Pensieve header: The “Heisenberg” invariant at $k=1$.

```
In[∘]:= PP_ = Identity; $k = 0; γ = γ; ℏ;
```

```
In[∘]:= Once[<< KnotTheory`];
```

The “Speedy” Engine

Internal Utilities

Canonical Form:

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
          Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CCF[x]];
        CF[ℰ_List] := CF /@ ℰ;
        CF[sd_SeriesData] := MapAt[CF, sd, 3];
        CF[ℰ_] := Module[
           {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ) _, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
           Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]
          ];
        CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
In[∘]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
          CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
        𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
        𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
In[∘]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
        {τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

In[∘]:=
```
U2l = {B_{i_}^{p_.} :> e^{-p ħ ɣ b_i}, B^{p_.} :> e^{-p ħ ɣ b}, T_{i_}^{p_.} :> e^{-p ħ t_i}, T^{p_.} :> e^{-p ħ t}, 𝒜_{i_}^{p_.} :> e^{p ɣ α_i}, 𝒜^{p_.} :> e^{p ɣ α}};
l2U = {e^{c_. b_i_+d_.} :> B_i^{-c/(ħ ɣ)} e^d, e^{c_. b+d_.} :> B^{-c/(ħ ɣ)} e^d,
   e^{c_. t_i_+d_.} :> T_i^{-c/ħ} e^d, e^{c_. t+d_.} :> T^{-c/ħ} e^d,
   e^{c_. α_i_+d_.} :> 𝒜_i^{c/ɣ} e^d, e^{c_. α+d_.} :> 𝒜^{c/ɣ} e^d,
   e^{ℰ_} :> e^{Expand@ℰ}};
```

Derivatives in the presence of exponentiated variables:

In[∘]:=
```
D_b[f_] := ∂_b f - ħ ɣ B ∂_B f; D_{b_i_}[f_] := ∂_{b_i} f - ħ ɣ B_i ∂_{B_i} f;
D_t[f_] := ∂_t f - ħ T ∂_T f; D_{t_i_}[f_] := ∂_{t_i} f - ħ T_i ∂_{T_i} f;
D_α[f_] := ∂_α f + ɣ 𝒜 ∂_𝒜 f; D_{α_i_}[f_] := ∂_{α_i} f + ɣ 𝒜_i ∂_{𝒜_i} f;
D_{v_}[f_] := ∂_v f; D_{{v_,0}}[f_] := f; D_{{}}[f_] := f; D_{{v_,n_Integer}}[f_] := D_v[D_{{v,n-1}}[f]];
D_{{l_List,ls___}}[f_] := D_{{ls}}[D_l[f]];
```

Finite Zips:

In[∘]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{{}}[P_] := P;
Zip_{ζs_}[Ps_List] := Zip_{ζs} /@ Ps;
Zip_{{ζ_,ζs___}}[P_] :=
 (collect[P // Zip_{{ζs}}, ζ] /. f_. ζ^{d_.} :> (D_{{ζ*,d}}[f])) /. ζ* → 0 /.
  ((ζ* /. {b → B, t → T, α → 𝒜}) → 1)
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = \mathbb{e}^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}|\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|\mathbb{e}^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

In[∘]:=
```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = P\mathbb{e}^{L+Q}$. Such zips regard all of $P\mathbb{e}^Q$ as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

In[∘]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
   zs = Table[ζ*, {ζ, ζs}];
   Zs = zs /. {b → B, t → T, α → 𝒜};
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   Zrule = Join[zrule,
     zrule /. r_Rule :→ ((U = r⟦1⟧ /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
   ζrule = Thread[ζs → ζs + ηs.lt];
   Q1 = Q /. (Zrule ⋃ ζrule);
   EEQ[ps___] := EEQ[ps] =
     (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
   CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
     Det[lt] (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
         Derivative[ps___][EQ][___] :→ EEQ[ps] /. _EQ → 1) ] ];
```

In[∘]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Join@@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} // QZip_{Join@@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

𝔼 morphisms with domain and range.

In[∘]:=

```
B_is_List[𝔼_d1_→r1_[L1_, Q1_, P1_], 𝔼_d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ B_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] // 𝔼_d2_→r2_[L2_, Q2_, P2_] :=
  B_r1⋂d2[𝔼_d1→r1[L1, Q1, P1], 𝔼_d2→r2[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_d1_→r1_[L1_, Q1_, P1_] 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼_(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_dr_[L_, Q_, P_]_$k_ := 𝔼_dr @@ 𝔼[L, Q, P]_$k;
𝔼_[ℰ___][i_] := {ℰ}[[i]];
R_i_,j_ := 𝔼_{}→{i,j}[0, (y_i - y_j) x_j, 1 + ϵ (x_i y_i - x_j y_i - x_j y_j) +
     ϵ^2 (Sum[r_If[#===i,0,1]&@k,If[#===i,0,1]&@l y_k x_l, {k, {i, j}}, {l, {i, j}}] +
        Sum[s_If[#===i,0,1]&@k,If[#===i,0,1]&@l,If[#===i,0,1]&@m,If[#===i,0,1]&@n y_k x_l y_m x_n,
         {k, {i, j}}, {l, {i, j}}, {m, {i, j}}, {n, {i, j}}]) + O[ϵ]^3] /.
  {r_1,0 → 0, r_1,1 → 1, r_0,0 → -1, r_0,1 → 0, s_0,0,0,0 → 1/2, s_1,0,1,0 → 0, s_1,0,0,0 → 1,
   s_0,0,1,0 → -1, s_1,1,1,1 → 1/2, s_1,1,1,0 → 1, s_1,0,1,1 → -1, s_0,0,1,1 → 1, s_0,1,0,1 → 1, s_0,1,1,0 → -1,
   s_1,0,0,1 → 1, s_1,1,0,0 → -1, s_0,0,0,1 → -1, s_0,1,0,0 → -1, s_0,1,1,1 → 1, s_1,1,0,1 → -1/2}
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

In[∘]:=

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_nisp,$k_Integer, Block[{i, j, k}, op_isp,$k = ℰ; op_nis,$k]];
      SD[op_isp, op_{is},$k]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

In[∘]:=

