only relevant such operation is the (disk) unzip u_e, the relevant commutative square is shown in Figure 13.

Each generator G of $\widetilde{wTF}$ has an associated skeleton denoted $S(G)$. The map Z sends each generator to an infinite sum of arrow diagrams on the skeleton $S(G)$, that is, $Z(G) \in \mathcal{A}^{sw}(S(G))$. The values of the crossings and the cap are computed explicitly in [BD2,BD3]; to refer to them we use the notation shown in Figure 14. In particular, the Z-value of a crossing of a black strand and a red string is the exponential e^a of an arrow a, to be interpreted as the power series, where a^n is shown in Figure 14.

Since Z is a circuit algebra homomorphism, given the values of Z on the generators, it is straightforward to compute Z of any w-foam F: if F is a circuit composition of some generators $\{G_i\}$, then $Z(F)$ is the same circuit composition of the values $Z(G_i)$.

To understand the diagrammatic statement, we need to discuss the Z-values of the vertices in some detail. By definition, $Z(\Upsilon) \in \mathcal{A}^{sw}(\Upsilon)$. Using iterative applications of the relation $\overrightarrow{VI}$, all arrow endings on the vertical strand of Υ can be moved to the bottom two strands. This induces an isomorphism $\mathcal{A}^{sw}(\Upsilon) \cong \mathcal{A}^{sw}(\uparrow\uparrow)$ [BD2]. Thus, $Z(\Upsilon)$ can be viewed as an

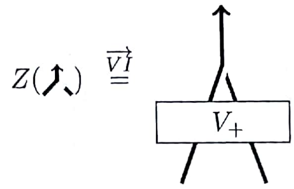


FIGURE 15. The definition of V_+.

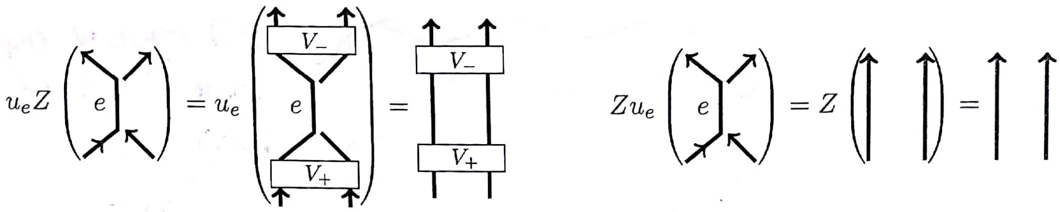


FIGURE 16. Visual proof of Lemma 3.2.

element of $\mathcal{A}^{sw}(\uparrow\uparrow)$ denoted by V_+, as shown in Figure 15. The arrow diagram $V_- = Z(\curlyvee) \in \mathcal{A}^{sw}(\uparrow\uparrow)$ is defined similarly.

Note ~~It is important to note~~ that $\mathcal{A}^{sw}(\uparrow\uparrow)$ is an algebra with multiplication given by vertical concatenation. (In fact, a Hopf algebra with coproduct $\Delta : \mathcal{A}^{sw}(\uparrow\uparrow) \to \mathcal{A}^{sw}(\uparrow\uparrow) \otimes \mathcal{A}^{sw}(\uparrow\uparrow)$. The coproduct Δ of an arrow diagram is a sum of all possible ways of attaching each of the connected component of the arrow graph – after removing the skeleton – to one of the tensor factor skeleta. Details on the Hopf algebra structure are in [BD2, Section 3.2].) Let us recall a useful fact from [BD2]:

Lemma 3.2. *In $\mathcal{A}^{sw}(\uparrow\uparrow)$, V_+ and V_- are multiplicative inverses, i.e. $V_+ \cdot V_- = 1 = \uparrow\uparrow$.*

Proof. This follows from the fact that Z is homomorphic with respect to the unzip operation, as shown in Figure 16. □

The following corollary is a crucial ingredient for the diagrammatic statement:

Corollary 3.3. $Z(\diamond) = \diamond$, *that is, the Z-value of this w-foam is trivial, a skeleton with no arrows.* □

3.3. **The diagrammatic statement.** Applying the homomorphic expansion Z to the topological statement of Section 2.3 gives rise to the following equation:

$$Z(\text{-}\!\!\uparrow\uparrow\!\!\text{-}) = Z(\text{-}\!\uparrow\!\text{-}) \# Z(\text{-}\!\uparrow\!\text{-}) = u^2 Z(\diamond)$$

Using the notation of Figure 14, we can compute each term of this individually to obtain the final form of the diagrammatic statement, as shown in Figure 17.

