Pensieve header: The PG version of the DoPeGDO programs.

tex

Here’s how the notebook \verb$TraditionalHopfStructure@.nb$ works. I started from a copy of the notebook you sent me, the notebook \verb$TraditionalHopfStructure.nb$. In the Mathematica {\it Cell$\to$Cell Tags} I enabled {\em Show Cell Tags}, I selected the whole notebook, removed all the tags that were already there, and added a {\tt pdf} tag to all cells. This tells \verb$Make.nb$ to create PDF files for all the cells and put the instructions to input them into the latex file \verb$TraditionalHopfStructure.tex$.

tex

But then I removed the {\tt pdf} tag from the ``Pensieve header'' cell because there is no need to include it in the resulting document, and I've inserted a text cells with tag {\tt tex} containing the paragraphs you are reading now. \verb$Make.nb$ simply copies cells with tag {\tt tex} into \verb$TraditionalHopfStructure.tex$, so it is easy to interlace latex with Mathematica. There's more information at \url{http://drorbn.net/AP/Projects/nb2tex/nb2tex.pdf}.

pdf

In[∘]:=
```
Once[<< KnotTheory`];
```

pdf

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

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ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

pdf

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

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ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

pdf

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

In[∘]:=
```
PP_ = Identity; $k = 1; γ = 1; ħ;
```

pdf

In[∘]:= $\mathtt{tKink}_1$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1 t_1,\ \hbar\, x_1 y_1,\ \frac{1}{\sqrt{T_1}} + \left(\frac{\hbar\, a_1}{\sqrt{T_1}} + \frac{\hbar\, a_1^2}{\sqrt{T_1}} - \frac{\hbar^3 x_1^2 y_1^2}{4\sqrt{T_1}}\right)\epsilon + O[\epsilon]^2\right]$

pdf

In[∘]:= $\mathtt{QZip}_{\{x_1,\xi_1,y_1,\eta_1,x_2,\xi_2,y_2,\eta_2\}}[\mathbb{E}\ @@\ (\mathtt{kR}_{1,2}\ \mathtt{km}_{2,1\to 5})]$

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Out[∘]= $\mathbb{E}\left[t\,\hbar\, a_2 + a_5\,\alpha_1 + a_5\,\alpha_2,\ 0,\ \frac{1}{T^2} + \frac{\hbar\, a_1 a_2\,\epsilon}{T^2} + O[\epsilon]^2\right]$

pdf

In[∘]:= $R_{1,2}\, R_{3,4}\, dm_{1,3\to5}$

pdf

Out[∘]= $\mathbb{E}_{\{1,3\}\to\{1,2,3,4,5\}}\Big[\hbar\, a_2\, b_1 + \hbar\, a_4\, b_3 + a_5\, \alpha_1 + a_5\, \alpha_3 + b_5\, \beta_1 + b_5\, \beta_3,\ \hbar\, x_2\, y_1 + \hbar\, x_4\, y_3 + y_5\, \eta_1 + \frac{y_5\, \eta_3}{\mathcal{A}_1} + \frac{x_5\, \xi_1}{\mathcal{A}_3} + \frac{(1 - B_5)\, \eta_3\, \xi_1}{\hbar} + x_5\, \xi_3,\ 1 + \Big(-\frac{1}{4}\hbar^3\, x_2^2\, y_1^2 - \frac{1}{4}\hbar^3\, x_4^2\, y_3^2 - \frac{y_5\, \beta_1\, \eta_3}{\mathcal{A}_1} - \frac{x_5\, \beta_3\, \xi_1}{\mathcal{A}_3} + a_5\, B_5\, \eta_3\, \xi_1 + \frac{\hbar\, x_5\, y_5\, \eta_3\, \xi_1}{\mathcal{A}_1\, \mathcal{A}_3} + \frac{(1 - 3\, B_5)\, y_5\, \eta_3^2\, \xi_1}{2\, \mathcal{A}_1} + \frac{(1 - 3\, B_5)\, x_5\, \eta_3\, \xi_1^2}{2\, \mathcal{A}_3} + \frac{(1 - 4\, B_5 + 3\, B_5^2)\, \eta_3^2\, \xi_1^2}{4\, \hbar}\Big)\, \epsilon + O[\epsilon]^2\Big]$

pdf

In[∘]:= $QZip_{\{x_1,\xi_1,y_1,\eta_1,x_3,\xi_3,y_3,\eta_3\}}[\mathbb{E}\ @@\ (kR_{1,2}\, kR_{3,4}\, kR_{5,6}\, km_{1,3\to5})]$

pdf

Out[∘]= $\mathbb{E}\Big[t\, \hbar\, a_2 + t\, \hbar\, a_4 + t\, \hbar\, a_6 + a_5\, \alpha_1 + a_5\, \alpha_3,\ \hbar\, x_6\, y_5,\ 1 + \Big(\hbar\, a_1\, a_2 + \hbar\, a_3\, a_4 + \hbar\, a_5\, a_6 - \frac{1}{4}\hbar^3\, x_6^2\, y_5^2\Big)\, \epsilon + O[\epsilon]^2\Big]$

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The “Speedy” Engine

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Internal Utilities

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Canonical Form:

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In[∘]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CCF[x]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := Module[
  {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
  Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]
 ];
CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

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The Kronecker δ:

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In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

pdf

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

pdf

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
          CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
        𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
        𝔼[L_, Q_, P_]$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

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Zip and Bind

pdf

Variables and their duals:

pdf

```
In[∘]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
        {τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)i;
```

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Upper to lower and lower to Upper:

