The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation. Continues pensieve://2016-06/.

```
In[•]:= Kδ /: Kδ_is__ := KroneckerDelta[1, Length[Union[{is}]]];
```

The Standard Representation

Following Inna Sysoeva's "Dimension *n* Representations of the Braid Group on *n* String", Section 5.

```
In[•]:= S_i_,j_[ξ_] := ξ /. v_j :> t_i v_j

In[•]:= {v1, v2, v3} // S1,3

Out[•]= {v1, v2, t1 v3}

In[•]:= Column@{R3l = {v1, v2, v3} // S1,2 // S1,3 // S2,3,
    R3r = {v1, v2, v3} // S2,3 // S1,3 // S1,2 ,
    R3l - R3r // Expand}

Out[•]= {v1, t1 v2, t1 t2 v3}
        {v1, t1 v2, t1 t2 v3}
        {0, 0, 0}
```

The Burau Representation

```
In[•]:= Inverse[{{0, t}, {1, 1 - t}}] /. t → t^-1 // Simplify // Transpose // MatrixForm

Out[•]//MatrixForm=
        ( 1 - t   t )
        (   1     0 )

In[•]:= B_i_,j_[ξ_] := ξ /. v_j :> (1 - t) v_i + t v_j

In[•]:= {v1, v2, v3} // B1,3

Out[•]= {v1, v2, (1 - t) v1 + t v3}

In[•]:= Column@{R3l = {v1, v2, v3} // B1,2 // B1,3 // B2,3,
    R3r = {v1, v2, v3} // B2,3 // B1,3 // B1,2 ,
    R3l - R3r // Expand}

Out[•]= {v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
        {v1, (1 - t) v1 + t v2, (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
        {0, 0, 0}
```

The Gassner Representation

```
In[•]:= G_i_,j_[ξ_] := ξ /. v_j :> (1 - t_i) v_i + t_i v_j
```

```
In[•]:= Column@{R3l = {v1, v2, v3} // G1,2 // G1,3 // G2,3,
          R3r = {v1, v2, v3} // G2,3 // G1,3 // G1,2,
          R3l - R3r // Expand}
```

$$\left\{v_1,\ \left(1-t_1\right) v_1+t_1 v_2,\ \left(1-t_1\right) v_1+t_1 \left(\left(1-t_2\right) v_2+t_2 v_3\right)\right\}$$

```
Out[•]= {v1, (1 - t1) v1 + t1 v2, (1 - t2) ((1 - t1) v1 + t1 v2) + t2 ((1 - t1) v1 + t1 v3)}
        {0, 0, 0}
```

```
In[•]:= Column@{OCl = {v1, v2, v3} // G1,2 // G1,3,
          OCr = {v1, v2, v3} // G1,3 // G1,2 ,
          OCl - OCr // Expand}
```

$$\left\{v_1,\ \left(1-t_1\right) v_1+t_1 v_2,\ \left(1-t_1\right) v_1+t_1 v_3\right\}$$

```
Out[•]= {v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
        {0, 0, 0}
```

```
In[•]:= Column@{UCl = {v1, v2, v3} // G1,3 // G2,3,
          UCr = {v1, v2, v3} // G2,3 // G1,3 ,
          UCl - UCr // Expand}
```

$$\left\{v_1,\ v_2,\ \left(1-t_1\right) v_1+t_1 \left(\left(1-t_2\right) v_2+t_2 v_3\right)\right\}$$

```
Out[•]= {v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3)}
        {0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

The Gassner-Plus Representation

```
In[•]:= GPi_,j_[ℱ_] := Expand[ℱ /. {uj :> (1 - ti) ui + ti uj,
            f_. vj :> f (1 - ti) vi + f ti vj + (ti - 1) (ti ∂ti f - tj ∂tj f) ui + f ti ui }]
```

```
In[•]:= GPchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3};
```

```
In[•]:= R3l = GPchecks // GP1,2 // GP1,3 // GP2,3
```

$$\begin{aligned}&\{f[t_1,t_2,t_3]\,v_1,\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,v_1-\\&\quad f[t_1,t_2,t_3]\,t_1\,v_1+f[t_1,t_2,t_3]\,t_1\,v_2+t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-\\&\quad t_1\,t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\\&\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,t_1\,t_2\,u_2+f[t_1,t_2,t_3]\,v_1-f[t_1,t_2,t_3]\,t_1\,v_1+\\&\quad f[t_1,t_2,t_3]\,t_1\,v_2-f[t_1,t_2,t_3]\,t_1\,t_2\,v_2+f[t_1,t_2,t_3]\,t_1\,t_2\,v_3+t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]-\\&\quad t_1\,t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]+t_1\,t_3\,u_2\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,t_2\,t_3\,u_2\,f^{(0,0,1)}[t_1,t_2,t_3]-\\&\quad t_1\,t_2\,u_2\,f^{(0,1,0)}[t_1,t_2,t_3]+t_1\,t_2^2\,u_2\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+\\&\quad t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\ u_1,\ u_1-t_1\,u_1+t_1\,u_2,\ u_1-t_1\,u_1+t_1\,u_2-t_1\,t_2\,u_2+t_1\,t_2\,u_3\}\end{aligned}$$