4. Understanding the Tensor Statement

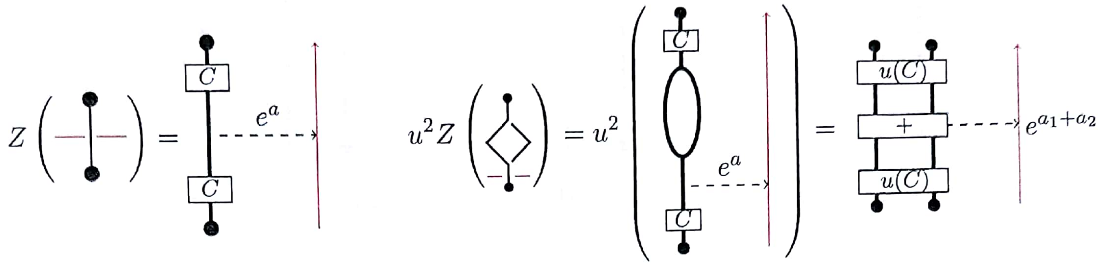
Ultimately we aim to give a proof of the Duflo isomorphism, which is a statement about finite dimensional Lie algebras. Up to this point, the spaces $\widetilde{wTF}$ and $\mathcal{A}^{sw}$ do not depend on

Postpone this remark to where it is needed.

mult. of the

will they depend on a Lie algebra later?

Here and elsewhere, u^2 looks too much like the square of an element rather than the square of an operation.



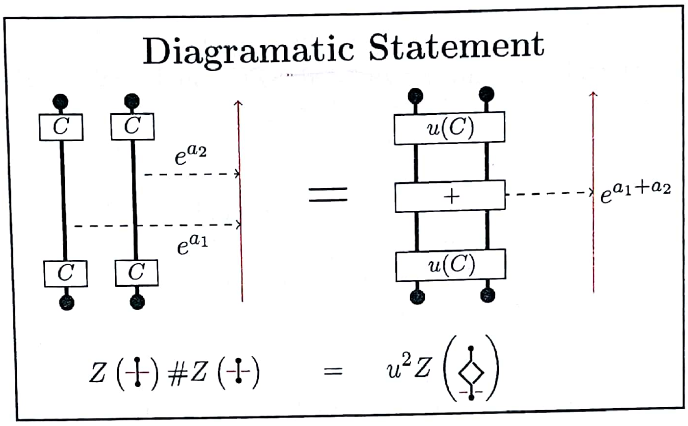


FIGURE 17. Computation and conclusion of the Diagrammatic Statement of "1+1 = 2" in $\mathcal{A}^{sw}$. Since tails commute, two caps on a strand can be combined into C^2, and the two unzipped caps can be combined as $u(C)^2 = u(C^2)$.

a Lie algebra, but from here on, we fix a finite dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic zero. For each such $\mathfrak{g}$, we define the *tensor interpretation* map

$$T : \mathcal{A}^{sw}(\vdots\,\uparrow) \to \hat{S}(\mathfrak{g}^*)_{\mathfrak{g}}^{\otimes k_1} \otimes \hat{U}(\mathfrak{g})^{\otimes k_2}.$$

Here $\mathcal{A}^{sw}(\vdots\,\uparrow)$ is the space of arrow diagrams on the skeleton of k_1 spheres and k_2 strings, where $k_1, k_2 \in \mathbb{N}$, $\hat{S}(\mathfrak{g}^*)$ is the degree completed symmetric algebra of the linear dual of $\mathfrak{g}$, the subscript $\mathfrak{g}$ denotes co-invariants under the co-adjoint action of $\mathfrak{g}$, and $\hat{U}(\mathfrak{g})$ is the degree completed universal enveloping algebra of $\mathfrak{g}$. (The tensor interpretation map map can be defined on all of $\mathcal{A}^{sw}$ but the target space is more complicated, and $\mathcal{A}^{sw}(\vdots\,\uparrow)$ is enough for our purposes.)

Before defining the map T, we describe the elements in $\mathcal{A}^{sw}(\vdots\,\uparrow)$ in more detail. For even more detail see [BD2, Section3].

