

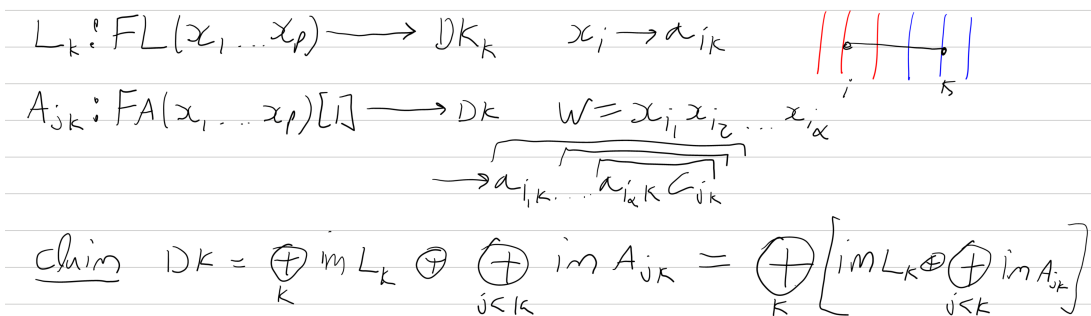
```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< FreeLie.m
```

FreeLie` implements / extends
 {*, +, **, \$SeriesShowDegree, <>, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords, AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, FreeLieFormatting, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, τ, ħ, ↦, ↪}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 240218.

```
In[*]:= Antipode[w_AW] := Antipode[w] = (-1)^Length[w] Reverse[w];
Antipode[expr_] := Expand[expr /. w_AW -> Antipode[w]]
```

Goal: Implement the “Emergent Drinfel’d-Kohno Algebra” EDK following



EDK[k_Integer, x_FL]: an element in the image of L_k .

EDK[j_Integer, k_Integer, w_AW]: an element in the image of $A_{j,k}$.

EDKBasis[ps, s, d] produces the basis of the EDK algebra with poles ps, strands 1,...,s, and at degree d.

```
In[*]:= EDKBasis[ps_List, s_Integer, d_Integer] := Flatten@{
  Table[EDK[j, k, #] & /@ AllWords[d - 1, ps], {j, 1, s - 1}, {k, j + 1, s}],
  Table[EDK[k, #] & /@ AllLyndonWords[d, ps], {k, s}]
}
```

```
In[*]:= EDKBasis[{x, y}, 3, 3]
```

```
Out[*]= {EDK[1, 2, AW[x, x]], EDK[1, 2, AW[x, y]], EDK[1, 2, AW[y, x]], EDK[1, 2, AW[y, y]],
  EDK[1, 3, AW[x, x]], EDK[1, 3, AW[x, y]], EDK[1, 3, AW[y, x]], EDK[1, 3, AW[y, y]],
  EDK[2, 3, AW[x, x]], EDK[2, 3, AW[x, y]], EDK[2, 3, AW[y, x]], EDK[2, 3, AW[y, y]],
  EDK[1, x̄xȳ], EDK[1, x̄ȳy], EDK[2, x̄xȳ], EDK[2, x̄ȳy], EDK[3, x̄xȳ], EDK[3, x̄ȳy]}
```

```
In[*]:= EDK[j_, k_, w_AW] /; j > k := EDK[k, j, Antipode[w]];
```

```
In[*]:= EDK[_, 0] = 0;
EDK /: EDK[jk_, x1_] + EDK[jk_, x2_] := EDK[jk, x1 + x2];
EDK /: c_ * EDK[jk_, x_] := EDK[jk, Expand[c x]];
EDK /: b[EDK[jk1_, c_ * (x1_LW | x1_AW)], EDK[jk2_, x2_]] :=
  Expand[c b[EDK[jk1, x1], EDK[jk2, x2]]];
EDK /: b[EDK[jk1_, x1_], EDK[jk2_, c_ * (x2_LW | x2_AW)]] :=
  Expand[c b[EDK[jk1, x1], EDK[jk2, x2]]];
EDK /: b[EDK[jk1_, x1_], EDK[jk2_, x2_Plus]] :=
  b[EDK[jk1, x1], EDK[jk2, #]] & /@ x2;
EDK /: b[EDK[jk1_, x1_Plus], EDK[jk2_, x2_]] := b[EDK[jk1, #], EDK[jk2, x2]] & /@ x1;
```

$$\left[A_{j_1 k_1}(w_1), A_{j_2 k_2}(w_2) \right] = 0 \text{ obvious}$$

```
In[*]:= EDK /: b[EDK[_, _, _], EDK[_, _, _]] := 0
```