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```
In[∘]:= U2l = {B_i_^p_. :> e^(-p ħ γ b_i), B^p_. :> e^(-p ħ γ b), T_i_^p_. :> e^(-p ħ t_i), T^p_. :> e^(-p ħ t), 𝒜_i_^p_. :> e^(p γ α_i), 𝒜^p_. :> e^(p γ α)};
        l2U = {e^(c_. b_i_ + d_.) :> B_i^(-c/(ħ γ)) e^d, e^(c_. b + d_.) :> B^(-c/(ħ γ)) e^d,
            e^(c_. t_i_ + d_.) :> T_i^(-c/ħ) e^d, e^(c_. t + d_.) :> T^(-c/ħ) e^d,
            e^(c_. α_i_ + d_.) :> 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :> 𝒜^(c/γ) e^d,
            e^ℰ_ :> e^(Expand@ℰ)};
```

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Derivatives in the presence of exponentiated variables:

pdf

```
In[∘]:= D_b[f_] := ∂_b f - ħ γ B ∂_B f; D_(b_i_)[f_] := ∂_(b_i) f - ħ γ B_i ∂_(B_i) f;
        D_t[f_] := ∂_t f - ħ T ∂_T f; D_(t_i_)[f_] := ∂_(t_i) f - ħ T_i ∂_(T_i) f;
        D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_(α_i_)[f_] := ∂_(α_i) f + γ 𝒜_i ∂_(𝒜_i) f;
        D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f; D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
        D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

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Finite Zips:

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```
In[∘]:= collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
        collect[ℰ_, ζ_] := Collect[ℰ, ζ];
        Zip_{}[P_] := P;
        Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
        Zip_{ζ_,ζs___}[P_] :=
          (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> (D_{ζ*,d}[f])) /. ζ* → 0 /.
           ((ζ* /. {b → B, t → T, α → 𝒜}) → 1)
```

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QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

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$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}| \left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|\mathbb{e}^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

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```
In[∘]:= QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_ζs[P /. (zrule ⋃ ζrule)]] ];
```

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LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

pdf

```
In[∘]:= LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    Zs = zs /. {b → B, t → T, α → 𝒜};
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    Zrule = Join[zrule,
      zrule /. r_Rule :> ((U = r[[1]] /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. (Zrule ⋃ ζrule);
    EEQ[ps___] := EEQ[ps] =
      (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
    CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
      Det[lt] (Zip_ζs[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
        Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ] ];
```

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In[∘]:=
```
B{}[L_, R_] := L R;
B{is__}[L_𝔼, R_𝔼] := Module[{n},
    Times[
      L /. Table[(v : b | B | t | T | a | x | y)i → vn@i, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)i → vn@i, {i, {is}}]
    ] // LZipJoin@@Table[{βn@i,τn@i,an@i},{i,{is}}] // QZipJoin@@Table[{ξn@i,yn@i},{i,{is}}] ];
Bis___[L_, R_] := B{is}[L, R];
```

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𝔼 morphisms with domain and range.

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In[∘]:=
```
Bis_List[𝔼d1_→r1_[L1_, Q1_, P1_], 𝔼d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ Bis[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] // 𝔼d2_→r2_[L2_, Q2_, P2_] :=
  Br1⋂d2[𝔼d1→r1[L1, Q1, P1], 𝔼d2→r2[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼d1_→r1_[L1_, Q1_, P1_] 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼dr_[L_, Q_, P_]$k_ := 𝔼dr @@ 𝔼[L, Q, P]$k;
𝔼_[ℰ___][i_] := {ℰ}〚i〛;
```

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𝔼[Λ]

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In[∘]:=
```
𝔼dr_[Λ_] := CF@
  Module[{L, Λ0 = Limit[Λ, ϵ → 0]}, 𝔼dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, ⅇ^(Λ-Λ0)]$k /. l2U]
```

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“Define” Code

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Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

pdf

```
In[∘]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
     SD[op_{nisp,$k_Integer}, Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
     SD[op_isp, op_{{is},$k}]; SD[op_{sis__}, op_{{sis}}];
    ] /. {SD → SetDelayed,
     isp → {is} /. {i → i_, j → j_, k → k_},
     nis → {is} /. {i → ii, j → jj, k → kk},
     nisp → {is} /. {i → ii_, j → jj_, k → kk_}
    }] ]]
```

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The Objects

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Symmetric Algebra Objects

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```
In[∘]:= sm_{i_,j_→k_} := 𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} := 𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_{i_} := 𝔼_{{}→{i}}[0];
sη_{i_} := 𝔼_{{i}→{}}[0];
```

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```
In[∘]:= sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sϒ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

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Booting Up QU

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```
In[∘]:= Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i], bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
```