```
Define[m_i,j→k = 𝔼_{i,j}→{k}[0, -ξ_i η_j + (η_i + η_j) y_k + (ξ_i + ξ_j) x_k, 1]]
 (*Heisenberg multiplication*)
```

In[∘]:=
```
AllMonomials[{}, 0] = {1};
AllMonomials[{}, d_Integer] /; d > 0 := {};
AllMonomials[{v_, vs___}, d_Integer] :=
  Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
```

In[∘]:=
```
Basis[js_List, m_] := Flatten@Outer[Times,
    AllMonomials[Table[y_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 1, m}]
```

In[∘]:= `Basis[{i, j}, {2}]`

Out[∘]= $\{x_i y_i, x_j y_i, x_i y_j, x_j y_j, x_i^2 y_i^2, x_i x_j y_i^2, x_j^2 y_i^2, x_i^2 y_i y_j, x_i x_j y_i y_j, x_j^2 y_i y_j, x_i^2 y_j^2, x_i x_j y_j^2, x_j^2 y_j^2\}$

In[∘]:=
```
GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}]
```

In[∘]:= `GenericCombination[Basis[{i, j}, {2}], c]`

Out[∘]= $c_1 x_i y_i + c_2 x_j y_i + c_5 x_i^2 y_i^2 + c_6 x_i x_j y_i^2 + c_7 x_j^2 y_i^2 + c_3 x_i y_j + c_4 x_j y_j + c_8 x_i^2 y_i y_j + c_9 x_i x_j y_i y_j + c_{10} x_j^2 y_i y_j + c_{11} x_i^2 y_j^2 + c_{12} x_i x_j y_j^2 + c_{13} x_j^2 y_j^2$

The docile R-matrix and its inverse

In[∘]:=
```
Once[DRules = {};]
```

In[∘]:=
```
R_{i_,j_} := 𝔼_{{}→{i,j}}[0, (y_i - y_j) (T - 1) x_j,
    1 + ϵ GenericCombination[Basis[{i, j}, {2}], c] + O[ϵ]^2] /. DRules
R̄_{i_,j_} := 𝔼_{{}→{i,j}}[0, (y_i - y_j) (T^-1 - 1) x_j,
    1 + ϵ GenericCombination[Basis[{i, j}, {2}], d] + O[ϵ]^2] /. DRules
CC_{i_} := 𝔼_{{}→{i}}[0, 0, √T + (α + β x_i y_i + γ x_i^2 y_i^2) ϵ + O[ϵ]^2] /. DRules
C̄C̄_{i_} := 𝔼_{{}→{i}}[0, 0, (√T)^-1 + (αi + βi x_i y_i + γi x_i^2 y_i^2) ϵ + O[ϵ]^2] /. DRules
Kink_{i_} := CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→i}
K̄īn̄k̄_{i_} := CC_3 R̄_{1,2} // m_{1,3→1} // m_{1,2→i}
```

Requirements on R and CC to nail them down

In[∘]:=
```
(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})
(R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2})
CC_1 C̄C̄_2 // m_{1,2→1}
Kink_1 ≡ (C̄C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})
```

Out[∘]=

$$\in c_3 x_2 y_1 - T \in c_3 x_2 y_1 + T \in c_1 x_3 y_1 - T^2 \in c_1 x_3 y_1 + \in c_2 x_3 y_1 - T \in c_2 x_3 y_1 + \in c_4 x_3 y_1 - T \in c_4 x_3 y_1 -$$
$$2 \in c_5 x_1 x_2 y_1^2 + 2 T \in c_5 x_1 x_2 y_1^2 + \in c_8 x_2^2 y_1^2 - 2 T \in c_8 x_2^2 y_1^2 + T^2 \in c_8 x_2^2 y_1^2 + \in c_{11} x_2^2 y_1^2 -$$
$$2 T \in c_{11} x_2^2 y_1^2 + T^2 \in c_{11} x_2^2 y_1^2 + 2 T \in c_5 x_1 x_3 y_1^2 - 2 T^2 \in c_5 x_1 x_3 y_1^2 + 2 \in c_7 x_2 x_3 y_1^2 - 2 T \in c_7 x_2 x_3 y_1^2 +$$
$$\in c_9 x_2 x_3 y_1^2 - 2 T \in c_9 x_2 x_3 y_1^2 + T^2 \in c_9 x_2 x_3 y_1^2 + \in c_{12} x_2 x_3 y_1^2 - 2 T \in c_{12} x_2 x_3 y_1^2 + T^2 \in c_{12} x_2 x_3 y_1^2 +$$
$$T^2 \in c_5 x_3^2 y_1^2 - 2 T^3 \in c_5 x_3^2 y_1^2 + T^4 \in c_5 x_3^2 y_1^2 + T \in c_6 x_3^2 y_1^2 - 2 T^2 \in c_6 x_3^2 y_1^2 + T^3 \in c_6 x_3^2 y_1^2 +$$
$$\in c_7 x_3^2 y_1^2 - 4 T \in c_7 x_3^2 y_1^2 + 3 T^2 \in c_7 x_3^2 y_1^2 + \in c_{10} x_3^2 y_1^2 - 2 T \in c_{10} x_3^2 y_1^2 + T^2 \in c_{10} x_3^2 y_1^2 + \in c_{13} x_3^2 y_1^2 -$$
$$2 T \in c_{13} x_3^2 y_1^2 + T^2 \in c_{13} x_3^2 y_1^2 - \in c_3 x_1 y_2 + T \in c_3 x_1 y_2 - \in c_3 x_2 y_2 + 2 T \in c_3 x_2 y_2 - T^2 \in c_3 x_2 y_2 +$$
$$T \in c_3 x_3 y_2 - T^2 \in c_3 x_3 y_2 - \in c_8 x_1^2 y_1 y_2 + T \in c_8 x_1^2 y_1 y_2 - 2 \in c_8 x_1 x_2 y_1 y_2 + 4 T \in c_8 x_1 x_2 y_1 y_2 -$$
$$2 T^2 \in c_8 x_1 x_2 y_1 y_2 + 2 T \in c_5 x_2^2 y_1 y_2 - 2 T^2 \in c_5 x_2^2 y_1 y_2 - \in c_8 x_2^2 y_1 y_2 + 4 T \in c_8 x_2^2 y_1 y_2 -$$
$$4 T^2 \in c_8 x_2^2 y_1 y_2 + T^3 \in c_8 x_2^2 y_1 y_2 + 2 T \in c_8 x_1 x_3 y_1 y_2 - 2 T^2 \in c_8 x_1 x_3 y_1 y_2 + 2 T \in c_6 x_2 x_3 y_1 y_2 -$$
$$2 T^2 \in c_6 x_2 x_3 y_1 y_2 - \in c_9 x_2 x_3 y_1 y_2 + 4 T \in c_9 x_2 x_3 y_1 y_2 - 3 T^2 \in c_9 x_2 x_3 y_1 y_2 + 2 \in c_{10} x_2 x_3 y_1 y_2 -$$
$$2 T \in c_{10} x_2 x_3 y_1 y_2 + 2 T \in c_7 x_3^2 y_1 y_2 - 2 T^2 \in c_7 x_3^2 y_1 y_2 + T^2 \in c_8 x_3^2 y_1 y_2 - 2 T^3 \in c_8 x_3^2 y_1 y_2 +$$
$$T^4 \in c_8 x_3^2 y_1 y_2 + T \in c_9 x_3^2 y_1 y_2 - 2 T^2 \in c_9 x_3^2 y_1 y_2 + T^3 \in c_9 x_3^2 y_1 y_2 - \in c_{11} x_1^2 y_2^2 + 2 T \in c_{11} x_1^2 y_2^2 -$$
$$T^2 \in c_{11} x_1^2 y_2^2 - 2 \in c_{11} x_1 x_2 y_2^2 + 6 T \in c_{11} x_1 x_2 y_2^2 - 6 T^2 \in c_{11} x_1 x_2 y_2^2 + 2 T^3 \in c_{11} x_1 x_2 y_2^2 -$$
$$\in c_{11} x_2^2 y_2^2 + 4 T \in c_{11} x_2^2 y_2^2 - 6 T^2 \in c_{11} x_2^2 y_2^2 + 4 T^3 \in c_{11} x_2^2 y_2^2 - T^4 \in c_{11} x_2^2 y_2^2 + 2 T \in c_{11} x_1 x_3 y_2^2 -$$
$$2 T^2 \in c_{11} x_1 x_3 y_2^2 + T \in c_{12} x_1 x_3 y_2^2 - T^2 \in c_{12} x_1 x_3 y_2^2 - \in c_{12} x_2 x_3 y_2^2 + 4 T \in c_{12} x_2 x_3 y_2^2 -$$
