Pensieve header: A Perturbed Alexander Invariant, with formulas from PABI-InBack.nb.

Programs

In[ ]:=
```
Once[<< KnotTheory`];
```

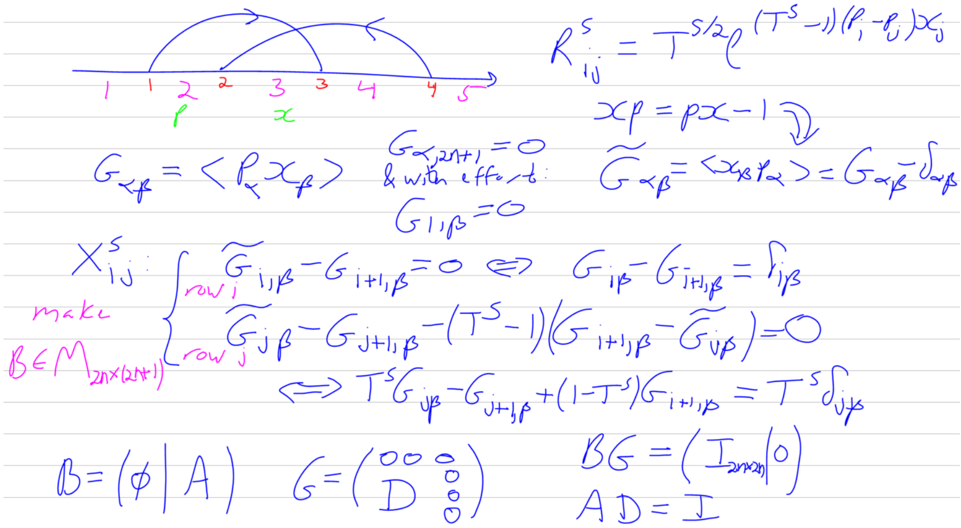
```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[ ]:=
```
PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
   X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
   X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
   X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
   X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
   X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
   X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
   X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
   X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
   X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
   X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
   X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
```

In[ ]:=
```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
    xs and a length 2n list of rotation numbers rots. Crossing
    sites are indexed 1 through 2n, and rots⟦k⟧ is the rotation
    between site k-1 and site k. RVK is also a casting operator
    converting to the RVK presentation from other knot presentations.";
```

In[ ]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {1}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x⟦4⟧, x⟦1⟧]  PositiveQ@x
                           Xm[x⟦2⟧, x⟦1⟧]  True        ];
   For[k = 1, k ≤ 2 n, ++k,
    If[FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
            Xp[k, l_] | Xm[l_, k] :> {l + 1, k + 1, -l},
            Xp[l_, k] | Xm[k, l_] :> (++rots⟦l⟧; {-l, k + 1, l + 1}),
            _Xp | _Xm :> {}
           }), {1}],
     Cases[front, k | -k] /. {k, -k} :> --rots⟦k⟧;
    ]
   ];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

Fast ρ_1




```
In[]:= OldPAI[K_] := Module[{Cs, Rs, n, B, A, c, s, i, j, Δ, G, g, p1},
  {Cs, Rs} = List @@ RVK[K]; n = Length[Cs];
  B = Table[0, 2 n, 2 n + 1];
  Do[s = If[Head[c] === Xp, 1, -1]; {i, j} = List @@ c;
   B[[{i, j}, {i, i + 1, j, j + 1}]] = (1   -1      0   0
                                        0   T^s - 1  1  -T^s),
   {c, Cs}];
  Δ = Det[A = B[[All, 2 ;;]]];
  G = Prepend[Table[0, 2 n]][Inverse[A]]; g_{α_,β_} := G[[α, β]];
  p1 = Plus[
    Sum[s = If[Head[c] === Xp, 1, -1]; {i, j} = List @@ c; Δ *= T^(-s/2);