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```
In[∘]:= Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^-1 ξ_i + ξ_j) x_k],
  bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^(-ϵ β_i) η_j) y_k]]
```

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Three types of inverses appear below!

$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.

P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element

of $V\otimes V$. As a map $\mathrm{P}:\mathbb{A}\otimes\mathbb{B}\to Q$.
$\overline{\mathrm{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

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In[∘]:=
```
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Sum_{k=1}^{$k+1} ((1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k) / (k (1 - e^{k γ ϵ ℏ}))],
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
        (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_j α_i / ℏ, η_j ξ_i / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
        (R_{1,2} // ((P_{{i,1},0})_{$k} (P_{{2,j},$k-1})_{$k}))[3]]]]
```

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In[∘]:=
```
R_{1,2} // P_{2,3}
```

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Out[∘]= $\mathbb{E}_{\{3\}\to\{1\}}\left[b_1\,\beta_3,\ y_1\,\eta_3,\ 1+O[\epsilon]^3\right]$

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In[∘]:=
```
(R_{1,2} // ((P_{{i,1},0})_2 (P_{{2,j},1})_2))[3]
```

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Out[∘]= $1+\left(-\frac{1}{8}\,\eta_j^2\,\xi_i^2-\frac{\eta_j^3\,\xi_i^3}{4\,\hbar}-\frac{\eta_j^4\,\xi_i^4}{16\,\hbar^2}\right)\epsilon^2+O[\epsilon]^3$

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In[∘]:=
```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{2,1},
  aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i, 1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
        ((aS̄_{{i},0})_{$k} // aS_i // (aS̄_{{i},$k-1})_{$k})[3]]]]
```

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In[∘]:=
```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{2,1},
  bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{2,1},
  aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{i,3},
  bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{3,i}]
```

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In[∘]:=
```
Define[
  dm_{i,j→k} = ((sϒ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // aS̄_3) (sϒ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
    (P_{1,-3} P_{3,-1} am_{2,-4→k} bm_{4,-2→k})]
```

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NB. We use the co-algebra structure B tensor A^{cop}. This has the benefit of making our algebra quasi-triangular in the traditional sense of the word.
Watch out: $\Delta_{i\to j,k}$ means j is to the RIGHT of strand k and dS looks like an S.

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In[∘]:=
```
Define[dσ_{i→j} = aσ_{i→j} bσ_{i→j},
 dϵ_i = sϵ_i, dη_i = sη_i,
 dS_i = sϒ_{i→1,1,2,2} // (bS_1 \overline{aS}_2) // dm_{2,1→i},
 \overline{dS}_i = sϒ_{i→1,1,2,2} // (\overline{bS}_1 aS_2) // dm_{2,1→i},
 dΔ_{i→j,k} = (bΔ_{i→1,3} aΔ_{i→4,2}) // (dm_{3,4→k} dm_{1,2→j}) ]
```

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In[∘]:=
```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} e^{-ħ ϵ a_i/2}]_{$k},
 \overline{C}_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} e^{ħ ϵ a_i/2}]_{$k},
 c_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/4} e^{-ħ ϵ a_i/4}]_{$k},
 \overline{c}_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/4} e^{ħ ϵ a_i/4}]_{$k},
 Kink_i = (R_{1,3} \overline{C}_2) // dm_{1,2→1} // dm_{1,3→i},
 \overline{Kink}_i = (\overline{R}_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i},
 ρ_i = (c_1 \overline{c}_3 dS_i) // dm_{1,i→i} // dm_{i,3→i}] (*ρ reverses a strand*)
```

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Note. $t = -\epsilon \mathrm{a} + \gamma \mathrm{b}$ and $b = t/\gamma + \epsilon \mathrm{a}/\gamma$

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In[∘]:=
```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i + β_i (ϵ a_i + t_i) / γ + ξ_i x_i + η_i y_i],
 t2b_i = 𝔼_{{i}→{i}}[α_i a_i + τ_i (-ϵ a_i + γ b_i) + ξ_i x_i + η_i y_i]]
```

pdf

In[∘]:=
```
𝔼_{{}→{1}}[0, 0, x_1] // dΔ_{1→1,2}
𝔼_{{}→{1}}[0, 0, x_1] // dS_1
𝔼_{{}→{1}}[0, 0, y_1] // dS_1
𝔼_{{}→{1}}[0, 0, x_1] // \overline{dS}_1
```