$$4 T^2 \in c_{12} x_2 x_3 y_2^2 + T^3 \in c_{12} x_2 x_3 y_2^2 + 2 \in c_{13} x_2 x_3 y_2^2 - 2 T \in c_{13} x_2 x_3 y_2^2 + T^2 \in c_{11} x_3^2 y_2^2 -$$
$$2 T^3 \in c_{11} x_3^2 y_2^2 + T^4 \in c_{11} x_3^2 y_2^2 + T \in c_{12} x_3^2 y_2^2 - 2 T^2 \in c_{12} x_3^2 y_2^2 + T^3 \in c_{12} x_3^2 y_2^2 + \in c_3 x_1 y_3 -$$
$$T \in c_3 x_1 y_3 - T \in c_3 x_2 y_3 + T^2 \in c_3 x_2 y_3 + \in c_8 x_1^2 y_1 y_3 - T \in c_8 x_1^2 y_1 y_3 - 2 T \in c_8 x_1 x_2 y_1 y_3 +$$
$$2 T^2 \in c_8 x_1 x_2 y_1 y_3 + T^2 \in c_8 x_2^2 y_1 y_3 - T^3 \in c_8 x_2^2 y_1 y_3 + 2 T \in c_{11} x_2^2 y_1 y_3 - 2 T^2 \in c_{11} x_2^2 y_1 y_3 +$$
$$2 T \in c_{12} x_2 x_3 y_1 y_3 - 2 T^2 \in c_{12} x_2 x_3 y_1 y_3 + 2 T \in c_{13} x_3^2 y_1 y_3 - 2 T^2 \in c_{13} x_3^2 y_1 y_3 - 2 T \in c_{11} x_1^2 y_2 y_3 +$$
$$2 T^2 \in c_{11} x_1^2 y_2 y_3 - 4 T \in c_{11} x_1 x_2 y_2 y_3 + 8 T^2 \in c_{11} x_1 x_2 y_2 y_3 - 4 T^3 \in c_{11} x_1 x_2 y_2 y_3 - 2 T \in c_{11} x_2^2 y_2 y_3 +$$
$$6 T^2 \in c_{11} x_2^2 y_2 y_3 - 6 T^3 \in c_{11} x_2^2 y_2 y_3 + 2 T^4 \in c_{11} x_2^2 y_2 y_3 - 2 T \in c_{12} x_1 x_3 y_2 y_3 + 2 T^2 \in c_{12} x_1 x_3 y_2 y_3 -$$
$$2 T \in c_{12} x_2 x_3 y_2 y_3 + 4 T^2 \in c_{12} x_2 x_3 y_2 y_3 - 2 T^3 \in c_{12} x_2 x_3 y_2 y_3 - 2 T \in c_{13} x_3^2 y_2 y_3 + 2 T^2 \in c_{13} x_3^2 y_2 y_3 +$$
$$\in c_{11} x_1^2 y_3^2 - T^2 \in c_{11} x_1^2 y_3^2 - 2 T^2 \in c_{11} x_1 x_2 y_3^2 + 2 T^3 \in c_{11} x_1 x_2 y_3^2 - T^2 \in c_{11} x_2^2 y_3^2 + 2 T^3 \in c_{11} x_2^2 y_3^2 -$$
$$T^4 \in c_{11} x_2^2 y_3^2 + T \in c_{12} x_1 x_3 y_3^2 - T^2 \in c_{12} x_1 x_3 y_3^2 - T^2 \in c_{12} x_2 x_3 y_3^2 + T^3 \in c_{12} x_2 x_3 y_3^2 == 0$$

Out[•]= $\mathbb{E}_{\{\}\to\{1,2\}}\Big[0, 0, 1 + \Big((c_1 + d_1 + d_3 - T d_3)\, x_1 y_1 + \frac{(-c_1 + T c_1 + c_2 + T d_2 + T d_4 - T^2 d_4)\, x_2 y_1}{T} + (c_5 + d_5 + d_8 - T d_8 + d_{11} - 2 T d_{11} + T^2 d_{11})\, x_1^2 y_1^2 + \frac{(-2 c_5 + 2 T c_5 + c_6 + T d_6 + T d_9 - T^2 d_9 + T d_{12} - 2 T^2 d_{12} + T^3 d_{12})\, x_1 x_2 y_1^2}{T} + \frac{(c_5 - 2 T c_5 + T^2 c_5 - c_6 + T c_6 + c_7 + T^2 d_7 + T^2 d_{10} - T^3 d_{10} + T^2 d_{13} - 2 T^3 d_{13} + T^4 d_{13})\, x_2^2 y_1^2}{T^2} + (c_3 + T d_3)\, x_1 y_2 + \frac{(-c_3 + T c_3 + c_4 + T^2 d_4)\, x_2 y_2}{T} + (c_8 + T d_8 + 2 T d_{11} - 2 T^2 d_{11})\, x_1^2 y_1 y_2 + \frac{(-2 c_8 + 2 T c_8 + c_9 + T^2 d_9 + 2 T^2 d_{12} - 2 T^3 d_{12})\, x_1 x_2 y_1 y_2}{T} + \frac{(c_8 - 2 T c_8 + T^2 c_8 - c_9 + T c_9 + c_{10} + T^3 d_{10} + 2 T^3 d_{13} - 2 T^4 d_{13})\, x_2^2 y_1 y_2}{T^2} + (c_{11} + T^2 d_{11})\, x_1^2 y_2^2 + \frac{(-2 c_{11} + 2 T c_{11} + c_{12} + T^3 d_{12})\, x_1 x_2 y_2^2}{T} + \frac{(c_{11} - 2 T c_{11} + T^2 c_{11} - c_{12} + T c_{12} + c_{13} + T^4 d_{13})\, x_2^2 y_2^2}{T^2}\Big) \in + O[\in]^2\Big]$

Out[•]= $\mathbb{E}_{\{\}\to\{1\}}\Big[0, 0, 1 + \Big(\frac{\alpha}{\sqrt{T}} + \sqrt{T}\,\alpha\mathbf{i} + \frac{\beta x_1 y_1}{\sqrt{T}} + \sqrt{T}\,\beta\mathbf{i}\, x_1 y_1 + \frac{(\gamma + T\gamma\mathbf{i})\, x_1^2 y_1^2}{\sqrt{T}}\Big) \in + O[\in]^2\Big]$

Out[•]= $\frac{1}{T^3}\big(T^2 \alpha \in - T^3 \alpha\mathbf{i} \in - T \beta \in + T^2 \beta \in + 2 \gamma \in - 4 T \gamma \in + 2 T^2 \gamma \in - T^{3/2} \in c_2 + T^{5/2} \in c_3 + 2 \sqrt{T} \in c_7 - 2 T^{5/2} \in c_{11} + T \beta \in x_1 y_1 - T^4 \beta\mathbf{i} \in x_1 y_1 - 4 \gamma \in x_1 y_1 + 4 T \gamma \in x_1 y_1 + T^{5/2} \in c_1 x_1 y_1 - T^{7/2} \in c_1 x_1 y_1 + T^{3/2} \in c_2 x_1 y_1 - T^{5/2} \in c_2 x_1 y_1 + T^{5/2} \in c_3 x_1 y_1 - T^{7/2} \in c_3 x_1 y_1 + T^{3/2} \in c_4 x_1 y_1 - T^{5/2} \in c_4 x_1 y_1 - 2 T^{3/2} \in c_6 x_1 y_1 - 4 \sqrt{T} \in c_7 x_1 y_1 + 2 T^{7/2} \in c_8 x_1 y_1 - T^{3/2} \in c_9 x_1 y_1 + T^{5/2} \in c_9 x_1 y_1 - 2 \sqrt{T} \in c_{10} x_1 y_1 + 4 T^{7/2} \in c_{11} x_1 y_1 + 2 T^{5/2} \in c_{12} x_1 y_1 + \gamma \in x_1^2 y_1^2 - T^5 \gamma\mathbf{i} \in x_1^2 y_1^2 + T^{5/2} \in c_5 x_1^2 y_1^2 - T^{9/2} \in c_5 x_1^2 y_1^2 + T^{3/2} \in c_6 x_1^2 y_1^2 - T^{7/2} \in c_6 x_1^2 y_1^2 + \sqrt{T} \in c_7 x_1^2 y_1^2 - T^{5/2} \in c_7 x_1^2 y_1^2 + T^{5/2} \in c_8 x_1^2 y_1^2 - T^{9/2} \in c_8 x_1^2 y_1^2 + T^{3/2} \in c_9 x_1^2 y_1^2 - T^{7/2} \in c_9 x_1^2 y_1^2 + \sqrt{T} \in c_{10} x_1^2 y_1^2 - T^{5/2} \in c_{10} x_1^2 y_1^2 + T^{5/2} \in c_{11} x_1^2 y_1^2 - T^{9/2} \in c_{11} x_1^2 y_1^2 + T^{3/2} \in c_{12} x_1^2 y_1^2 - T^{7/2} \in c_{12} x_1^2 y_1^2 + \sqrt{T} \in c_{13} x_1^2 y_1^2 - T^{5/2} \in c_{13} x_1^2 y_1^2\big) == 0$