     s (2 g_ii g_ij - g_ii g_jj - g_ij g_ji + (1 - T^s) (g_ij^2 - g_ij g_jj)), {c, Cs}],
    Sum[Δ *= T^(Rs[[k]]/2); Rs[[k]] g_kk, {k, 2 n}] ];
  Factor@{Δ, T (Δ^2 p1 - T Δ ∂_T Δ)/(T - 1)^2} ];
```

pdf

In[*]:=
```
PAI[K_] := Module[{Cs, r, n, B, A, c, s, i, j, Δ, G, g, p1},
  {Cs, r} = List @@ RVK[K] /. {Xp[i_, j_] :> {+1, i, j}, Xm[i_, j_] :> {-1, i, j}};
  n = Length[Cs];
  B = Table[0, 2 n, 2 n + 1];
  Do[{s, i, j} = c;
   B[[{i, j}, {i, i + 1, j, j + 1}]] = ( 1   -1    0   0
                                         0  T^s-1  1  -T^s ), {c, Cs}];
  A = B[[All, 2 ;;]];
  Δ = T^((Total[r]-Total[First/@Cs])/2) Det[A];
  G = Prepend[Table[0, 2 n]][Inverse[A]]; g_{α_,β_} := G[[α, β]];
  p1 = Sum[{s, i, j} = c;
    s (2 g_ii g_ij - g_ii g_jj - g_ij g_ji + (1 - T^s) (g_ij^2 - g_ij g_jj) - g_ij + g_jj - 1 / 2), {c, Cs}];
  p1 += Sum[r[[k]] (g_kk - 1 / 2), {k, 2 n}] ;
  Factor@{Δ, T Δ^2 p1 / (T - 1)^2}];
```

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In[*]:= `Table[Expand@PAI[K], {K, AllKnots[{3, 6}]}] // Column`

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Out[*]=

$\left\{-1+\frac{1}{T}+T,\ \frac{1}{T}+T\right\}$

$\left\{3-\frac{1}{T}-T,\ 0\right\}$

$\left\{1+\frac{1}{T^2}-\frac{1}{T}-T+T^2,\ \frac{2}{T^3}+\frac{3}{T}+3T+2T^3\right\}$

$\left\{-3+\frac{2}{T}+2T,\ -4+\frac{5}{T}+5T\right\}$

$\left\{5-\frac{2}{T}-2T,\ -4+\frac{1}{T}+T\right\}$

$\left\{-3-\frac{1}{T^2}+\frac{3}{T}+3T-T^2,\ -4+\frac{1}{T^3}-\frac{4}{T^2}+\frac{4}{T}+4T-4T^2+T^3\right\}$

$\left\{5+\frac{1}{T^2}-\frac{3}{T}-3T+T^2,\ 0\right\}$

In[*]:= `Sum[Simplify[Alexander[K][T] == PAI[K][[1]]], {K, AllKnots[{3, 7}]}]`

Out[*]= 14 True

In[*]:= `Timing@Sum[Simplify[Alexander[K][T] == PAI[K][[1]]], {K, AllKnots[{3, 10}]}]`

Out[*]= {139.063, 249 True}

In[*]:=
```
K = GST48;
Factor@Alexander[K][T]
```

Out[*]= $-\frac{\left(-1+2T-T^2-T^3+2T^4-T^5+T^8\right)\left(-1+T^3-2T^4+T^5+T^6-2T^7+T^8\right)}{T^8}$

pdf

In[∘]:=
```
Timing@PAI[GST48]
```

pdf

Out[∘]=

$$\left\{54.875,\ \left\{-\frac{\left(-1+2\,T-T^2-T^3+2\,T^4-T^5+T^8\right)\left(-1+T^3-2\,T^4+T^5+T^6-2\,T^7+T^8\right)}{T^8},\right.\right.$$

$$\frac{1}{T^{15}}\left(5-18\,T+33\,T^2-32\,T^3+2\,T^4+42\,T^5-62\,T^6-8\,T^7+166\,T^8-242\,T^9+108\,T^{10}+132\,T^{11}-226\,T^{12}+148\,T^{13}-11\,T^{14}-36\,T^{15}-11\,T^{16}+148\,T^{17}-226\,T^{18}+132\,T^{19}+108\,T^{20}-242\,T^{21}+166\,T^{22}-8\,T^{23}-62\,T^{24}+42\,T^{25}+2\,T^{26}-32\,T^{27}+33\,T^{28}-18\,T^{29}+5\,T^{30}\right)\Big\}\Big\}$$