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (x_1 + x_2) - \hbar\, a_2\, x_1\, \epsilon + \frac{1}{2}\hbar^2\, a_2^2\, x_1\, \epsilon^2 + O[\epsilon]^3\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1 + (\hbar\, x_1 - \hbar\, a_1\, x_1)\ \epsilon + \left(-\frac{1}{2}\hbar^2\, x_1 + \hbar^2\, a_1\, x_1 - \frac{1}{2}\hbar^2\, a_1^2\, x_1\right)\epsilon^2 + O[\epsilon]^3\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{y_1}{B_1} + O[\epsilon]^3\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1 - \hbar\, a_1\, x_1\, \epsilon - \frac{1}{2}\left(\hbar^2\, a_1^2\, x_1\right)\epsilon^2 + O[\epsilon]^3\right]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, (1 + \epsilon\, a_1\, \hbar)\, x_1]$ // dS_1

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, -x_1 + \left(\frac{\hbar^2 x_1}{2} - \hbar^2 a_1 x_1 + \frac{1}{2}\hbar^2 a_1^2 x_1\right)\epsilon^2 + O[\epsilon]^3\right]$

pdf

In[∘]:= $((-1 + \hbar)\, x_1 + (1 - \hbar)\, a_1\, x_1)$ // Expand

pdf

Out[∘]= $-x_1 + \hbar\, x_1 + a_1\, x_1 - \hbar\, a_1\, x_1$

pdf

In[∘]:= $t2b_1\, t2b_2$ // $P_{2,1}$

pdf

Out[∘]= $\mathbb{E}_{\{1,2\}\to\{\}}\left[\frac{\alpha_2\,\tau_1}{\hbar}, \frac{\eta_1\,\xi_2}{\hbar}, 1 + \left(\frac{\eta_1^2\,\xi_2^2}{4\hbar} - \frac{\tau_1\,\tau_2}{\hbar}\right)\epsilon + O[\epsilon]^2\right]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, y_1]$ // $b\Delta_{1\to1,2}$
$\mathbb{E}_{\{\}\to\{1\}}[0, 0, y_1]$ // $d\Delta_{1\to1,2}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, (B_2\, y_1 + y_2) + O[\epsilon]^2]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, (B_2\, y_1 + y_2) + O[\epsilon]^2]$

pdf

In[∘]:= $(R_{1,2}$ // $bS_1) \equiv \overline{R}_{1,2}$
$(R_{1,2}$ // $aS_2) \equiv \overline{R}_{1,2}$

pdf

Out[∘]= True

pdf

Out[∘]= True

pdf

$\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $d\Delta_{1\to1,2}$

pdf

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}[0, 0, (x_1 + x_2) - \hbar\, a_2\, x_1\,\epsilon + O[\epsilon]^2]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $a\Delta_{1\to1,2}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, (x_1 + x_2) - \hbar\, a_1\, x_2\,\epsilon + O[\epsilon]^2]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // $(\overline{aS})_1$
$\mathbb{E}_{\{\}\to\{1\}}[0, 0, x_1]$ // aS_1

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, -x_1 + (\hbar\, x_1 - \hbar\, a_1\, x_1)\,\epsilon + O[\epsilon]^2]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}[0, 0, -x_1 - \hbar\, a_1\, x_1\,\epsilon + O[\epsilon]^2]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, b_1 y_2]$ // $bm_{1,2\to 1}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, b_1 y_1 - y_1 \in + O[\in]^2\right]$

pdf

In[∘]:= $a\Delta_{i\to 1,2}$ // aS_1 // $am_{1,2\to 1}$
$a\Delta_{i\to 1,2}$ // aS_2 // $am_{1,2\to 1}$

pdf

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0, 0, 1 + O[\in]^2\right]$

pdf

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0, 0, 1 + O[\in]^2\right]$

pdf

In[∘]:= $a\Delta_{1\to 1,2}$

pdf

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}\left[a_1 \alpha_1 + a_2 \alpha_1, x_1 \xi_1 + x_2 \xi_1, 1 + \left(-\hbar a_1 x_2 \xi_1 + \frac{1}{2} \hbar x_1 x_2 \xi_1^2\right) \in + O[\in]^2\right]$

pdf

Testing

pdf

co-associativity

pdf

In[∘]:= $(d\Delta_{1\to 1,2}$ // $d\Delta_{2\to 2,3}) \equiv (d\Delta_{1\to 2,3}$ // $d\Delta_{2\to 1,2})$

pdf

Out[∘]= True

pdf

algebra morphism

pdf

In[∘]:= $\left(d\Delta_{i\to 1,2}\, d\Delta_{j\to 3,4}\right.$ // $dm_{1,3\to i}$ // $\left.dm_{2,4\to j}\right) \equiv \left(dm_{i,j\to k}\right.$ // $\left.d\Delta_{k\to i,j}\right)$

pdf

Out[∘]= True

pdf

associativity

pdf

In[∘]:= $(dm_{1,2\to k}$ // $dm_{k,3\to k}) \equiv (dm_{2,3\to k}$ // $dm_{1,k\to k})$

pdf

Out[∘]= True

pdf

antipode

pdf

In[∘]:= $d\Delta_{i\to 1,2}$ // dS_1 // $dm_{1,2\to 1}$
$d\Delta_{i\to 1,2}$ // dS_2 // $dm_{1,2\to 1}$

pdf

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}[0, 0, 1 + O[\epsilon]^2]$

pdf

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}[0, 0, 1 + O[\epsilon]^2]$

pdf

quasi-triangular axioms

pdf

In[∘]:= ($R_{1,3}$ // $d\Delta_{1\to 1,2}$) ≡ ($R_{1,3}$ $R_{2,4}$ // $dm_{3,4\to 3}$)
($R_{1,3}$ // $d\Delta_{3\to 2,3}$) ≡ ($R_{1,3}$ $R_{0,2}$ // $dm_{1,0\to 1}$)
($d\Delta_{i\to k,j}$ $R_{1,2}$ // $dm_{j,1\to 1}$ // $dm_{k,2\to 2}$) ≡ ($R_{1,2}$ $d\Delta_{i\to j,k}$ // $dm_{1,j\to 1}$ // $dm_{2,k\to 2}$)

pdf

Out[∘]= True

pdf

Out[∘]= True

pdf

Out[∘]= True

pdf

In[∘]:= ($R_{1,2}$ // aS_2) ≡ ($\overline{R}_{1,2}$)

pdf

Out[∘]= True

pdf

In[∘]:= ($R_{1,2}$ // dS_1) ≡ ($\overline{R}_{1,2}$)
($R_{1,2}$ // $\overline{dS}_2$) ≡ ($\overline{R}_{1,2}$)

pdf

Out[∘]= True

pdf

Out[∘]= True

pdf

In[∘]:= $QQ_{s\_,r\_}$:= $R_{11,22}$ $R_{33,44}$ // $dm_{11,44\to s}$ // $dm_{22,33\to r}$
$\overline{QQ}_{s\_,r\_}$:= $\overline{R}_{22,11}$ $\overline{R}_{44,33}$ // $dm_{11,44\to s}$ // $dm_{22,33\to r}$

pdf