```
In[•]:= eqns = Join[
  Thread[CoefficientRules[Coefficient[
        (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3)〚3〛 - (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)〚
          3〛 // Normal, ϵ], {x1, x2, x3, y1, y2, y3}]〚;; , 2〛 == 0],
  Thread[CoefficientRules[Coefficient[(R1,2 R̄3,4 // m1,3→1 // m2,4→2)〚3〛 // Normal, ϵ],
      {x1, x2, y1, y2}]〚;; , 2〛 == 0],
  Thread[CoefficientRules[Coefficient[(CC1 C̄C̄2 // m1,2→1)〚3〛 // Normal, ϵ], {x1, y1}]〚
     ;; , 2〛 == 0],
  Thread[CoefficientRules[Coefficient[(Kink1)〚3〛 - (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)〚3〛 // Normal,
       ϵ], {x1, y1}]〚;; , 2〛 == 0]
 ]
```

Out[•]=

$$\{-c_8+T c_8==0,\ c_8-T c_8==0,\ -c_{11}+2 T c_{11}-T^2 c_{11}==0,\ -2 T c_{11}+2 T^2 c_{11}==0,$$
$$c_{11}-T^2 c_{11}==0,\ -2 c_5+2 T c_5==0,\ -2 c_8+4 T c_8-2 T^2 c_8==0,\ -2 T c_8+2 T^2 c_8==0,$$
$$-2 c_{11}+6 T c_{11}-6 T^2 c_{11}+2 T^3 c_{11}==0,\ -4 T c_{11}+8 T^2 c_{11}-4 T^3 c_{11}==0,\ -2 T^2 c_{11}+2 T^3 c_{11}==0,$$
$$2 T c_5-2 T^2 c_5==0,\ 2 T c_8-2 T^2 c_8==0,\ 2 T c_{11}-2 T^2 c_{11}+T c_{12}-T^2 c_{12}==0,\ -2 T c_{12}+2 T^2 c_{12}==0,$$
$$T c_{12}-T^2 c_{12}==0,\ -c_3+T c_3==0,\ c_3-T c_3==0,\ c_8-2 T c_8+T^2 c_8+c_{11}-2 T c_{11}+T^2 c_{11}==0,$$
$$2 T c_5-2 T^2 c_5-c_8+4 T c_8-4 T^2 c_8+T^3 c_8==0,\ T^2 c_8-T^3 c_8+2 T c_{11}-2 T^2 c_{11}==0,$$
$$-c_{11}+4 T c_{11}-6 T^2 c_{11}+4 T^3 c_{11}-T^4 c_{11}==0,\ -2 T c_{11}+6 T^2 c_{11}-6 T^3 c_{11}+2 T^4 c_{11}==0,$$
$$-T^2 c_{11}+2 T^3 c_{11}-T^4 c_{11}==0,\ 2 c_7-2 T c_7+c_9-2 T c_9+T^2 c_9+c_{12}-2 T c_{12}+T^2 c_{12}==0,$$
$$2 T c_6-2 T^2 c_6-c_9+4 T c_9-3 T^2 c_9+2 c_{10}-2 T c_{10}==0,\ 2 T c_{12}-2 T^2 c_{12}==0,$$
$$-c_{12}+4 T c_{12}-4 T^2 c_{12}+T^3 c_{12}+2 c_{13}-2 T c_{13}==0,\ -2 T c_{12}+4 T^2 c_{12}-2 T^3 c_{12}==0,$$
$$-T^2 c_{12}+T^3 c_{12}==0,\ c_3-T c_3==0,\ -c_3+2 T c_3-T^2 c_3==0,\ -T c_3+T^2 c_3==0,$$
$$T^2 c_5-2 T^3 c_5+T^4 c_5+T c_6-2 T^2 c_6+T^3 c_6+c_7-4 T c_7+3 T^2 c_7+c_{10}-2 T c_{10}+T^2 c_{10}+c_{13}-$$
$$2 T c_{13}+T^2 c_{13}==0,\ 2 T c_7-2 T^2 c_7+T^2 c_8-2 T^3 c_8+T^4 c_8+T c_9-2 T^2 c_9+T^3 c_9==0,$$
$$2 T c_{13}-2 T^2 c_{13}==0,\ T^2 c_{11}-2 T^3 c_{11}+T^4 c_{11}+T c_{12}-2 T^2 c_{12}+T^3 c_{12}==0,$$
$$-2 T c_{13}+2 T^2 c_{13}==0,\ T c_1-T^2 c_1+c_2-T c_2+c_4-T c_4==0,\ T c_3-T^2 c_3==0,$$
$$c_5+d_5+d_8-T d_8+d_{11}-2 T d_{11}+T^2 d_{11}==0,\ c_8+T d_8+2 T d_{11}-2 T^2 d_{11}==0,$$
$$c_{11}+T^2 d_{11}==0,\ 2 c_5-\frac{2 c_5}{T}+\frac{c_6}{T}+d_6+d_9-T d_9+d_{12}-2 T d_{12}+T^2 d_{12}==0,$$
$$2 c_8-\frac{2 c_8}{T}+\frac{c_9}{T}+T d_9+2 T d_{12}-2 T^2 d_{12}==0,\ 2 c_{11}-\frac{2 c_{11}}{T}+\frac{c_{12}}{T}+T^2 d_{12}==0,\ c_1+d_1+d_3-T d_3==0,$$
$$c_3+T d_3==0,\ c_5+\frac{c_5}{T^2}-\frac{2 c_5}{T}-\frac{c_6}{T^2}+\frac{c_6}{T}+\frac{c_7}{T^2}+d_7+d_{10}-T d_{10}+d_{13}-2 T d_{13}+T^2 d_{13}==0,$$
$$c_8+\frac{c_8}{T^2}-\frac{2 c_8}{T}-\frac{c_9}{T^2}+\frac{c_9}{T}+\frac{c_{10}}{T^2}+T d_{10}+2 T d_{13}-2 T^2 d_{13}==0,$$
$$c_{11}+\frac{c_{11}}{T^2}-\frac{2 c_{11}}{T}-\frac{c_{12}}{T^2}+\frac{c_{12}}{T}+\frac{c_{13}}{T^2}+T^2 d_{13}==0,\ c_1-\frac{c_1}{T}+\frac{c_2}{T}+d_2+d_4-T d_4==0,$$
$$c_3-\frac{c_3}{T}+\frac{c_4}{T}+T d_4==0,\ \frac{\gamma}{\sqrt{T}}+\sqrt{T}\,\gamma i==0,\ \frac{\beta}{\sqrt{T}}+\sqrt{T}\,\beta i==0,\ \frac{\alpha}{\sqrt{T}}+\sqrt{T}\,\alpha i==0,$$
$$\frac{\gamma}{T^3}-T^2\gamma i+\frac{c_5}{\sqrt{T}}-T^{3/2} c_5+\frac{c_6}{T^{3/2}}-\sqrt{T}\,c_6+\frac{c_7}{T^{5/2}}-\frac{c_7}{\sqrt{T}}+\frac{c_8}{\sqrt{T}}-T^{3/2} c_8+\frac{c_9}{T^{3/2}}-\sqrt{T}\,c_9+\frac{c_{10}}{T^{5/2}}-\frac{c_{10}}{\sqrt{T}}+$$
$$\frac{c_{11}}{\sqrt{T}}-T^{3/2} c_{11}+\frac{c_{12}}{T^{3/2}}-\sqrt{T}\,c_{12}+\frac{c_{13}}{T^{5/2}}-\frac{c_{13}}{\sqrt{T}}==0,\ \frac{\beta}{T^2}-T\beta i-\frac{4\gamma}{T^3}+\frac{4\gamma}{T^2}+\frac{c_1}{\sqrt{T}}-\sqrt{T}\,c_1+\frac{c_2}{T^{3/2}}-\frac{c_2}{\sqrt{T}}+$$
$$\frac{c_3}{\sqrt{T}}-\sqrt{T}\,c_3+\frac{c_4}{T^{3/2}}-\frac{c_4}{\sqrt{T}}-\frac{2 c_6}{T^{3/2}}-\frac{4 c_7}{T^{5/2}}+2\sqrt{T}\,c_8-\frac{c_9}{T^{3/2}}+\frac{c_9}{\sqrt{T}}-\frac{2 c_{10}}{T^{5/2}}+4\sqrt{T}\,c_{11}+\frac{2 c_{12}}{\sqrt{T}}==0,$$