In[∘]:=
```
Alex2[K_] := Module[{Cs, r, n, rot, w, B, A, c, s, i, j, Δ, G, g, a2},
   {Cs, r} = List @@ RVK[K] /. {Xp[i_, j_] :> {+1, i, j}, Xm[i_, j_] :> {-1, i, j}};
   n = Length[Cs]; rot = Total[r]; w = Total[First /@ Cs];
   B = Table[0, 2 n, 2 n + 1];
   Do[{s, i, j} = c;
    B[[{i, j}, {i, i + 1, j, j + 1}]] = {{1, -1, 0, 0}, {0, T^s - 1, 1, -T^s}}, {c, Cs}];
   Δ = T^((rot - w)/2) Det[A = B[[All, 2 ;;]]];
   G = Prepend[Table[0, 2 n]][Inverse[A]]; g_{α_, β_} := G[[α, β]];
   a2 = Sum[{s, i, j} = c; s (g_{ij} - g_{jj}), {c, Cs}] + (rot + w) / 2;
   Expand@Together@{Δ, T ∂_T Δ, Δ a2} ];
MatrixForm[Alex2 /@ AllKnots[{3, 8}]]
```

Out[∘]//MatrixForm=

$-1+\frac{1}{T}+T$	$-\frac{1}{T}+T$	$-\frac{1}{T}+T$
$3-\frac{1}{T}-T$	$\frac{1}{T}-T$	$\frac{1}{T}-T$
$1+\frac{1}{T^2}-\frac{1}{T}-T+T^2$	$-\frac{2}{T^2}+\frac{1}{T}-T+2\,T^2$	$-\frac{2}{T^2}+\frac{1}{T}-T+2\,T^2$
$-3+\frac{2}{T}+2\,T$	$-\frac{2}{T}+2\,T$	$-\frac{2}{T}+2\,T$
$5-\frac{2}{T}-2\,T$	$\frac{2}{T}-2\,T$	$\frac{2}{T}-2\,T$
$-3-\frac{1}{T^2}+\frac{3}{T}+3\,T-T^2$	$\frac{2}{T^2}-\frac{3}{T}+3\,T-2\,T^2$	$\frac{2}{T^2}-\frac{3}{T}+3\,T-2\,T^2$
$5+\frac{1}{T^2}-\frac{3}{T}-3\,T+T^2$	$-\frac{2}{T^2}+\frac{3}{T}-3\,T+2\,T^2$	$-\frac{2}{T^2}+\frac{3}{T}-3\,T+2\,T^2$
$-1+\frac{1}{T^3}-\frac{1}{T^2}+\frac{1}{T}+T-T^2+T^3$	$-\frac{3}{T^3}+\frac{2}{T^2}-\frac{1}{T}+T-2\,T^2+3\,T^3$	$-\frac{3}{T^3}+\frac{2}{T^2}-\frac{1}{T}+T-2\,T^2+3\,T^3$
$-5+\frac{3}{T}+3\,T$	$-\frac{3}{T}+3\,T$	$-\frac{3}{T}+3\,T$
$3+\frac{2}{T^2}-\frac{3}{T}-3\,T+2\,T^2$	$-\frac{4}{T^2}+\frac{3}{T}-3\,T+4\,T^2$	$-\frac{4}{T^2}+\frac{3}{T}-3\,T+4\,T^2$
$-7+\frac{4}{T}+4\,T$	$-\frac{4}{T}+4\,T$	$-\frac{4}{T}+4\,T$
$5+\frac{2}{T^2}-\frac{4}{T}-4\,T+2\,T^2$	$-\frac{4}{T^2}+\frac{4}{T}-4\,T+4\,T^2$	$-\frac{4}{T^2}+\frac{4}{T}-4\,T+4\,T^2$
$-7-\frac{1}{T^2}+\frac{5}{T}+5\,T-T^2$	$\frac{2}{T^2}-\frac{5}{T}+5\,T-2\,T^2$	$\frac{2}{T^2}-\frac{5}{T}+5\,T-2\,T^2$
$9+\frac{1}{T^2}-\frac{5}{T}-5\,T+T^2$	$-\frac{2}{T^2}+\frac{5}{T}-5\,T+2\,T^2$	$-\frac{2}{T^2}+\frac{5}{T}-5\,T+2\,T^2$
$7-\frac{3}{T}-3\,T$	$\frac{3}{T}-3\,T$	$\frac{3}{T}-3\,T$
$3-\frac{1}{T^3}+\frac{3}{T^2}-\frac{3}{T}-3\,T+3\,T^2-T^3$	$\frac{3}{T^3}-\frac{6}{T^2}+\frac{3}{T}-3\,T+6\,T^2-3\,T^3$	$\frac{3}{T^3}-\frac{6}{T^2}+\frac{3}{T}-3\,T+6\,T^2-3\,T^3$

$$
\begin{pmatrix}
9-\frac{4}{T}-4T & \frac{4}{T}-4T & \frac{4}{T}-4T \\