```
In[•]:= R3r = GPchecks // GP2,3 // GP1,3 // GP1,2; R3l - R3r
```

```
Out[•]= {0, 0, 0, 0, 0, 0}
```

```
In[•]:= OCl = GPchecks // GP1,2 // GP1,3
```

$$\begin{aligned}&\{f[t_1,t_2,t_3]\,v_1,\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,v_1-\\&\quad f[t_1,t_2,t_3]\,t_1\,v_1+f[t_1,t_2,t_3]\,t_1\,v_2+t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-\\&\quad t_1\,t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\\&\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,v_1-f[t_1,t_2,t_3]\,t_1\,v_1+f[t_1,t_2,t_3]\,t_1\,v_3+\\&\quad t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+\\&\quad t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\ u_1,\ u_1-t_1\,u_1+t_1\,u_2,\ u_1-t_1\,u_1+t_1\,u_3\}\end{aligned}$$

```
In[•]:= OCr = GPchecks // GP1,3 // GP1,2; OCl - OCr
Out[•]= {0, 0, 0, 0, 0, 0}
```

Question. Does GP factor through G? How?

The End(G) Representation

```
EGi_,j_[ℰ_] := Expand[ℰ /. {uj → (1 - ti) ui + ti uj, wi → wi + (1 - ti^-1) wj, wj → ti^-1 wj}];
EGchecks = Flatten@Table[ui wj, {i, 3}, {j, 3}]
```

$$\{u_1 w_1,\ u_1 w_2,\ u_1 w_3,\ u_2 w_1,\ u_2 w_2,\ u_2 w_3,\ u_3 w_1,\ u_3 w_2,\ u_3 w_3\}$$

```
Short[R3l = EGchecks // EG1,2 // EG1,3 // EG2,3, 10]
```

$$\Big\{u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},\ \frac{u_1 w_3}{t_1 t_2},$$
$$u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$
$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3,\ -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$
$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$
$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$
$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$
$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$
$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\Big\}$$

```
R3r = EGchecks // EG2,3 // EG1,3 // EG1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

```
(# → Collect[EGi,j[#], u_ w_, Simplify]) & /@ {uk wj, uk wi, uj wk, uj wi}
```

$$\Big\{u_k w_j \to \frac{u_k w_j}{t_i},\ u_k w_i \to u_k w_i + \left(1 - \frac{1}{t_i}\right) u_k w_j,\ u_j w_k \to (1 - t_i)\, u_i w_k + t_i u_j w_k,$$
$$u_j w_i \to (1 - t_i)\, u_i w_i + t_i u_j w_i - \frac{(-1 + t_i)^2 u_i w_j}{t_i} + (-1 + t_i)\, u_j w_j\Big\}$$

The End(G)+c Representation

Is there topology behind this representation?

```
EGc_{i_,j_}[ℰ_] := Expand[ℰ /. {
      c_i → c_i - (1 - t_i^-1) u_i w_j, c_j → c_j + (1 - t_i^-1) u_i w_j,
      u_j → (1 - t_i) u_i + t_i u_j,
      w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
EGcchecks = {c_1, c_2, c_3, u_1 w_1, u_1 w_2, u_1 w_3, u_2 w_1, u_2 w_2, u_2 w_3, u_3 w_1, u_3 w_2, u_3 w_3};
(# → Collect[EGc_{1,2}[#], u_ w_, Simplify]) & /@ EGcchecks
```

$$\left\{c_1 \to c_1 + \left(-1 + \frac{1}{t_1}\right) u_1 w_2,\ c_2 \to c_2 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\right.$$
$$c_3 \to c_3,\ u_1 w_1 \to u_1 w_1 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\ u_1 w_2 \to \frac{u_1 w_2}{t_1},\ u_1 w_3 \to u_1 w_3,$$
$$u_2 w_1 \to (1 - t_1) u_1 w_1 + t_1 u_2 w_1 - \frac{(-1 + t_1)^2 u_1 w_2}{t_1} + (-1 + t_1) u_2 w_2,\ u_2 w_2 \to \left(-1 + \frac{1}{t_1}\right) u_1 w_2 + u_2 w_2,$$
$$\left.u_2 w_3 \to (1 - t_1) u_1 w_3 + t_1 u_2 w_3,\ u_3 w_1 \to u_3 w_1 + \left(1 - \frac{1}{t_1}\right) u_3 w_2,\ u_3 w_2 \to \frac{u_3 w_2}{t_1},\ u_3 w_3 \to u_3 w_3\right\}$$

```
u_1 w_2 + c_2 // EGc_{1,2}
```