Claim $[L_{k_1}(w_1), A_{j_2, k_2}(w_2)]$

$x \ y \ z \ 1 \ 2$
 $\rightarrow x y z$
 $\rightarrow a_{x2} a_{y2} a_{z2} c_{12}$
 \parallel
 $(-1)^3 a_{z1} a_{y1} a_{x1} c_{12}$

$k_1 \notin \{j_2, k_2\}$

$[L(w_1), A(w_2)] = 0$

$k_1 = j_2$ $k_1 = k_2$

$[L_{k_1}(w_1), A_{j_2, k_2}(w_2)] = A_{j_2, k_2}(w_2 \bar{w}_1)$

$[L_{k_1}(w_1), A_{j_2, k_2}(w_2)] = A_{j_2, k_2}(w_1 w_2)$

$= -A_{j_2, k_2}(w_2 w_1)$

```

In[ ] := EDK /: b[EDK[k1_, w1_LW], EDK[j2_, k2_, w2_AW]] := Which[
  k1 == j2, -EDK[j2, k2, w2 ** L[w1]],
  k1 == k2, EDK[j2, k2, L[w1] ** w2],
  True, 0
];
EDK /: b[EDK[j2_, k2_, w2_AW], EDK[k1_, w1_LW]] :=
  Expand[-b[EDK[k1, w1], EDK[j2, k2, w2]]]
  
```

```

In[ ] := b[EDK[1, LW[x, y]], EDK[1, 2, AW[z, z, x, y]]]
  
```

```

Out[ ] := AW[x, y, z, z, x, y] - AW[y, x, z, z, x, y]
  
```

In[*]:= Cases[{AW[1], LW[2], Yu[3]}, x_LW | x_AW]

Out[*]=

{AW[1], 2}

Claim $[L_{k_1}(w_1), L_{k_2}(w_2)]$

$(k_1 < k_2)$

$k_1 = k_2 (=k)$
 $[L_k(w_1), L_k(w_2)] = L_k(w_1, w_2)$

$\sum_{i_1, i_2} A_{k_1, k_2}(x_{i_2})$

$k_1 < k_2$

(i) $[L_{k_1}(x_{i_1}), L_{k_2}(x_{i_2})] = [a_{i_1 k_1}, a_{i_2 k_2}] = \sum_{i_1, i_2} A_{i_1, i_2} [a_{i_2 k_2}, c_{k_1, k_2}]$

(ii) $[L_{k_1}(w_1, w'_1), L_{k_2}(w_2)]$ should be $[L_{k_1}(w_1), [L_{k_1}(w'_1), L_{k_2}(w_2)]] - [L_{k_1}(w'_1), [L_{k_1}(w_1), L_{k_2}(w_2)]]$

(iii) $[L_{k_1}(w_1), L_{k_2}(w_2, w'_2)]$ should be $-[L_{k_2}(w'_2), [L_{k_1}(w_1), L_{k_2}(w_2)]] + [L_{k_2}(w_2), [L_{k_1}(w_1), L_{k_2}(w'_2)]]$

Prop $[L_{k_1}(w_1), L_{k_2}(w_2)] = A_{k_1, k_2} \left(\sum_i (a_i w_2) x_i \overline{(a_i w_1)} \right)$

Satisfies the above three conditions

In[*]:= AllLyndonWords[4, {x, y, z}] /. w_LW => Union[List@@w]

Out[*]=

{x, y}, {x, z}, {x, y}, {x, y, z}, {x, y, z}, {x, z}, {x, y, z}, {x, y}, {x, y, z},
 {x, y, z}, {x, y, z}, {x, y, z}, {x, y, z}, {x, y, z}, {x, z}, {y, z}, {y, z}, {y, z}

```
EDK /: b[EDK[k1_, w1_LW], EDK[k2_, w2_LW]] := Which[
  k1 == k2, EDK[k1, b[w1, w2]],
  k1 < k2, EDK[k1, k2, Sum[tau[LW@p, w2] ** AW[p] ** Antipode[tau[LW@p, w1]],
    {p, Union[List@@w1] ∩ Union[List@@w2]}]],
  k2 < k1, EDK[k2, k1, Sum[-tau[LW@p, w1] ** AW[p] ** Antipode[tau[LW@p, w2]],
    {p, Union[List@@w1] ∩ Union[List@@w2]}]]
]
```