4.1. **Elements of $\mathcal{A}^{sw}(\vdots\,\uparrow)$.** The space $\mathcal{A}^{sw}(\vdots\,\uparrow)$ is linearly generated by a relatively "easy" set of arrow diagrams, modulo the relations $\overrightarrow{STU}$ (including TC) and CP. Recall that $\overrightarrow{IHX}$ and $\overrightarrow{AS}$ are implied by $\overrightarrow{STU}$, note that RI is implied by CP here, and $\overrightarrow{VI}$ is not relevant as there are no vertices. With this in mind, a linear generator of $\mathcal{A}^{sw}(\vdots\,\uparrow)$ has the following properties:

(1) Only arrow heads lie on red strings; this is true for every element of $\mathcal{A}^{sw}(\vdots\,\uparrow)$ by definition.

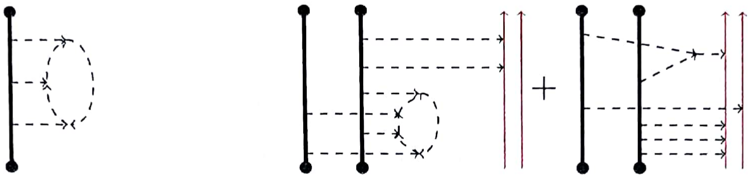


FIGURE 18. An example wheel with three spokes, and an example element in $\mathcal{A}^{sw}(\uparrow\uparrow\uparrow\uparrow)$ with $k_1 = k_2 = 2$.

(2) Only arrow tails lie on capped black strands: if an arrow diagram has an arrow head on a capped strand, one can use iterated $\overrightarrow{STU}$'s and finally CP to express it as a linear combination of arrow diagrams that don't.

(3) When the skeleton is removed, any connected component of the uni-trivalent arrow graph belongs to one of the following two classes:
   (a) *Wheels*: an oriented cycle of arrows with a finite number of incoming arrows, or "spokes", with all tails lying on capped strands. (The 2-in-1-out property implies that all univalent ends must be arrow tails.) See Figure 18 for an example. Note that reversing the orientation of an even wheel yields an equivalent arrow diagram, while reversing the orientation of an odd wheel produces its negative. Hence, from now on we assume all wheels are oriented clockwise.
   (b) *Binary trees* oriented toward a single head, with all tails on caped black strands and the head on a red string. A special case is a single arrow. See Figure 18 for examples.

4.2. **The Tensor Intpretation Map T.** For a fixed finite dimensional Lie algebra $\mathfrak{g}$, we now define the *tensor interpretation* map $T : \mathcal{A}^{sw}(\uparrow\uparrow\uparrow\uparrow) \to \hat{S}(\mathfrak{g}_{\mathfrak{g}}^*)^{\otimes k_1} \otimes \hat{U}(\mathfrak{g})^{\otimes k_2}$. The idea is that trivalent arrow vertices "represent" the Lie bracket in $\mathfrak{g}$, and the relations in $\mathcal{A}^{sw}$ translate to Lie algebra axioms and enveloping algebra or symmetric algebra relations.

Denote the structure tensor of the Lie bracket of $\mathfrak{g}$ by $[,]_{\mathfrak{g}} \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. Given a basis $\{b_1, \cdots, b_m\}$ for $\mathfrak{g}$ and the dual basis $\{b_1^*, \cdots, b_m^*\}$ for $\mathfrak{g}^*$, write $[b_i, b_j] = \sum_{k=1}^m c_{i,j,k} b_k$ for structure constants $c_{i,j,k} \in \mathbb{K}$. Then

$$[,]_{\mathfrak{g}} = \sum_{i=1}^{n} c_{i,j,k}\, b_i^* \otimes b_j^* \otimes b_k.$$

Similarly, let $\mathrm{id} \in \mathfrak{g}^* \otimes \mathfrak{g}$ denote the identity map, given by $\mathrm{id} = \sum_{i=1}^m b_i^* \otimes b_i$.

For an arrow diagram D we define $T(D)$ as follows, and as shown in Figure 19:

(1) Place a copy of $\mathrm{id} \in \mathfrak{g}^* \otimes \mathfrak{g}$ on every arrow from a black capped strand to a red strand.
(2) Whenever two trivalent arrow vertices share an edge, divide the edge in half so each vertex can be viewed as having three edges "all to themselves".
(3) Place the structure tensor of the bracket $[,]_{\mathfrak{g}}$ on all trivalent arrow vertices.
(4) The edges that were divided in Step (2) now have an element of $\mathfrak{g}^*$ meeting an element of $\mathfrak{g}$. Contract these by evaluating the element of $\mathfrak{g}^*$ on the element of $\mathfrak{g}$ to get a constant coefficient. Multiply the constants together. This is illustrated on examples in Figures 20 and 21.
(5) What remains is a linear combination of diagrams with elements of $\mathfrak{g}^*$ along the black capped strands and elements of $\mathfrak{g}$ along the red strings. Multiplying along the orientation of the strands produces an element in the tensor algebra $\mathcal{T}\mathfrak{g}^{*\otimes k_1} \otimes \mathcal{T}\mathfrak{g}^{\otimes k_2}$.