$$\frac{\alpha}{T}-\alpha i-\frac{\beta}{T^2}+\frac{\beta}{T}+\frac{2\gamma}{T^3}-\frac{4\gamma}{T^2}+\frac{2\gamma}{T}-\frac{c_2}{T^{3/2}}+\frac{c_3}{\sqrt{T}}+\frac{2 c_7}{T^{5/2}}-\frac{2 c_{11}}{\sqrt{T}}==0\}$$

In[∘]:= **Solve[eqns, {α, β, γ, αi, βi, γi} ⋃ Table[c_j, {j, 13}] ⋃ Table[d_j, {j, 13}]]**

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
$$\left\{\left\{\alpha \to \frac{c_2}{2\sqrt{T}},\ \alpha i \to -\frac{c_2}{2\,T^{3/2}},\ \beta \to -\frac{c_9}{\sqrt{T}},\ \beta i \to \frac{c_9}{T^{3/2}},\ \gamma \to 0,\ \gamma i \to 0,\ c_3 \to 0,\ c_4 \to -T\,c_1 - c_2,\ c_5 \to 0,\ c_7 \to -\frac{1}{2}\times(1-T)\,c_9,\ c_8 \to 0,\ c_{10} \to -T\,c_6 - \frac{1}{2}\times(-1+3T)\,c_9,\ c_{11} \to 0,\ c_{12} \to 0,\ c_{13} \to 0,\ d_1 \to -c_1,\ d_2 \to -\frac{c_2}{T^2},\ d_3 \to 0,\ d_4 \to \frac{c_1}{T} + \frac{c_2}{T^2},\ d_5 \to 0,\ d_6 \to -\frac{c_6}{T} - \frac{(-1+T)\,c_9}{T^2},\ d_7 \to -\frac{(1-T)\,c_9}{2\,T^3},\ d_8 \to 0,\ d_9 \to -\frac{c_9}{T^2},\ d_{10} \to \frac{c_6}{T^2} - \frac{(-1-T)\,c_9}{2\,T^3},\ d_{11} \to 0,\ d_{12} \to 0,\ d_{13} \to 0\right\}\right\}$$

In[∘]:= **{DRules} = Solve[eqns, {α, β, γ, αi, βi, γi} ⋃ Table[c_j, {j, 13}] ⋃ Table[d_j, {j, 13}]]**

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
$$\left\{\left\{\alpha \to \frac{c_2}{2\sqrt{T}},\ \alpha i \to -\frac{c_2}{2\,T^{3/2}},\ \beta \to -\frac{c_9}{\sqrt{T}},\ \beta i \to \frac{c_9}{T^{3/2}},\ \gamma \to 0,\ \gamma i \to 0,\ c_3 \to 0,\ c_4 \to -T\,c_1 - c_2,\ c_5 \to 0,\ c_7 \to -\frac{1}{2}\times(1-T)\,c_9,\ c_8 \to 0,\ c_{10} \to -T\,c_6 - \frac{1}{2}\times(-1+3T)\,c_9,\ c_{11} \to 0,\ c_{12} \to 0,\ c_{13} \to 0,\ d_1 \to -c_1,\ d_2 \to -\frac{c_2}{T^2},\ d_3 \to 0,\ d_4 \to \frac{c_1}{T} + \frac{c_2}{T^2},\ d_5 \to 0,\ d_6 \to -\frac{c_6}{T} - \frac{(-1+T)\,c_9}{T^2},\ d_7 \to -\frac{(1-T)\,c_9}{2\,T^3},\ d_8 \to 0,\ d_9 \to -\frac{c_9}{T^2},\ d_{10} \to \frac{c_6}{T^2} - \frac{(-1-T)\,c_9}{2\,T^3},\ d_{11} \to 0,\ d_{12} \to 0,\ d_{13} \to 0\right\}\right\}$$

In[∘]:= $\{R_{1,2},\ \overline{R}_{1,2},\ CC_1,\ \overline{CC}_1\}$

Out[∘]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[0,\ (-1+T)\,x_2\,(y_1-y_2),\ 1 + \Big(c_1 x_1 y_1 + c_2 x_2 y_1 + c_6 x_1 x_2 y_1^2 - \frac{1}{2}\times(1-T)\,c_9 x_2^2 y_1^2 + (-T c_1 - c_2)\,x_2 y_2 + c_9 x_1 x_2 y_1 y_2 + \Big(-T c_6 - \frac{1}{2}\times(-1+3T)\,c_9\Big)\,x_2^2 y_1 y_2\Big)\,\epsilon + O[\epsilon]^2\Big],$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[0,\ \Big(-1+\frac{1}{T}\Big)\,x_2\,(y_1-y_2),\ 1 + \Big(-c_1 x_1 y_1 - \frac{c_2 x_2 y_1}{T^2} + \Big(-\frac{c_6}{T} - \frac{(-1+T)\,c_9}{T^2}\Big)\,x_1 x_2 y_1^2 - \frac{(1-T)\,c_9 x_2^2 y_1^2}{2\,T^3} + \Big(\frac{c_1}{T} + \frac{c_2}{T^2}\Big)\,x_2 y_2 - \frac{c_9 x_1 x_2 y_1 y_2}{T^2} + \Big(\frac{c_6}{T^2} - \frac{(-1-T)\,c_9}{2\,T^3}\Big)\,x_2^2 y_1 y_2\Big)\,\epsilon + O[\epsilon]^2\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \sqrt{T} + \Big(\frac{c_2}{2\sqrt{T}} - \frac{c_9 x_1 y_1}{\sqrt{T}}\Big)\,\epsilon + O[\epsilon]^2\Big],\ \mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{1}{\sqrt{T}} + \Big(-\frac{c_2}{2\,T^{3/2}} + \frac{c_9 x_1 y_1}{T^{3/2}}\Big)\,\epsilon + O[\epsilon]^2\Big]\Big\}$$

In[∘]:= $(R_{1,2}\, R_{4,3}\, R_{5,6}$ // $m_{1,4\to 1}$ // $m_{2,5\to 2}$ // $m_{3,6\to 3}) \equiv (R_{2,3}\, R_{4,5}\, R_{1,6}$ // $m_{1,4\to 1}$ // $m_{2,5\to 2}$ // $m_{3,6\to 3})$
$(R_{1,2}\, \overline{R}_{3,4}$ // $m_{1,3\to 1}$ // $m_{2,4\to 2})$
$CC_1\, \overline{CC}_2$ // $m_{1,2\to 1}$
$Kink_1 \equiv (\overline{CC}_3\, R_{1,2}$ // $m_{1,3\to 1}$ // $m_{1,2\to 1})$

Out[∘]= True

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, 1 + O[\epsilon]^2]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, 1 + O[\epsilon]^2]$

Out[∘]= True

OC fails

In[∘]:= $(R_{1,2}\, R_{4,3}$ // $m_{1,4\to 1}) \equiv (R_{1,3}\, R_{4,2}$ // $m_{1,4\to 1})$ // Simplify

Out[∘]= $(-1 + T)\, \epsilon\, y_1\, (c_1\, (x_2 - x_3) + c_9\, x_2\, x_3\, (-y_2 + y_3)) == 0$

R2 braid-like

In[∘]:= $R_{1,2}\, \overline{R}_{3,4}$ // $m_{1,3\to 1}$ // $m_{2,4\to 2}$