-5-\frac{2}{T^2}+\frac{5}{T}+5T-2T^2 & \frac{4}{T^2}-\frac{5}{T}+5T-4T^2 & \frac{4}{T^2}-\frac{5}{T}+5T-4T^2 \\
5-\frac{1}{T^3}+\frac{3}{T^2}-\frac{4}{T}-4T+3T^2-T^3 & \frac{3}{T^3}-\frac{6}{T^2}+\frac{4}{T}-4T+6T^2-3T^3 & \frac{3}{T^3}-\frac{6}{T^2}+\frac{4}{T}-4T+6T^2-3T^3 \\
-7-\frac{2}{T^2}+\frac{6}{T}+6T-2T^2 & \frac{4}{T^2}-\frac{6}{T}+6T-4T^2 & \frac{4}{T^2}-\frac{6}{T}+6T-4T^2 \\
-5+\frac{1}{T^3}-\frac{3}{T^2}+\frac{5}{T}+5T-3T^2+T^3 & -\frac{3}{T^3}+\frac{6}{T^2}-\frac{5}{T}+5T-6T^2+3T^3 & -\frac{3}{T^3}+\frac{6}{T^2}-\frac{5}{T}+5T-6T^2+3T^3 \\
9+\frac{2}{T^2}-\frac{6}{T}-6T+2T^2 & -\frac{4}{T^2}+\frac{6}{T}-6T+4T^2 & -\frac{4}{T^2}+\frac{6}{T}-6T+4T^2 \\
7-\frac{1}{T^3}+\frac{3}{T^2}-\frac{5}{T}-5T+3T^2-T^3 & \frac{3}{T^3}-\frac{6}{T^2}+\frac{5}{T}-5T+6T^2-3T^3 & \frac{3}{T^3}-\frac{6}{T^2}+\frac{5}{T}-5T+6T^2-3T^3 \\
-7+\frac{1}{T^3}-\frac{3}{T^2}+\frac{6}{T}+6T-3T^2+T^3 & -\frac{3}{T^3}+\frac{6}{T^2}-\frac{6}{T}+6T-6T^2+3T^3 & -\frac{3}{T^3}+\frac{6}{T^2}-\frac{6}{T}+6T-6T^2+3T^3 \\
-9-\frac{2}{T^2}+\frac{7}{T}+7T-2T^2 & \frac{4}{T^2}-\frac{7}{T}+7T-4T^2 & \frac{4}{T^2}-\frac{7}{T}+7T-4T^2 \\
13+\frac{1}{T^2}-\frac{7}{T}-7T+T^2 & -\frac{2}{T^2}+\frac{7}{T}-7T+2T^2 & -\frac{2}{T^2}+\frac{7}{T}-7T+2T^2 \\
11+\frac{2}{T^2}-\frac{7}{T}-7T+2T^2 & -\frac{4}{T^2}+\frac{7}{T}-7T+4T^2 & -\frac{4}{T^2}+\frac{7}{T}-7T+4T^2 \\
-11-\frac{2}{T^2}+\frac{8}{T}+8T-2T^2 & \frac{4}{T^2}-\frac{8}{T}+8T-4T^2 & \frac{4}{T^2}-\frac{8}{T}+8T-4T^2 \\
11+\frac{3}{T^2}-\frac{8}{T}-8T+3T^2 & -\frac{6}{T^2}+\frac{8}{T}-8T+6T^2 & -\frac{6}{T^2}+\frac{8}{T}-8T+6T^2 \\
-9+\frac{1}{T^3}-\frac{4}{T^2}+\frac{8}{T}+8T-4T^2+T^3 & -\frac{3}{T^3}+\frac{8}{T^2}-\frac{8}{T}+8T-8T^2+3T^3 & -\frac{3}{T^3}+\frac{8}{T^2}-\frac{8}{T}+8T-8T^2+3T^3 \\
11-\frac{1}{T^3}+\frac{4}{T^2}-\frac{8}{T}-8T+4T^2-T^3 & \frac{3}{T^3}-\frac{8}{T^2}+\frac{8}{T}-8T+8T^2-3T^3 & \frac{3}{T^3}-\frac{8}{T^2}+\frac{8}{T}-8T+8T^2-3T^3 \\
13-\frac{1}{T^3}+\frac{5}{T^2}-\frac{10}{T}-10T+5T^2-T^3 & \frac{3}{T^3}-\frac{10}{T^2}+\frac{10}{T}-10T+10T^2-3T^3 & \frac{3}{T^3}-\frac{10}{T^2}+\frac{10}{T}-10T+10T^2-3T^3 \\
1+\frac{1}{T^3}-\frac{1}{T^2}-T^2+T^3 & -\frac{3}{T^3}+\frac{2}{T^2}-2T^2+3T^3 & -\frac{3}{T^3}+\frac{2}{T^2}-2T^2+3T^3 \\
3+\frac{1}{T^2}-\frac{2}{T}-2T+T^2 & -\frac{2}{T^2}+\frac{2}{T}-2T+2T^2 & -\frac{2}{T^2}+\frac{2}{T}-2T+2T^2 \\
-5-\frac{1}{T^2}+\frac{4}{T}+4T-T^2 & \frac{2}{T^2}-\frac{4}{T}+4T-2T^2 & \frac{2}{T^2}-\frac{4}{T}+4T-2T^2
\end{pmatrix}
$$