In[∘]:= $QQ_{1,2}$ $\overline{QQ}_{3,4}$ // $dm_{1,3\to 1}$ // $dm_{2,4\to 2}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, 1 + O[\epsilon]^2]$

pdf

Drinfeld element u

pdf

In[∘]:= $u_{i\_}$:= $R_{11,22}$ // dS_{22} // $dm_{22,11\to i}$
$\overline{u}_{i\_}$:= $\overline{R}_{11,22}$ // $\overline{dS}_{22}$ // $dm_{22,11\to i}$
$\overline{uu}_{i\_}$:= $\overline{R}_{11,22}$ // $\overline{dS}_{22}$ // $dm_{11,22\to i}$
$\overline{u2}_{i\_}$:= $\overline{R}_{11,22}$ // dS_{11} // $dm_{11,22\to i}$
$\overline{u3}_{i\_}$:= $R_{11,22}$ // dS_{11} // dS_{11} // $dm_{22,11\to i}$

pdf

In[∘]:= $u_i\,\overline{u}_j$ // $dm_{i,j\to i}$
$u_i\,\overline{uu}_j$ // $dm_{i,j\to i}$
$u_i\,\overline{u2}_j$ // $dm_{i,j\to i}$
$u_i\,\overline{u3}_j$ // $dm_{i,j\to i}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i-\hbar\,a_i\,B_i\,\epsilon+O[\epsilon]^2\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i-\hbar\,a_i\,B_i\,\epsilon+O[\epsilon]^2\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

pdf

In[∘]:= (u_1 // dS_1)
$R_{11,22}$ // dS_{22} // $dm_{11,22\to i}$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[-\hbar\,a_1\,b_1,\,-\frac{\hbar\,x_1\,y_1}{B_1},\,1+\left(\frac{\hbar^2\,x_1\,y_1}{B_1}-\frac{\hbar^2\,a_1\,x_1\,y_1}{B_1}-\frac{3\,\hbar^3\,x_1^2\,y_1^2}{4\,B_1^2}\right)\epsilon+\left(-\frac{\hbar^3\,x_1\,y_1}{2\,B_1}+\frac{\hbar^3\,a_1\,x_1\,y_1}{B_1}-\frac{\hbar^3\,a_1^2\,x_1\,y_1}{2\,B_1}+\frac{5\,\hbar^4\,x_1^2\,y_1^2}{2\,B_1^2}-\frac{5\,\hbar^4\,a_1\,x_1^2\,y_1^2}{2\,B_1^2}+\frac{\hbar^4\,a_1^2\,x_1^2\,y_1^2}{2\,B_1^2}-\frac{67\,\hbar^5\,x_1^3\,y_1^3}{36\,B_1^3}+\frac{3\,\hbar^5\,a_1\,x_1^3\,y_1^3}{4\,B_1^3}+\frac{9\,\hbar^6\,x_1^4\,y_1^4}{32\,B_1^4}\right)\epsilon^2+O[\epsilon]^3\Big]$$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\Big[-\hbar\,a_i\,b_i,\,-\frac{\hbar\,x_i\,y_i}{B_i},\,1+\left(\frac{\hbar^2\,x_i\,y_i}{B_i}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right)\epsilon+\left(-\frac{\hbar^3\,x_i\,y_i}{2\,B_i}+\frac{\hbar^3\,a_i\,x_i\,y_i}{B_i}-\frac{\hbar^3\,a_i^2\,x_i\,y_i}{2\,B_i}+\frac{5\,\hbar^4\,x_i^2\,y_i^2}{2\,B_i^2}-\frac{5\,\hbar^4\,a_i\,x_i^2\,y_i^2}{2\,B_i^2}+\frac{\hbar^4\,a_i^2\,x_i^2\,y_i^2}{2\,B_i^2}-\frac{67\,\hbar^5\,x_i^3\,y_i^3}{36\,B_i^3}+\frac{3\,\hbar^5\,a_i\,x_i^3\,y_i^3}{4\,B_i^3}+\frac{9\,\hbar^6\,x_i^4\,y_i^4}{32\,B_i^4}\right)\epsilon^2+O[\epsilon]^3\Big]$$

pdf

In[∘]:= $\left(\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,B_1^{-1}\left(1+\epsilon\,a_1\,\hbar+\frac{\epsilon^2}{2}\,a_1^2\,\hbar^2\right)\right]\,u_2\text{ // }dm_{1,2\to 1}\right)\equiv(u_1\text{ // }dS_1)$

pdf

Out[∘]= True

pdf

In[∘]:= u_1

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[-\hbar\, a_1\, b_1,\ -\frac{\hbar\, x_1\, y_1}{B_1},\ B_1+\left(-\hbar\, a_1\, B_1-\hbar^2\, x_1\, y_1-\hbar^2\, a_1\, x_1\, y_1-\frac{3\,\hbar^3\, x_1^2\, y_1^2}{4\, B_1}\right)\in+\left(\frac{1}{2}\,\hbar^2\, a_1^2\, B_1-\frac{1}{2}\,\hbar^3\, x_1\, y_1+\frac{1}{2}\,\hbar^3\, a_1^2\, x_1\, y_1-\frac{\hbar^4\, x_1^2\, y_1^2}{2\, B_1}+\frac{\hbar^4\, a_1\, x_1^2\, y_1^2}{4\, B_1}+\frac{\hbar^4\, a_1^2\, x_1^2\, y_1^2}{2\, B_1}-\frac{13\,\hbar^5\, x_1^3\, y_1^3}{36\, B_1^2}+\frac{3\,\hbar^5\, a_1\, x_1^3\, y_1^3}{4\, B_1^2}+\frac{9\,\hbar^6\, x_1^4\, y_1^4}{32\, B_1^3}\right)\in^2+O[\in]^3\right]$$

pdf

q

pdf

In[∘]:= $(u_1\ //\ dS_1)\ \overline{u3}_2\ //\ dm_{1,2\to 1}$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{1}{B_1}+\frac{\hbar\, a_1 \in}{B_1}+O[\in]^2\right]$$

pdf

In[∘]:= $(u_1\ //\ d\Delta_{1\to 2,1})\ \equiv\ \left(\overline{QQ}_{1,2}\ u_3\ u_4\ //\ dm_{1,3\to 1}\ //\ dm_{2,4\to 2}\right)$

pdf

Out[∘]= True

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{i\}}[0,\ 0,\ x_i]\ //\ dS_i$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ -x_i+(\hbar\, x_i-\hbar\, a_i\, x_i)\in+O[\in]^2\right]$$

pdf

In[∘]:= $Kink_1$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, b_1,\ \hbar\, x_1\, y_1,\ \frac{1}{\sqrt{B_1}}+\left(\frac{\hbar\, a_1}{2\,\sqrt{B_1}}-\frac{\hbar^3\, x_1^2\, y_1^2}{4\,\sqrt{B_1}}\right)\in+O[\in]^2\right]$$

pdf

In[∘]:= $(u_1 \,//\, dS_1)\, u_2 \,//\, dm_{1,2\to 1}$
$(u_1 \,//\, dS_1)\, u_2 \,//\, dm_{2,1\to 1}$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[-2\hbar\, a_1 b_1,\ \frac{(-\hbar-\hbar B_1)\, x_1 y_1}{B_1^2},\ B_1+\left(-\hbar\, a_1 B_1+\frac{a_1\left(-2\hbar^2-\hbar^2 B_1\right) x_1 y_1}{B_1}+\frac{\left(-7\hbar^3-8\hbar^3 B_1-3\hbar^3 B_1^2\right) x_1^2 y_1^2}{4 B_1^3}\right)\epsilon+O[\epsilon]^2\right]$$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[-2\hbar\, a_1 b_1,\ \frac{(-\hbar-\hbar B_1)\, x_1 y_1}{B_1^2},\ B_1+\left(-\hbar\, a_1 B_1+\frac{a_1\left(-2\hbar^2-\hbar^2 B_1\right) x_1 y_1}{B_1}+\frac{\left(-7\hbar^3-8\hbar^3 B_1-3\hbar^3 B_1^2\right) x_1^2 y_1^2}{4 B_1^3}\right)\epsilon+O[\epsilon]^2\right]$$