$\overline{R}_{1,2}\, R_{3,4}$ // $m_{1,3\to 1}$ // $m_{2,4\to 2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, 1 + O[\epsilon]^2]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, 1 + O[\epsilon]^2]$

‘naked’ R2 cyclic (!!):

In[∘]:= $R_{3,2}\, \overline{R}_{1,4}$ // $m_{1,3\to 1}$ // $m_{2,4\to 2}$
$R_{1,4}\, \overline{R}_{3,2}$ // $m_{1,3\to 1}$ // $m_{2,4\to 2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0, 0, 1 + \frac{(c_9 - T\, c_9)\, x_2\, y_1\, \epsilon}{T^2} + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0, 0, 1 + \frac{(c_9 - T\, c_9)\, x_2\, y_1\, \epsilon}{T} + O[\epsilon]^2\right]$

Proper R2 cyclic:

In[∘]:= $(CC_3\, R_{5,2}\, \overline{R}_{1,4}$ // $m_{1,3\to 1}$ // $m_{1,5\to 1}$ // $m_{2,4\to 2}) \equiv CC_1\, \mathbb{E}_{\{\}\to\{2\}}[0, 0, 1 + O[\epsilon]^2]$
$(\overline{CC}_4\, R_{1,6}\, \overline{R}_{3,2}$ // $m_{1,3\to 1}$ // $m_{2,4\to 2}$ // $m_{2,6\to 2}) \equiv \overline{CC}_2\, \mathbb{E}_{\{\}\to\{1\}}[0, 0, 1 + O[\epsilon]^2]$

Out[∘]= True

Out[∘]= True

R3:

In[∘]:= $(R_{1,2}\, R_{4,3}\, R_{5,6}$ // $m_{1,4\to 1}$ // $m_{2,5\to 2}$ // $m_{3,6\to 3}) \equiv (R_{2,3}\, R_{4,5}\, R_{1,6}$ // $m_{1,4\to 1}$ // $m_{2,5\to 2}$ // $m_{3,6\to 3})$

Out[∘]= True

Pairwise equality of the four kinks:

In[∘]:= $\overline{\text{Kink}}_1 \equiv (\overline{CC}_3\, R_{1,2} \,//\, m_{1,3\to 1} \,//\, m_{1,2\to 1})$

$\overline{\text{Kink}}_1 \equiv (\overline{CC}_3\, \overline{R}_{1,2} \,//\, m_{2,3\to 2} \,//\, m_{2,1\to 1})$

Out[∘]= True

Out[∘]= True

Trefoils?:

In[∘]:= $\overline{\text{Kink}}_8\, \overline{\text{Kink}}_9\, \overline{\text{Kink}}_{10}\, \overline{CC}_7\, R_{1,4}\, R_{5,2}\, R_{3,6} \,//\, m_{1,2\to 1} \,//\, m_{1,3\to 1} \,//\, m_{1,7\to 1} \,//\, m_{1,4\to 1} \,//\, m_{1,5\to 1} \,//\, m_{1,6\to 1} \,//\, m_{1,8\to 1} \,//\, m_{1,9\to 1} \,//\, m_{1,10\to 1}$

$\overline{\text{Kink}}_8\, \overline{\text{Kink}}_9\, \overline{\text{Kink}}_{10}\, CC_7\, R_{4,1}\, R_{2,5}\, R_{6,3} \,//\, m_{1,2\to 1} \,//\, m_{1,3\to 1} \,//\, m_{1,7\to 1} \,//\, m_{1,4\to 1} \,//\, m_{1,5\to 1} \,//\, m_{1,6\to 1} \,//\, m_{1,8\to 1} \,//\, m_{1,9\to 1} \,//\, m_{1,10\to 1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{T}{1-T+T^2} + \frac{(c_2 - T c_2 + T^3 c_2 - T^4 c_2 - T c_9 + 2 T^2 c_9 - 3 T^3 c_9 + 2 T^4 c_9)\, \epsilon}{1 - 3T + 6T^2 - 7T^3 + 6T^4 - 3T^5 + T^6} + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{T}{1-T+T^2} + \frac{(c_2 - T c_2 + T^3 c_2 - T^4 c_2 - T c_9 + 2 T^2 c_9 - 3 T^3 c_9 + 2 T^4 c_9)\, \epsilon}{1 - 3T + 6T^2 - 7T^3 + 6T^4 - 3T^5 + T^6} + O[\epsilon]^2\right]$

Unfortunately? This docile invariant does not see the difference between the mirror trefoils. Perhaps it is actually determined by Alexander.

In[∘]:= $\text{Kink}_8\, \text{Kink}_9\, \text{Kink}_{10}\, CC_4\, \overline{R}_{1,5}\, \overline{R}_{6,2}\, \overline{R}_{3,7} \,//\, m_{1,2\to 1} \,//\, m_{1,3\to 1} \,//\, m_{1,4\to 1} \,//\, m_{1,5\to 1} \,//\, m_{1,6\to 1} \,//\, m_{1,7\to 1} \,//\, m_{1,8\to 1} \,//\, m_{1,9\to 1} \,//\, m_{1,10\to 1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{T}{1-T+T^2} + \frac{(c_2 - T c_2 + T^3 c_2 - T^4 c_2 - 2 c_9 + 3 T c_9 - 2 T^2 c_9 + T^3 c_9)\, \epsilon}{1 - 3T + 6T^2 - 7T^3 + 6T^4 - 3T^5 + T^6} + O[\epsilon]^2\right]$

In[∘]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
  n = Length@pd; rots = Table[0, {2 n}];
  xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]  PositiveQ@x
                          Xm[x[[2]], x[[1]]]  True       ];
  For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
    front = Flatten[front /. k → (xs /. {
          Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
          Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]]; {1 - l, k + 1, l})
         })],
    Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
   ]];
  RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
rot[i_, 0] := 𝔼{}→{i}[0, 0, 1];
rot[i_, n_] := Module[{j},
  rot[i, n] = If[n > 0, rot[i, n - 1] CCj, rot[i, n + 1] CC̄j] // mi,j→i];
```

