Pensieve Header: Trying to remember how FreeLie works....

Preliminaries

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
        << FreeLie.m
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop,
  cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⌐, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.
```

```
In[•]:= AllLyndonWords[3, {x, y}] // FullForm
Out[•]//FullForm=
List[LW[x, x, y], LW[x, y, y]]
```

```
In[•]:= t1 = Total[AllLyndonWords[3, {x, y}]]
Out[•]=
x x y + x y y
```

```
In[•]:= t2 = LS[t1]
Out[•]=
LS[0, 0, x x y + x y y, ...]
```

```
In[•]:= t2[6]
Out[•]=
0
```

```
In[•]:= t2 // FullForm
Out[•]//FullForm=
LieSeries[LieSeries$13]
```

```
In[•]:= LieSeries[t1]
$RecursionLimit: Recursion depth of 4096 exceeded.
Out[•]=
```

```
In[•]:= LieSeries$13[3] // FullForm
Out[•]//FullForm=
Plus[LW[x, x, y], LW[x, y, y]]
```

```
In[•]:= Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V ;
UnitarityEqn = (V ** (V // dA[x] // dA[y]) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs[x] = 1 / 2; βs[y] = 0;
SeriesSolve[{α, β, γ, κ}, (ħ^-1 R4Eqn) && UnitarityEqn && CapEqn];
{V, κ}
```

SeriesSolve: In degree 1 arbitrarily setting {κs[x] → 0}.

SeriesSolve: In degree 3 arbitrarily setting {αs[x, y, y] → 0}.

```
Out[•]=
```

$$\left\{\text{Es}\left[\left\langle \text{x} \to \text{LS}\left[0, -\frac{\overline{\text{x y}}}{24}, 0, \ldots\right], \text{y} \to \text{LS}\left[\frac{\overline{\text{x}}}{2}, -\frac{\overline{\text{x y}}}{12}, 0, \ldots\right]\right\rangle, \text{CWS}\left[0, -\frac{\widehat{\text{x y}}}{48}, 0, \ldots\right]\right], \text{CWS}\left[0, -\frac{\widehat{\text{x x}}}{96}, 0, \ldots\right]\right\}$$

```
In[•]:= logF = V[[1]] // dσ[{x, y} → {y, x}]
logF@{4}
```

```
Out[•]=
```

$$\left\langle \text{x} \to \text{LS}\left[\frac{\overline{\text{y}}}{2}, \frac{\overline{\text{x y}}}{12}, 0, \ldots\right], \text{y} \to \text{LS}\left[0, \frac{\overline{\text{x y}}}{24}, 0, \ldots\right]\right\rangle$$

```
Out[•]=
```

$$\left\langle \text{x} \to \text{LS}\left[\frac{\overline{\text{y}}}{2}, \frac{\overline{\text{x y}}}{12}, 0, -\frac{1}{720}\overline{\text{x}\,\overline{\text{x}\,\overline{\text{x y}}}} + \frac{1}{720}\overline{\text{x}\,\overline{\overline{\text{x y}}\,\text{y}}} - \frac{\overline{\overline{\overline{\text{x y}}\,\text{y}}\,\text{y}}}{5760}, \ldots\right], \text{y} \to \text{LS}\left[0, \frac{\overline{\text{x y}}}{24}, 0, -\frac{\overline{\text{x}\,\overline{\text{x}\,\overline{\text{x y}}}}}{1440} + \frac{7\,\overline{\text{x}\,\overline{\overline{\text{x y}}\,\text{y}}}}{5760} - \frac{7\,\overline{\overline{\overline{\text{x y}}\,\text{y}}\,\text{y}}}{5760}, \ldots\right]\right\rangle$$

```
In[•]:= AWExp_d_[ℒ_] := Module[{t = AW[]},
  AW[] + Sum[t = Expand[t ** ℒ / k] /. a_AW /; Length[a] > d :→ 0, {k, d}]
 ]
```

Testing