pdf

In[∘]:= $(u_1 \,//\, dS_1)$
u_2

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[-\hbar\, a_1 b_1,\ -\frac{\hbar\, x_1 y_1}{B_1},\ 1+\left(\frac{\hbar^2 x_1 y_1}{B_1}-\frac{\hbar^2 a_1 x_1 y_1}{B_1}-\frac{3\hbar^3 x_1^2 y_1^2}{4 B_1^2}\right)\epsilon+O[\epsilon]^2\right]$$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{2\}}\left[-\hbar\, a_2 b_2,\ -\frac{\hbar\, x_2 y_2}{B_2},\ B_2+\left(-\hbar\, a_2 B_2-\hbar^2 x_2 y_2-\hbar^2 a_2 x_2 y_2-\frac{3\hbar^3 x_2^2 y_2^2}{4 B_2}\right)\epsilon+O[\epsilon]^2\right]$$

pdf

In[∘]:= $R_{1,2}$
$\overline{R}_{1,2}$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 b_1,\ \hbar\, x_2 y_1,\ 1-\frac{1}{4}\left(\hbar^3 x_2^2 y_1^2\right)\epsilon+O[\epsilon]^2\right]$$

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1,2\}}\left[-\hbar\, a_2 b_1,\ -\frac{\hbar\, x_2 y_1}{B_1},\ 1+\left(-\frac{\hbar^2 a_2 x_2 y_1}{B_1}-\frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2}\right)\epsilon+O[\epsilon]^2\right]$$

pdf

In[∘]:= C_1

pdf

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \sqrt{B_1}-\frac{1}{2}\left(\hbar\, a_1\sqrt{B_1}\right)\epsilon+O[\epsilon]^2\right]$$

pdf

The Knot Tensors

pdf

In[∘]:=
```
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
  \overline{kR}_{i,j} = \overline{R}_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
  km_{i,j→k} = ((t2b_i t2b_j) // dm_{i,j→k} // b2t_k ) /. {t_k → t, T_k → T, τ_{i|j} → 0},
  kC_i = (C_i // b2t_i ) /. T_i → T,
  \overline{kC}_i = (\overline{C}_i // b2t_i ) /. T_i → T,
  kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
  \overline{kKink}_i = \overline{Kink}_i // b2t_i /. {t_i → t, T_i → T}]
```

pdf

In[∘]:=
```
Define[BS_{i,j→k} =
   C_3 C_4 dΔ_{i→l1,r1} dΔ_{j→l2,r2} // \overline{dS}_{r1} // dS_{r2} // dm_{l1,3→k} // dm_{k,r2→k} // dm_{k,r1→k} // dm_{k,4→k} // dm_{k,l2→k}]
Define[tBS_{i,j→k} = (t2b_i t2b_j) // C_3 C_4 dΔ_{i→l1,r1} dΔ_{j→l2,r2} // \overline{dS}_{r1} // dS_{r2} // dm_{l1,3→k} // dm_{k,r2→k} //
     dm_{k,r1→k} // dm_{k,4→k} // dm_{k,l2→k} // b2t_k]
Define[tm_{i,j→k} = t2b_i // t2b_j // dm_{i,j→k} // b2t_k]
Define[tΔ_{i→j,k} = t2b_i // dΔ_{i→j,k} // b2t_j // b2t_k]
Define[tS_i = t2b_i // dS_i // b2t_i]
Define[\overline{tS}_i = t2b_i // \overline{dS}_i // b2t_i]
Define[tR_{i,j} = R_{i,j} // b2t_i // b2t_j, \overline{tR}_{i,j} = \overline{R}_{i,j} // b2t_i // b2t_j]
Define[tC_i = C_i // b2t_i , \overline{tC}_i = \overline{C}_i // b2t_i ]
Define[tKink_i = Kink_i // b2t_i , \overline{tKink}_i = \overline{Kink}_i // b2t_i]
```

pdf

In[∘]:=
```
R_{1,3} R_{2,6} // dm_{3,6→3}
R_{1,3} // dΔ_{1→2,1}
```

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\ \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\ 1 + \left(-\frac{1}{4}\hbar^3 B_2^2 x_3^2 y_1^2 - \frac{1}{4}\hbar^3 x_3^2 y_2^2\right)\epsilon + O[\epsilon]^2\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\ \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\ 1 + \left(-\frac{1}{4}\hbar^3 B_2^2 x_3^2 y_1^2 - \frac{1}{4}\hbar^3 x_3^2 y_2^2\right)\epsilon + O[\epsilon]^2\right]$

pdf

In[∘]:= $\mathrm{tR}_{1,2}$ // $\overline{\mathrm{tS}}_1$ // $\overline{\mathrm{tS}}_1$ // $\mathrm{tm}_{1,2\to1}$
$\left(\mathrm{tR}_{1,2} \mathbin{//} \overline{\mathrm{tS}}_1 \mathbin{//} \overline{\mathrm{tS}}_1 \mathbin{//} \mathrm{tm}_{1,2\to1} \mathbin{//} \mathrm{tS}_1\right) \mathbb{E}_{\{\}\to\{2\}}[0, 0, T_2 (1 - 2 \epsilon \hbar a_1)] \mathbin{//} \mathrm{tm}_{1,2\to1}$
$\left(\mathrm{tR}_{1,2} \mathbin{//} \overline{\mathrm{tS}}_1 \mathbin{//} \overline{\mathrm{tS}}_1 \mathbin{//} \mathrm{tm}_{2,1\to1}\right) \mathbb{E}_{\{\}\to\{2\}}[0, 0, T_2 (1 - 2 \epsilon \hbar a_1)] \mathbin{//} \mathrm{tm}_{1,2\to1}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar a_1 t_1, \hbar x_1 y_1, 1 + \left(\hbar a_1^2 + \hbar^2 x_1 y_1 - \frac{1}{4} \hbar^3 x_1^2 y_1^2\right) \epsilon + O[\epsilon]^2\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar a_1 t_1, \hbar x_1 y_1, 1 + \left(\hbar a_1^2 - \hbar^2 x_1 y_1 - \frac{1}{4} \hbar^3 x_1^2 y_1^2\right) \epsilon + O[\epsilon]^2\right]$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar a_1 t_1, \hbar x_1 y_1, 1 + \left(\hbar a_1^2 - \hbar^2 x_1 y_1 - \frac{1}{4} \hbar^3 x_1^2 y_1^2\right) \epsilon + O[\epsilon]^2\right]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, x_2] \mathbin{//} \mathrm{dS}_2$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, -x_2 - \hbar a_2 x_2 \epsilon + O[\epsilon]^2\right]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2] \mathbin{//} \overline{\mathrm{dS}}_2$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, -\frac{y_2}{B_2} + O[\epsilon]^2\right]$