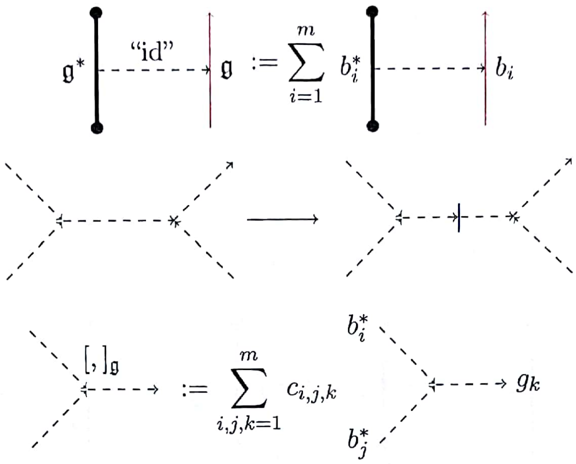
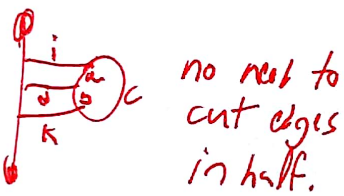


FIGURE 19. Steps (1), (2) and (3) in computing T.



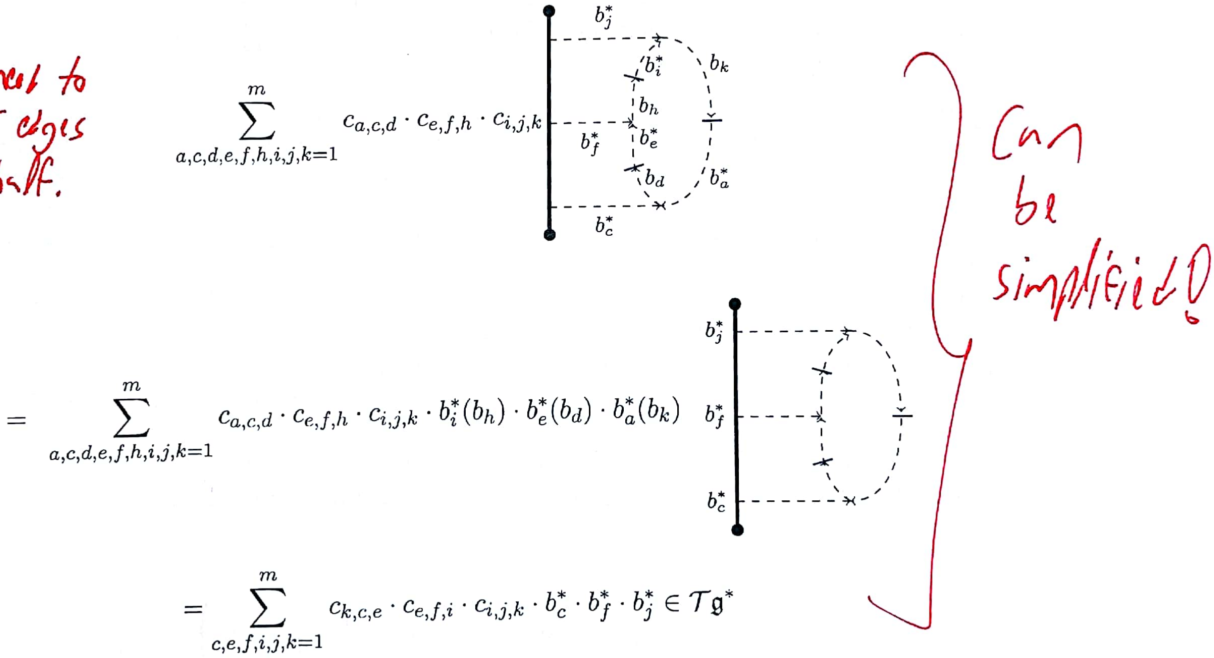


FIGURE 20. Example computation of T for a wheel with three spokes.