In[∘]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ζ, done, st, cx, ζ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ζ = 𝔼_{}→{0}[0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ζ1 = Switch[Head[cx],
     Xp, (R_{i,j} OverBar[Kink_k]) // m_{j,k→j},
     Xm, (OverBar[R]_{i,j} Kink_k) // m_{j,k→j}
    ];
   ζ1 = (rot[k, rots[[i]]] ζ1) // m_{k,i→i}; rots[[i]] = 0;
   ζ1 = (ζ1 rot[k, rots[[i + 1]]]) // m_{i,k→i}; rots[[i + 1]] = 0;
   ζ1 = (rot[k, rots[[j]]] ζ1) // m_{k,j→j}; rots[[j]] = 0;
   ζ1 = (ζ1 rot[k, rots[[j + 1]]]) // m_{j,k→j}; rots[[j + 1]] = 0;
   ζ *= ζ1;
   If[MemberQ[done, i], ζ = ζ // m_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ζ = ζ // m_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ζ = ζ // m_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ζ = ζ // m_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ζ (*/. {x_0→x,y_0→y,a_0→a}*))
 ]
```

In[∘]:=
```
NewBit[K_] := Module[{Alex = Alexander[K][T]},
  T^3 Alex^3/(T - 1) Coefficient[Z[K][[3]], ϵ] // Factor]
```

In[∘]:=
```
NewBit /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\left\{-c_2 - T^3 c_2 + 2 c_9 - T c_9 + T^2 c_9,\ -(1+T)\times\left(1-3T+T^2\right)(c_2-c_9),\ -\frac{2 c_2 - T c_2 + 2 T^2 c_2 + 2 T^5 c_2 - T^6 c_2 + 2 T^7 c_2 - 4 c_9 + 3 T c_9 - 5 T^2 c_9 + 3 T^3 c_9 - 3 T^4 c_9 + T^5 c_9 - T^6 c_9}{T^2},\ -4 c_2 + 2 T c_2 + 2 T^2 c_2 - 4 T^3 c_2 + 9 c_9 - 11 T c_9 + 7 T^2 c_9 - T^3 c_9\right\}$$

```
In[•]:= (*Two knots with equal Alexander, new bit does not agree*)
Alexander[Knot[6, 1]] == Alexander[Knot[9, 46]]
eq = (NewBit[Knot[6, 1]] == NewBit[Knot[9, 46]]);
eq /. c9 → 0
eq /. c2 → 0
```