```
In[•]:= d = 4
```

```
Out[•]=
```

4

In[•]:=
```
lhs = Sum[AW1 @@ Table[x, k] / k!, {k, 0, d}] // FA[x → x + y]
```

Out[•]=

$$\begin{aligned}
&AW_1[\,] + AW_1[x] + AW_1[y] + \frac{1}{2} AW_1[x, x] + \frac{1}{2} AW_1[x, y] + \frac{1}{2} AW_1[y, x] + \\
&\frac{1}{2} AW_1[y, y] + \frac{1}{6} AW_1[x, x, x] + \frac{1}{6} AW_1[x, x, y] + \frac{1}{6} AW_1[x, y, x] + \frac{1}{6} AW_1[x, y, y] + \\
&\frac{1}{6} AW_1[y, x, x] + \frac{1}{6} AW_1[y, x, y] + \frac{1}{6} AW_1[y, y, x] + \frac{1}{6} AW_1[y, y, y] + \\
&\frac{1}{24} AW_1[x, x, x, x] + \frac{1}{24} AW_1[x, x, x, y] + \frac{1}{24} AW_1[x, x, y, x] + \frac{1}{24} AW_1[x, x, y, y] + \\
&\frac{1}{24} AW_1[x, y, x, x] + \frac{1}{24} AW_1[x, y, x, y] + \frac{1}{24} AW_1[x, y, y, x] + \frac{1}{24} AW_1[x, y, y, y] + \\
&\frac{1}{24} AW_1[y, x, x, x] + \frac{1}{24} AW_1[y, x, x, y] + \frac{1}{24} AW_1[y, x, y, x] + \frac{1}{24} AW_1[y, x, y, y] + \\
&\frac{1}{24} AW_1[y, y, x, x] + \frac{1}{24} AW_1[y, y, x, y] + \frac{1}{24} AW_1[y, y, y, x] + \frac{1}{24} AW_1[y, y, y, y]
\end{aligned}$$

In[•]:=
```
rhs1 = AWExp_d[Plus @@ ι[Ad[x /. logF[[1]]][LW@x]]@{d}] /. AW → AW1
```

Out[•]=

$$\begin{aligned}
&AW_1[\,] + AW_1[x] + \frac{1}{2} AW_1[x, x] - \frac{1}{2} AW_1[x, y] + \frac{1}{2} AW_1[y, x] + \frac{1}{6} AW_1[x, x, x] - \\
&\frac{1}{3} AW_1[x, x, y] + \frac{1}{6} AW_1[x, y, x] + \frac{1}{8} AW_1[x, y, y] + \frac{1}{6} AW_1[y, x, x] - \\
&\frac{1}{4} AW_1[y, x, y] + \frac{1}{8} AW_1[y, y, x] + \frac{1}{24} AW_1[x, x, x, x] - \frac{1}{8} AW_1[x, x, x, y] + \\
&\frac{1}{24} AW_1[x, x, y, x] + \frac{1}{12} AW_1[x, x, y, y] + \frac{1}{24} AW_1[x, y, x, x] - \frac{1}{24} AW_1[x, y, x, y] - \\
&\frac{1}{48} AW_1[x, y, y, y] + \frac{1}{24} AW_1[y, x, x, x] - \frac{1}{8} AW_1[y, x, x, y] + \frac{1}{24} AW_1[y, x, y, x] + \\
&\frac{1}{16} AW_1[y, x, y, y] + \frac{1}{24} AW_1[y, y, x, x] - \frac{1}{16} AW_1[y, y, x, y] + \frac{1}{48} AW_1[y, y, y, x]
\end{aligned}$$

In[•]:=
```
rhs2 = AWExp_d[Plus @@ ι[Ad[y /. logF[[2]]][LW@y]]@{d}] /. AW → AW2
```

Out[•]=

$$\begin{aligned}
&AW_2[\,] + AW_2[y] + \frac{1}{2} AW_2[y, y] + \frac{1}{24} AW_2[x, y, y] - \frac{1}{12} AW_2[y, x, y] + \\
&\frac{1}{24} AW_2[y, y, x] + \frac{1}{6} AW_2[y, y, y] + \frac{1}{48} AW_2[x, y, y, y] - \frac{1}{48} AW_2[y, x, y, y] - \\
&\frac{1}{48} AW_2[y, y, x, y] + \frac{1}{48} AW_2[y, y, y, x] + \frac{1}{24} AW_2[y, y, y, y]
\end{aligned}$$

In[•]:=
```
rhs = (Expand[rhs1 * rhs2] // m_{1,2→1}) /. AW_1[w___] /; Length@{w} > d :→ 0
```

Out[•]=

$$\begin{aligned}
&AW_1[\,] + AW_1[x] + AW_1[y] + \frac{1}{2} AW_1[x, x] + \frac{1}{2} AW_1[x, y] + \frac{1}{2} AW_1[y, x] + \\
&\frac{1}{2} AW_1[y, y] + \frac{1}{6} AW_1[x, x, x] + \frac{1}{6} AW_1[x, x, y] + \frac{1}{6} AW_1[x, y, x] + \frac{1}{6} AW_1[x, y, y] + \\
&\frac{1}{6} AW_1[y, x, x] + \frac{1}{6} AW_1[y, x, y] + \frac{1}{6} AW_1[y, y, x] + \frac{1}{6} AW_1[y, y, y] + \\
&\frac{1}{24} AW_1[x, x, x, x] + \frac{1}{24} AW_1[x, x, x, y] + \frac{1}{24} AW_1[x, x, y, x] + \frac{1}{24} AW_1[x, x, y, y] + \\
&\frac{1}{24} AW_1[x, y, x, x] + \frac{1}{24} AW_1[x, y, x, y] + \frac{1}{24} AW_1[x, y, y, x] + \frac{1}{24} AW_1[x, y, y, y] + \\
&\frac{1}{24} AW_1[y, x, x, x] + \frac{1}{24} AW_1[y, x, x, y] + \frac{1}{24} AW_1[y, x, y, x] + \frac{1}{24} AW_1[y, x, y, y] + \\
&\frac{1}{24} AW_1[y, y, x, x] + \frac{1}{24} AW_1[y, y, x, y] + \frac{1}{24} AW_1[y, y, y, x] + \frac{1}{24} AW_1[y, y, y, y]
\end{aligned}$$

In[•]:=
```
lhs == rhs
```

Out[•]=
```
True
```