pdf

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2] \mathbin{//} \overline{\mathrm{dS}}_2 \mathbin{//} \overline{\mathrm{dS}}_2$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, y_2 + \hbar y_2 \epsilon + O[\epsilon]^2\right]$

pdf

In[∘]:= $\mathrm{tm}_{i,j\to k}$
$\mathrm{tR}_{i,j}$
$\overline{\mathrm{tR}}_{i,j}$
tC_i
$\overline{\mathrm{tC}}_i$
tKink_i
$\overline{\mathrm{tKink}}_i$
$\mathrm{t\Delta}_{i\to j,k}$
tS_i

pdf

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k \alpha_i + a_k \alpha_j + t_k \tau_i + t_k \tau_j, y_k \eta_i + \frac{y_k \eta_j}{\mathcal{A}_i} + \frac{x_k \xi_i}{\mathcal{A}_j} + \frac{(1 - T_k) \eta_j \xi_i}{\hbar} + x_k \xi_j,\right.$
$1 + \left(2 a_k T_k \eta_j \xi_i + \frac{\hbar x_k y_k \eta_j \xi_i}{\mathcal{A}_i \mathcal{A}_j} + \frac{(1 - 3 T_k) y_k \eta_j^2 \xi_i}{2 \mathcal{A}_i} + \frac{(1 - 3 T_k) x_k \eta_j \xi_i^2}{2 \mathcal{A}_j} + \frac{\left(1 - 4 T_k + 3 T_k^2\right) \eta_j^2 \xi_i^2}{4 \hbar}\right) \epsilon +$
$\left. O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j\, t_i,\ \hbar\, x_j\, y_i,\ 1+\left(\hbar\, a_i\, a_j-\frac{1}{4}\hbar^3 x_j^2 y_i^2\right)\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\, a_j\, t_i,\ -\frac{\hbar\, x_j\, y_i}{T_i},\ 1+\left(-\hbar\, a_i\, a_j-\frac{\hbar^2 a_i\, x_j\, y_i}{T_i}-\frac{\hbar^2 a_j\, x_j\, y_i}{T_i}-\frac{3\hbar^3 x_j^2 y_i^2}{4 T_i^2}\right)\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{T_i}-\hbar\, a_i\sqrt{T_i}\,\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{T_i}}+\frac{\hbar\, a_i\,\epsilon}{\sqrt{T_i}}+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, t_i,\ \hbar\, x_i\, y_i,\ \frac{1}{\sqrt{T_i}}+\left(\frac{\hbar\, a_i}{\sqrt{T_i}}+\frac{\hbar\, a_i^2}{\sqrt{T_i}}-\frac{\hbar^3 x_i^2 y_i^2}{4\sqrt{T_i}}\right)\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\, a_i\, t_i,\ -\frac{\hbar\, x_i\, y_i}{T_i},\ \sqrt{T_i}+\left(-\hbar\, a_i\sqrt{T_i}-\hbar\, a_i^2\sqrt{T_i}-\frac{2\hbar^2 a_i\, x_i\, y_i}{\sqrt{T_i}}-\frac{3\hbar^3 x_i^2 y_i^2}{4 T_i^{3/2}}\right)\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+t_j\,\tau_i+t_k\,\tau_i,\ y_j\,\eta_i+T_j\, y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,\right.$
$\left.1+\left(-\hbar\, a_j\, T_j\, y_k\,\eta_i+\frac{1}{2}\hbar\, T_j\, y_j\, y_k\,\eta_i^2-\hbar\, a_j\, x_k\,\xi_i+\frac{1}{2}\hbar\, x_j\, x_k\,\xi_i^2\right)\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-t_i\,\tau_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{T_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\, T_i},\right.$
$1+\left(\frac{\hbar\, y_i\,\mathcal{A}_i\,\eta_i}{T_i}-\frac{\hbar\, a_i\, y_i\,\mathcal{A}_i\,\eta_i}{T_i}-\frac{\hbar\, y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2T_i^2}-\hbar\, a_i\, x_i\,\mathcal{A}_i\,\xi_i+\right.$
$\frac{2a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{T_i}-\frac{\hbar\, x_i\, y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{T_i}+\frac{(-\mathcal{A}_i+T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{T_i}+\frac{y_i\left(3\mathcal{A}_i^2-T_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2T_i^2}-$
$\left.\left.\frac{1}{2}\hbar\, x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\left(3\mathcal{A}_i^2-T_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2T_i}+\frac{\left(-3\mathcal{A}_i^2+4T_i\,\mathcal{A}_i^2-T_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\hbar\, T_i^2}\right)\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i\}}\left[a_i\, t_i,\ x_i\, y_i,\ \frac{1}{\sqrt{T_i}}+\left(\frac{a_i}{\sqrt{T_i}}+\frac{a_i^2}{\sqrt{T_i}}-\frac{x_i^2 y_i^2}{4\sqrt{T_i}}\right)\epsilon+O[\epsilon]^2\right]$

pdf

Out[•]= $\mathbb{E}_{\{\}\to\{i\}}\left[-a_i\, t_i,\ -\frac{x_i\, y_i}{T_i},\ \sqrt{T_i}+\left(-a_i\sqrt{T_i}-a_i^2\sqrt{T_i}-\frac{2a_i\, x_i\, y_i}{\sqrt{T_i}}-\frac{3x_i^2 y_i^2}{4T_i^{3/2}}\right)\epsilon+O[\epsilon]^2\right]$

pdf

In[∘]:=
```
RVK[pd_PD] := PP_RVK@Module[{n, xs, x, rots, front = {0}, k},
    n = Length@pd; rots = Table[0, {2 n}];
    xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]  PositiveQ@x
                          { Xm[x[[2]], x[[1]]]  True        ];
    For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → (xs /. {
              Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
              Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]]; {1 - l, k + 1, l})
             })],
      Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
     ]];
    RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