Out[•]= True

Out[•]= True

Out[•]= $5 c_9 - 11 T c_9 - T^2 c_9 + 3 T^3 c_9 == 7 c_9 - 21 T c_9 + 9 T^2 c_9 + T^3 c_9$

```
In[•]:= Factor[NewBit /@ AllKnots[{3, 7}] /. c2 → 0]
```

Out[•]= $$\Big\{(2 - T + T^2)\, c_9,\ (1 + T) \times (1 - 3 T + T^2)\, c_9,\ \frac{(4 - 3 T + 5 T^2 - 3 T^3 + 3 T^4 - T^5 + T^6)\, c_9}{T^2},$$
$$-(-9 + 11 T - 7 T^2 + T^3)\, c_9,\ (5 - 11 T - T^2 + 3 T^3)\, c_9,\ \frac{(3 - 12 T + 16 T^2 - 12 T^3 + 4 T^4 - 2 T^6 + T^7)\, c_9}{T^2},$$
$$\frac{(1 + T) \times (2 - 3 T + 2 T^2) \times (1 - 3 T + 5 T^2 - 3 T^3 + T^4)\, c_9}{T^2},$$
$$\frac{(6 - 5 T + 9 T^2 - 7 T^3 + 9 T^4 - 6 T^5 + 6 T^6 - 3 T^7 + 3 T^8 - T^9 + T^{10})\, c_9}{T^4},$$
$$-(-23 + 36 T - 24 T^2 + 5 T^3)\, c_9,\ \frac{(-1 + 7 T - 13 T^2 + 24 T^3 - 32 T^4 + 35 T^5 - 27 T^6 + 17 T^7)\, c_9}{T^2},$$
$$4 \times (-2 + 11 T - 17 T^2 + 10 T^3)\, c_9,\ -\frac{(-17 + 41 T - 65 T^2 + 65 T^3 - 49 T^4 + 25 T^5 - 9 T^6 + T^7)\, c_9}{T^2},$$
$$\frac{(3 - 22 T + 53 T^2 - 53 T^3 + 25 T^4 - T^5 - 4 T^6 + T^7)\, c_9}{T^2},$$
$$\frac{(2 - 13 T + 27 T^2 - 9 T^3 - 31 T^4 + 33 T^5 - 13 T^6 + 2 T^7)\, c_9}{T^2}\Big\}$$

```
In[•]:= Kink1
```

Out[•]= $$\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{1}{\sqrt{T}} - \frac{c_2\, \epsilon}{2 T^{3/2}} + O[\epsilon]^2\right]$$

```
In[•]:= CC1
```

Out[•]= $$\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \sqrt{T} + \left(\frac{c_2}{2\sqrt{T}} - \frac{c_9 x_1 y_1}{\sqrt{T}}\right)\epsilon + O[\epsilon]^2\right]$$