Testing anti-symmetry and Jacobi for EDK:

```
In[*]:= bas = EDKBasis[{x, y, z}, 3, 4]
Union@Table[{α, β} = αβ; b[α, β] + b[β, α], {αβ, Subsets[bas, {2}]}]

Out[*]=
{EDK[1, 2, AW[x, x, x]], EDK[1, 2, AW[x, x, y]], EDK[1, 2, AW[x, x, z]],
EDK[1, 2, AW[x, y, x]], EDK[1, 2, AW[x, y, y]], EDK[1, 2, AW[x, y, z]],
EDK[1, 2, AW[x, z, x]], EDK[1, 2, AW[x, z, y]], EDK[1, 2, AW[x, z, z]],
EDK[1, 2, AW[y, x, x]], EDK[1, 2, AW[y, x, y]], EDK[1, 2, AW[y, x, z]],
EDK[1, 2, AW[y, y, x]], EDK[1, 2, AW[y, y, y]], EDK[1, 2, AW[y, y, z]],
EDK[1, 2, AW[y, z, x]], EDK[1, 2, AW[y, z, y]], EDK[1, 2, AW[y, z, z]],
EDK[1, 2, AW[z, x, x]], EDK[1, 2, AW[z, x, y]], EDK[1, 2, AW[z, x, z]],
EDK[1, 2, AW[z, y, x]], EDK[1, 2, AW[z, y, y]], EDK[1, 2, AW[z, y, z]],
EDK[1, 2, AW[z, z, x]], EDK[1, 2, AW[z, z, y]], EDK[1, 2, AW[z, z, z]],
EDK[1, 3, AW[x, x, x]], EDK[1, 3, AW[x, x, y]], EDK[1, 3, AW[x, x, z]],
EDK[1, 3, AW[x, y, x]], EDK[1, 3, AW[x, y, y]], EDK[1, 3, AW[x, y, z]],
EDK[1, 3, AW[x, z, x]], EDK[1, 3, AW[x, z, y]], EDK[1, 3, AW[x, z, z]],
EDK[1, 3, AW[y, x, x]], EDK[1, 3, AW[y, x, y]], EDK[1, 3, AW[y, x, z]],
EDK[1, 3, AW[y, y, x]], EDK[1, 3, AW[y, y, y]], EDK[1, 3, AW[y, y, z]],
EDK[1, 3, AW[y, z, x]], EDK[1, 3, AW[y, z, y]], EDK[1, 3, AW[y, z, z]],
EDK[1, 3, AW[z, x, x]], EDK[1, 3, AW[z, x, y]], EDK[1, 3, AW[z, x, z]],
EDK[1, 3, AW[z, y, x]], EDK[1, 3, AW[z, y, y]], EDK[1, 3, AW[z, y, z]],
EDK[1, 3, AW[z, z, x]], EDK[1, 3, AW[z, z, y]], EDK[1, 3, AW[z, z, z]],
EDK[2, 3, AW[x, x, x]], EDK[2, 3, AW[x, x, y]], EDK[2, 3, AW[x, x, z]],
EDK[2, 3, AW[x, y, x]], EDK[2, 3, AW[x, y, y]], EDK[2, 3, AW[x, y, z]],
EDK[2, 3, AW[x, z, x]], EDK[2, 3, AW[x, z, y]], EDK[2, 3, AW[x, z, z]],
EDK[2, 3, AW[y, x, x]], EDK[2, 3, AW[y, x, y]], EDK[2, 3, AW[y, x, z]],
EDK[2, 3, AW[y, y, x]], EDK[2, 3, AW[y, y, y]], EDK[2, 3, AW[y, y, z]],
EDK[2, 3, AW[y, z, x]], EDK[2, 3, AW[y, z, y]], EDK[2, 3, AW[y, z, z]],
EDK[2, 3, AW[z, x, x]], EDK[2, 3, AW[z, x, y]], EDK[2, 3, AW[z, x, z]],
EDK[2, 3, AW[z, y, x]], EDK[2, 3, AW[z, y, y]], EDK[2, 3, AW[z, y, z]],
EDK[2, 3, AW[z, z, x]], EDK[2, 3, AW[z, z, y]], EDK[2, 3, AW[z, z, z]],
EDK[1, x x x̄y], EDK[1, x x x̄z], EDK[1, x x̄y y], EDK[1, x x̄y z],
EDK[1, x x̄z y], EDK[1, x x̄z z], EDK[1, x̄y x̄z], EDK[1, x̄y y y], EDK[1, x̄y y z],
EDK[1, x̄y z y], EDK[1, x̄y z z], EDK[1, x̄z y y], EDK[1, x̄z y z], EDK[1, x̄z z y],
EDK[1, x̄z z z], EDK[1, y y y z], EDK[1, y y z z], EDK[1, y z z z], EDK[2, x x x̄y],
EDK[2, x x x̄z], EDK[2, x x̄y y], EDK[2, x x̄y z], EDK[2, x x̄z y], EDK[2, x x̄z z],
EDK[2, x̄y x̄z], EDK[2, x̄y y y], EDK[2, x̄y y z], EDK[2, x̄y z y], EDK[2, x̄y z z],
EDK[2, x̄z y y], EDK[2, x̄z y z], EDK[2, x̄z z y], EDK[2, x̄z z z], EDK[2, y y y z],
EDK[2, y y z z], EDK[2, y z z z], EDK[3, x x x̄y], EDK[3, x x x̄z], EDK[3, x x̄y y],
```