pdf

In[∘]:=
```
rot[i_, 0] := E_{{}→{i}}[0, 0, 1];
rot[i_, n_] := Module[{j},
   rot[i, n] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] kC̄_j] // km_{i,j→i}];
```

pdf

In[∘]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{}→{0}[0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
     Xp, (kR_{i,j} kKink‾_k) // km_{j,k→j},
     Xm, (kR‾_{i,j} kKink_k) // km_{j,k→j}
    ];
   ξ1 = (rot[k, rots[[i]]] ξ1) // km_{k,i→i}; rots[[i]] = 0;
   ξ1 = (ξ1 rot[k, rots[[i + 1]]]) // km_{i,k→i}; rots[[i + 1]] = 0;
   ξ1 = (rot[k, rots[[j]]] ξ1) // km_{k,j→j}; rots[[j]] = 0;
   ξ1 = (ξ1 rot[k, rots[[j + 1]]]) // km_{j,k→j}; rots[[j + 1]] = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // km_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ξ = ξ // km_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ξ = ξ // km_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ξ = ξ // km_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ /. {x_0 → x, y_0 → y, a_0 → a})
 ]
```

pdf

In[∘]:=
```
Z@Knot[3, 1]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

pdf

Out[∘]=

$$\mathbb{E}_{\{\}\to\{0\}}\left[0,\,0,\,\frac{T}{1-T+T^2}+\left(\frac{a\,\left(-2\,T\,\hbar+2\,T^3\,\hbar\right)}{1-2\,T+3\,T^2-2\,T^3+T^4}+\frac{-2\,T\,\hbar+3\,T^2\,\hbar-2\,T^3\,\hbar+T^4\,\hbar}{1-3\,T+6\,T^2-7\,T^3+6\,T^4-3\,T^5+T^6}+\frac{x\,y\,\left(-2\,T\,\hbar^2-2\,T^2\,\hbar^2\right)}{1-2\,T+3\,T^2-2\,T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]$$

pdf

In[∘]:= $R_{1,2}\, R_{3,4}$ // $dm_{1,3\to5}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{2,4,5\}}\left[a_2 b_5 + a_4 b_5,\ x_2 y_5 + x_4 y_5,\ 1 + \left(-a_2 x_4 y_5 - \frac{1}{4} x_2^2 y_5^2 - \frac{1}{4} x_4^2 y_5^2\right) \epsilon + O[\epsilon]^2\right]$

pdf

In[∘]:= $\overline{kR}_{1,2}\, \overline{kR}_{3,4}$ // $tm_{1,4\to5}$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{2,3,5\}}\left[-t\, a_2 - t\, a_5,\ -\frac{x_5 y_3}{T} - \frac{x_2 y_5}{T},\right.$
$\left. 1 + \left(-a_2 a_5 - a_3 a_5 - \frac{a_3 x_5 y_3}{T} - \frac{a_5 x_5 y_3}{T} - \frac{3 x_5^2 y_3^2}{4 T^2} - \frac{a_2 x_2 y_5}{T} - \frac{a_5 x_2 y_5}{T} - \frac{3 x_2^2 y_5^2}{4 T^2}\right) \epsilon + O[\epsilon]^2\right]$

pdf

$\overline{kR}_{1,2}\, \overline{kR}_{3,4}$ // $tm_{1,4\to5}$

pdf

In[∘]:=

```
(*Working Casimir, not unique!*)
Define[ωi = E{}→{i}[0, 0, Series[y e^(ϵ a) x + (e^(ϵ (a+1)) + e^(-ϵ a) T)/(e^ϵ - 1) - (T + 1) ϵ^-1, {ϵ, 0, 3}] /.
    {a → ai, T → Ti, x → xi, y → yi}]
]
ωsq = ω1 ω2 // tm1,2→1;
ωcub = ωsq ω2 // tm1,2→1;
ω4 = ωcub ω2 // tm1,2→1;
(*Cleaned versions*)
ωc = ω1[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ωsqc = ωsq[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ωcubc = ωcub[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ω4c = ω4[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
```