



June 20, 2023

Dror Bar-Natan – 2023 Progress Report for the Chu Family Foundation

Dear Members of the Chu Family,

As requested when the grant was offered, this is a report on my research progress over the first year of being supported by the Chu Family Foundation.

I did not make progress with what might have been my primary goal for the year – a writeup of my results with Roland van der Veen on the polynomial-time computability of the full portfolio of operations associated with the solvable approximation of quantum groups, and the polynomial-time computability of the associated knot and tangle invariants. I remain stuck in the same pedagogical place as a year ago: the computations work, but they take “two step” procedure which is hard to motivate. And so I find myself having to introduce ugliness into what is otherwise a very beautiful picture, and over the last year I kept preferring to wait for a better language to arise. It didn't, and perhaps it is time to give up and tolerate a bit of inelegance.

I did make progress, along with several collaborators, on several other aspects of the same project and on several other related and unrelated projects:

- Along with Roland van der Veen, we finished the writeup of a very simple introduction to ρ_1 , the simplest of the family of poly-time-computable knot invariants that are a part of our “main project” as above. See our paper “[A Perturbed Alexander Invariant](#)” ([arXiv:2206.12298](#), to appear in *Quantum Topology*). I'm very proud of that paper! The results are as elegant as they can be, completely hiding the hard work that went into producing them.
- Also along with Roland van der Veen, we've commenced a study of the symmetry properties of ρ_1 . This was supposed to be easy but turned out to be surprisingly difficult. It seems that we first need to understand the symmetry properties of the “Green function” $g_{\alpha\beta}$, which is an Alexander-level (and hence “classical”) quantity. But it turns out that the palindromicity property of the Alexander polynomial is understood mostly from a “Seifert” perspective and not from the edge-edge perspective that is necessary for $g_{\alpha\beta}$, and thus we need to re-understand bits of the Alexander theory. We've made progress but we are not done.
- Along with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich, we've constructed a homomorphic expansion for the Goldman-Turaev Lie bi-algebra of curves in a punctured plane using the Kontsevich integral for tangles in a “Pole Dancing Studio” – a room with a few vertical lines (the “poles”) removed. A paper on the subject is near completion.



- Along with Zsuzsanna Dancso and Tamara Hogan, we are working on a continuation paper, in which we hope to explain how the Kashiwara-Vergne equations (first related with the Goldman-Turaev Lie bialgebras by Alekseev, Kawazumi, Kuno, and Naef) arise when trying to construct an expansion for “emergent” tangles in a pole dancing studio.
- Along with Jessica Liu we’ve found and implemented (on a computer) extensions of the Tristram-Levine and Kashaev signatures of knots to tangles, using a novel “pushforward” operation for quadratic forms (I’m very proud of it!). This project was detailed in a talk I gave in Providence in May 2023 (available at <http://drorbn.net/icerm23>) and is in initial phases of writeup for publication.
- Along with Itai Bar-Natan (my son), Iva Halacheva, and Nancy Scherich, we have been able to “square root” the best-known computation time for finite type invariants, from roughly n^d to roughly $n^{d/2}$, for invariants of type d evaluated on n -crossing knots. I am very much surprised by this result – had anyone told me about it two years ago, I would not have believed it. A paper on the subject is near completion.

Over the last year I have acknowledged support from the Chu Family Foundation on two accepted papers (“[Over Then Under Tangles](#)”, with Zsuzsanna Dancso and Roland van der Veen, [arXiv:2007.09828](https://arxiv.org/abs/2007.09828), to appear in the *Journal of Knot Theory and its Ramifications* and “[A Perturbed Alexander Invariant](#)”, with Roland van der Veen, [arXiv:2206.12298](https://arxiv.org/abs/2206.12298), to appear in *Quantum Topology*) and in 6 invited lectures (in [Les Diablerets](#), [Oaxaca](#), [Los Angeles](#), [Toronto](#), [Providence](#), and [Ottawa](#)).

I wish to thank the Chu Family Foundation for their support, which enabled what had been an excellent year, and I’m looking forward to the second year of support and hoping that it will be just as good or maybe better.

Sincerely,

Dror Bar-Natan.

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