EDK [3, $\overline{x x y z}$], EDK [3, $\overline{x x z y}$], EDK [3, $\overline{x x z z}$], EDK [3, $\overline{x y x z}$], EDK [3, $\overline{x y y y}$],
 EDK [3, $\overline{x y y z}$], EDK [3, $\overline{x y z y}$], EDK [3, $\overline{x y z z}$], EDK [3, $\overline{x z y y}$], EDK [3, $\overline{x z y z}$],
 EDK [3, $\overline{x z z y}$], EDK [3, $\overline{x z z z}$], EDK [3, $\overline{y y y z}$], EDK [3, $\overline{y y z z}$], EDK [3, $\overline{y z z z}$]

Out[]=

{0}

```
In[ ]:= bas = EDKBasis[{x, y, z}, 3, 4]
Union@Table[{α, β, γ} = αβγ;
  b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]], {αβγ, Subsets[bas, {3}]}
]
```

Out[]=

{EDK [1, 2, AW[x, x, x]], EDK [1, 2, AW[x, x, y]], EDK [1, 2, AW[x, x, z]],
 EDK [1, 2, AW[x, y, x]], EDK [1, 2, AW[x, y, y]], EDK [1, 2, AW[x, y, z]],
 EDK [1, 2, AW[x, z, x]], EDK [1, 2, AW[x, z, y]], EDK [1, 2, AW[x, z, z]],
 EDK [1, 2, AW[y, x, x]], EDK [1, 2, AW[y, x, y]], EDK [1, 2, AW[y, x, z]],
 EDK [1, 2, AW[y, y, x]], EDK [1, 2, AW[y, y, y]], EDK [1, 2, AW[y, y, z]],
 EDK [1, 2, AW[y, z, x]], EDK [1, 2, AW[y, z, y]], EDK [1, 2, AW[y, z, z]],
 EDK [1, 2, AW[z, x, x]], EDK [1, 2, AW[z, x, y]], EDK [1, 2, AW[z, x, z]],
 EDK [1, 2, AW[z, y, x]], EDK [1, 2, AW[z, y, y]], EDK [1, 2, AW[z, y, z]],
 EDK [1, 2, AW[z, z, x]], EDK [1, 2, AW[z, z, y]], EDK [1, 2, AW[z, z, z]],
 EDK [1, 3, AW[x, x, x]], EDK [1, 3, AW[x, x, y]], EDK [1, 3, AW[x, x, z]],
 EDK [1, 3, AW[x, y, x]], EDK [1, 3, AW[x, y, y]], EDK [1, 3, AW[x, y, z]],
 EDK [1, 3, AW[x, z, x]], EDK [1, 3, AW[x, z, y]], EDK [1, 3, AW[x, z, z]],
 EDK [1, 3, AW[y, x, x]], EDK [1, 3, AW[y, x, y]], EDK [1, 3, AW[y, x, z]],
 EDK [1, 3, AW[y, y, x]], EDK [1, 3, AW[y, y, y]], EDK [1, 3, AW[y, y, z]],
 EDK [1, 3, AW[y, z, x]], EDK [1, 3, AW[y, z, y]], EDK [1, 3, AW[y, z, z]],
 EDK [1, 3, AW[z, x, x]], EDK [1, 3, AW[z, x, y]], EDK [1, 3, AW[z, x, z]],
 EDK [1, 3, AW[z, y, x]], EDK [1, 3, AW[z, y, y]], EDK [1, 3, AW[z, y, z]],
 EDK [1, 3, AW[z, z, x]], EDK [1, 3, AW[z, z, y]], EDK [1, 3, AW[z, z, z]],
 EDK [2, 3, AW[x, x, x]], EDK [2, 3, AW[x, x, y]], EDK [2, 3, AW[x, x, z]],
 EDK [2, 3, AW[x, y, x]], EDK [2, 3, AW[x, y, y]], EDK [2, 3, AW[x, y, z]],