$$c_2 + u_1 w_2$$

```
(# → Simplify[EGc_{1,2}[#] /. {u_i_ w_i_ → 1, u_i_ w_j_ /; i ≠ j :> 0}]) & /@ EGcchecks
```

$$\{c_1 \to c_1,\ c_2 \to c_2,\ c_3 \to c_3,\ u_1 w_1 \to 1,\ u_1 w_2 \to 0,\ u_1 w_3 \to 0,$$
$$u_2 w_1 \to 0,\ u_2 w_2 \to 1,\ u_2 w_3 \to 0,\ u_3 w_1 \to 0,\ u_3 w_2 \to 0,\ u_3 w_3 \to 1\}$$

```
Short[R3l = EGcchecks // EGc_{1,2} // EGc_{1,3} // EGc_{2,3}, 10]
```

$$\left\{c_1 - u_1 w_2 + \frac{u_1 w_2}{t_1} - u_1 w_3 + \frac{u_1 w_3}{t_1},\ c_2 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2},\right.$$
$$c_3 + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},\ u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},$$
$$\frac{u_1 w_3}{t_1 t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$
$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3,\ -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$
$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$
$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$
$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$
$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$
$$\left.\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\right\}$$

```
R3r = EGcchecks // EGc_{2,3} // EGc_{1,3} // EGc_{1,2}; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

The Turbo-Gassner Representation

```
In[•]:= TG_{i_,j_}[ℰ_] := Expand[ℰ /. {
          f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i) v_i + t_i v_j,
            (1 - t_i^-1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
            Kδ_{k,i} f (u_j - u_i) u_i w_j],
          u_j → (1 - t_i) u_i + t_i u_j,
          w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
        TGchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};

In[•]:= Short[R3l = TGchecks // TG_{1,2} // TG_{1,3} // TG_{2,3}, 10]

Out[•]//Short= {f[t1, t2, t3] v1 - f[t1, t2, t3] u1^2 w2 + f[t1, t2, t3] u1 u2 w2 -
    f[t1, t2, t3] u1^2 w3 + f[t1, t2, t3] u1 u3 w3 - (t3 u1^2 w3 f^(0,0,1)[t1, t2, t3])/t2 +
    <<15>> + t2 u1 u2 w3 f^(0,1,0)[t1, t2, t3] - u1^2 w2 f^(1,0,0)[t1, t2, t3] +
    t1 u1^2 w2 f^(1,0,0)[t1, t2, t3] - u1^2 w3 f^(1,0,0)[t1, t2, t3] + t1 u1^2 w3 f^(1,0,0)[t1, t2, t3],
   f[t1, t2, t3] v1 - <<1>> t1 v1 + <<63>> + t1^2 u1 u2 w3 f^(1,0,0)[t1, t2, t3], <<5>>, <<1>>, w3/(t1 t2)}

In[•]:= R3r = TGchecks // TG_{2,3} // TG_{1,3} // TG_{1,2}; R3l - R3r

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= Short[OCl = TGchecks // TG_{1,2} // TG_{1,3}]

Out[•]//Short= {<<18>> + t1 <<2>> f^<<1>>[t1, t2, t3], <<1>>, <<6>>, <<1>>/<<1>>}

In[•]:= OCr = TGchecks // TG_{1,3} // TG_{1,2}; OCl - OCr

Out[•]= {0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 + f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3,
  -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 + f[t1, t2, t3] u1 u3 w2 - f[t1, t2, t3] t1 u1 u3 w2,
  0, 0, 0, 0, 0, 0}
```

A Finite-Rank Turbo-Gassner Representation

```
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
FTG_{i_,j_}[ℰ_] := Expand[ℰ /. {
      f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i - η[i]) v_i + (t_i + η[i]) v_j,
        (t_i Coefficient[f, η[i]] - t_j Coefficient[f, η[j]]) ×
          (1 - t_i^-1) (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
        Kδ_{k,i} (f /. _η → 0) (u_j - u_i) u_i w_j],
      u_j → (1 - t_i) u_i + t_i u_j,
      w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];

ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
FTGchecks = {ff v1, ff v2, ff v3, u1, u2, u3, w1, w2, w3};
```