Example 4.1. See Figures 20 and 21 for sample computations of T for two arrow diagrams.

Proposition 4.2. *T descends to a well defined map on $\mathcal{A}^{sw}(\uparrow\uparrow\Uparrow) \to (\hat{S}\mathfrak{g}^*)_{\mathfrak{g}}^{\otimes k_1} \otimes (\hat{U}\mathfrak{g})^{\otimes k_2}$, where $(S\mathfrak{g}^*)_{\mathfrak{g}}$ denotes invariants under the co-adjoint action.*

to

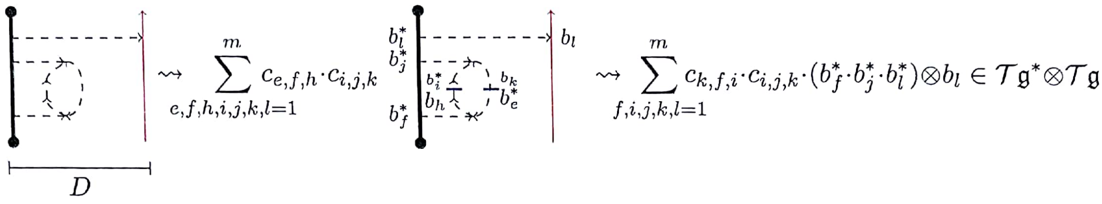
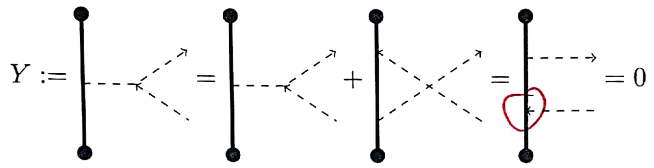


FIGURE 21. An example for computing T.



here and elsewhere: very hard to see the arrow heads.

FIGURE 22. A single trivalent vertex on a twice-capped strand is zero.

Proof. Given and arrow diagram D, the algorithm above gives an element of $(\mathcal{T}\mathfrak{g}^*)^{\otimes k1} \otimes (\mathcal{T}\mathfrak{g})^{\otimes k_2}$. We need to show that, modulo the relations of $\mathcal{A}^{sw}(\text{↑↑↑})$, we obtain a well-defined map into $(\hat{S}\mathfrak{g}^*)_{\mathfrak{g}}^{\otimes k_1} \otimes (\hat{U}\mathfrak{g})^{\otimes k_2}$.

Along the red strings, the $\overrightarrow{STU}$ relation translates to the defining relation $[g, h] = gh - hg$ of $U\mathfrak{g}$. Along the capped black strands, there are two further CP relations at the caps.

Due to the first CP relation only tails lie on the capped strands, and the TC relation then translates to the defining relation $\phi\psi = \psi\phi$ of $S\mathfrak{g}^*$. We need to analyze the effect of the second CP relation. First, we assume that only one arrow tail ends on a given capped black strand.

For diagrams which have a single arrow tail ending on a double-capped strand, the second cap relation is equivalent to stating that an arrow diagram where a single trivalent arrow vertex is attached to a capped strand is zero in $\mathcal{A}^{sw}(\text{↑↑↑})$, as shown in Figure 22.

Using the basis $\{b_1, \cdots b_m\}$ and $\{b_1^*, \cdots b_m^*\}$ as before, apply T to the arrow diagram Y of Figure 22. We only know part of the diagram Y, but we can compute the corresponding part of the image. In particular, the structure tensor $\sum_{i,j,k}$ is placed on the single trivalent vertex, the index j does not appear anywhere else, and there are no other factors on the shown capped strand:

$$T(Y) = \sum_{i,j,k} c_{i,j,k} \cdot \underline{\hspace{3em}} \cdot b_j^* \otimes \underline{\hspace{3em}} = \left(\sum_j c_{i,j,k} b_j^* \right) \otimes \sum_{i,k,\ldots} \underline{\hspace{3em}}$$

Since $Y = 0$ and hence $T(Y) = 0$ for all such diagrams Y, we need

$$\sum_j c_{i,j,k} b_j^* = 0 \text{ for all } i, k.$$

A short calculation shows that this is precisely the defining relation for the co-invariant space for the co-adjoint action. In other words the first tensor factor of $T(Y)$ has a well-defined value in $(\mathfrak{g}^*)_{\mathfrak{g}}$.