 EDK [2, 3, AW[x, z, x]], EDK [2, 3, AW[x, z, y]], EDK [2, 3, AW[x, z, z]],
 EDK [2, 3, AW[y, x, x]], EDK [2, 3, AW[y, x, y]], EDK [2, 3, AW[y, x, z]],
 EDK [2, 3, AW[y, y, x]], EDK [2, 3, AW[y, y, y]], EDK [2, 3, AW[y, y, z]],
 EDK [2, 3, AW[y, z, x]], EDK [2, 3, AW[y, z, y]], EDK [2, 3, AW[y, z, z]],
 EDK [2, 3, AW[z, x, x]], EDK [2, 3, AW[z, x, y]], EDK [2, 3, AW[z, x, z]],
 EDK [2, 3, AW[z, y, x]], EDK [2, 3, AW[z, y, y]], EDK [2, 3, AW[z, y, z]],
 EDK [2, 3, AW[z, z, x]], EDK [2, 3, AW[z, z, y]], EDK [2, 3, AW[z, z, z]],
 EDK [1, x $\overline{x x y}$], EDK [1, x $\overline{x x z}$], EDK [1, x $\overline{x y y}$], EDK [1, x $\overline{x y z}$],
 EDK [1, x $\overline{x z y}$], EDK [1, x $\overline{x z z}$], EDK [1, $\overline{x y x z}$], EDK [1, $\overline{x y y y}$], EDK [1, x $\overline{y y z}$],
 EDK [1, x $\overline{y z y}$], EDK [1, x $\overline{y z z}$], EDK [1, $\overline{x z y y}$], EDK [1, $\overline{x z y z}$], EDK [1, $\overline{x z z y}$],

EDK[1, $\overline{xz z z}$], EDK[1, $\overline{y y yz}$], EDK[1, $\overline{y yz z}$], EDK[1, $\overline{yz z z}$], EDK[2, $\overline{x x xy}$],
 EDK[2, $\overline{x x xz}$], EDK[2, $\overline{x xy y}$], EDK[2, $\overline{x x yz}$], EDK[2, $\overline{x xzy}$], EDK[2, $\overline{x xzz}$],
 EDK[2, $\overline{xy xz}$], EDK[2, $\overline{xy y y}$], EDK[2, $\overline{x y yz}$], EDK[2, $\overline{x yz y}$], EDK[2, $\overline{x yzz}$],
 EDK[2, $\overline{xzy y}$], EDK[2, $\overline{xz yz}$], EDK[2, $\overline{xz z y}$], EDK[2, $\overline{xz z z}$], EDK[2, $\overline{y y yz}$],
 EDK[2, $\overline{y yz z}$], EDK[2, $\overline{yz z z}$], EDK[3, $\overline{x x xy}$], EDK[3, $\overline{x x xz}$], EDK[3, $\overline{x xy y}$],
 EDK[3, $\overline{x x yz}$], EDK[3, $\overline{x xzy}$], EDK[3, $\overline{x xzz}$], EDK[3, $\overline{xy xz}$], EDK[3, $\overline{xy y y}$],
 EDK[3, $\overline{x y yz}$], EDK[3, $\overline{x yz y}$], EDK[3, $\overline{x yz z}$], EDK[3, $\overline{xzy y}$], EDK[3, $\overline{xz yz}$],
 EDK[3, $\overline{xz z y}$], EDK[3, $\overline{xz z z}$], EDK[3, $\overline{y y yz}$], EDK[3, $\overline{y yz z}$], EDK[3, $\overline{yz z z}$]