```
{v1, v2} // FTG1,2 // Column
```

$$v_1 - u_1^2 w_2 + u_1 u_2 w_2$$
$$v_1 - t_1 v_1 + t_1 v_2 - v_1 \eta[1] + v_2 \eta[1]$$

```
Short[R3l = FTGchecks // FTG1,2 // FTG1,3 // FTG2,3, 10]
```

$$\left\{f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + f_1 t_1 u_1^2 w_2 - f_2 t_2 u_1^2 w_2 + \frac{f_2 t_2 u_1^2 w_2}{t_1} + f_0 u_1 u_2 w_2 - f_0 u_1^2 w_3 - f_1 u_1^2 w_3 + f_2 u_1^2 w_3 - \frac{f_2 u_1^2 w_3}{t_1} + f_1 t_1 u_1^2 w_3 - f_2 t_2 u_1^2 w_3 + \frac{f_2 t_2 u_1^2 w_3}{t_1} - \frac{f_3 t_3 u_1^2 w_3}{t_2} + \frac{f_3 t_3 u_1^2 w_3}{t_1 t_2} - f_2 u_1 u_2 w_3 + f_2 t_2 u_1 u_2 w_3 - f_3 t_3 u_1 u_2 w_3 + \frac{f_3 t_3 u_1 u_2 w_3}{t_2} + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \text{<<7>>}, \frac{w_3}{t_1 t_2}\right\}$$

```
R3r = FTGchecks // FTG2,3 // FTG1,3 // FTG1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

```
Short[OCl = FTGchecks // FTG1,2 // FTG1,3]
```

$$\left\{f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + f_1 t_1 u_1^2 w_2 - f_2 t_2 u_1^2 w_2 + \text{<<10>>} + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \text{<<1>>}, \text{<<48>>} + \text{<<1>>}, \text{<<3>>}, \text{<<1>>}, \frac{w_2}{t_1}, \frac{w_3}{t_1}\right\}$$

```
Short[OCr = FTGchecks // FTG1,3 // FTG1,2]
```

$$\left\{f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + f_1 t_1 u_1^2 w_2 - f_2 t_2 u_1^2 w_2 + \text{<<10>>} + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \text{<<1>>}, \text{<<54>>} + \text{<<1>>}, \text{<<3>>}, \text{<<1>>}, \frac{w_2}{t_1}, \frac{w_3}{t_1}\right\}$$

```
OCl - OCr
```

$$\{0, -f_0 u_1 u_2 w_3 + f_0 t_1 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - f_0 t_1 u_1 u_3 w_3, -f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - f_0 t_1 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0\}$$

The Turbo-Burau Representation

```
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
TBi_,j_[ℱ_] :=
  Expand[ℱ /. {
    f_. vk_ :> Plus[f vk /. vj → (1 - t - η[i]) vi + (t + η[i]) vj,
      (t - 1) (Coefficient[f, η[i]] - Coefficient[f, η[j]]) (uk /. uj → (1 - t) ui + t uj) ui wj,
      Kδk,i (f /. _η → 0) (uj - ui) ui wj],
    uj → (1 - t) ui + t uj,
    wi → wi + (1 - t^-1) wj, wj → t^-1 wj}];
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
checks = {ff v1, ff v2, ff v3, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};
```

```
Short[R3l = checks // TB1,2 // TB1,3 // TB2,3, 10]
```

$$\left\{f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + t f_1 u_1^2 w_2 + f_2 u_1^2 w_2 - t f_2 u_1^2 w_2 + f_0 u_1 u_2 w_2 - f_0 u_1^2 w_3 - f_1 u_1^2 w_3 + t f_1 u_1^2 w_3 + 2 f_2 u_1^2 w_3 - \frac{f_2 u_1^2 w_3}{t} - t f_2 u_1^2 w_3 - f_3 u_1^2 w_3 + \frac{f_3 u_1^2 w_3}{t} - f_2 u_1 u_2 w_3 + t f_2 u_1 u_2 w_3 + f_3 u_1 u_2 w_3 - t f_3 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \text{<<9>>}, \frac{w_3}{t^2}\right\}$$

```
R3r = checks // TB2,3 // TB1,3 // TB1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

```
Short[OCl = checks // TB1,2 // TB1,3]
```

$$\left\{\text{<<20>>} + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \text{<<54>>} + t f_3 v_2 \eta[3], \text{<<7>>}, \frac{\text{<<1>>}}{t}, \frac{w_3}{t}\right\}$$

```
OCr = checks // TB1,3 // TB1,2; OCl - OCr
```

$$\{0, -f_0 u_1 u_2 w_3 + t f_0 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - t f_0 u_1 u_3 w_3, -f_0 u_1 u_2 w_2 + t f_0 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - t f_0 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0, 0, 0\}$$