Out[] =
 {0}

σ

$\Phi^{\sigma[1,23,4]} \rightarrow \Phi^{\sigma[\{1\},\{2,3\},\{4\}]} \rightarrow \sigma[\{1\}, \{2, 3\}, \{4\}] [\Phi]$
 $\Phi^{\sigma[2,3,1]}$
 $\Phi^{\sigma[1,x23,4]}$
 $\Delta[i \rightarrow jk]$
 EDK[j1, sum of Lie words] + EDK[j2, sum of Lie words] +
 EDK[j, k, sum of Associative words] +
 p2s

```
In[ ] :=
σ /:  $\mathcal{E}_{s-\sigma} := s[\mathcal{E}]$ ;
σ[lft___, i_Integer, rgt___] := σ[lft, IntegerDigits[i], rgt];
σ[___][0] = 0;
x_Plus //  $s_{-\sigma} := s[\#] \& /@ x$ ;
EDK[jk_, x_Plus] //  $s_{-\sigma} := s[EDK[jk, \#]] \& /@ x$ ;
EDK[jk_, c_ * (w_LW | w_AW)] //  $s_{-\sigma} := Expand[c * s[EDK[jk, w]]]$ ;
EDK[k_, LW[x_]] //  $s_{-\sigma} := Sum[EDK[\alpha, LW[x]], \{\alpha, s[[k]]\}]$ ;
EDK[k_, w_LW] //  $s_{-\sigma} := b@@(s[EDK[k, \#]] \& /@ LyndonFactorization[w])$ ;
EDK[j_, k_, AW[]] //  $s_{-\sigma} := Sum[EDK[\alpha, \beta, AW[]], \{\alpha, s[[j]]\}, \{\beta, s[[k]]\}]$ ;
EDK[j_, k_, AW[x_, w___]] //  $s_{-\sigma} := b[s@EDK[k, LW@x], s@EDK[j, k, AW[w]]]$ 
```

```
In[*]:= bas = EDKBasis[{x, y}, 2, 1];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[12,3]}$ , { $\alpha$ , bas}]
```

```
Out[*]=
```

```
EDK[1, 2, AW[]]  $\rightarrow$  EDK[1, 3, AW[]] + EDK[2, 3, AW[]]
EDK[1,  $\overline{x}$ ]  $\rightarrow$  EDK[1,  $\overline{x}$ ] + EDK[2,  $\overline{x}$ ]
EDK[1,  $\overline{y}$ ]  $\rightarrow$  EDK[1,  $\overline{y}$ ] + EDK[2,  $\overline{y}$ ]
EDK[2,  $\overline{x}$ ]  $\rightarrow$  EDK[3,  $\overline{x}$ ]
EDK[2,  $\overline{y}$ ]  $\rightarrow$  EDK[3,  $\overline{y}$ ]
```

```
In[*]:= bas = EDKBasis[{x, y}, 2, 2];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[12,3]}$ , { $\alpha$ , bas}]
```

```
Out[*]=
```

```
EDK[1, 2, AW[x]]  $\rightarrow$  EDK[1, 3, AW[x]] + EDK[2, 3, AW[x]]
EDK[1, 2, AW[y]]  $\rightarrow$  EDK[1, 3, AW[y]] + EDK[2, 3, AW[y]]
EDK[1,  $\overline{xy}$ ]  $\rightarrow$  EDK[1,  $\overline{xy}$ ] + EDK[2,  $\overline{xy}$ ]
EDK[2,  $\overline{xy}$ ]  $\rightarrow$  EDK[3,  $\overline{xy}$ ]
```

```
In[*]:= bas = EDKBasis[{x, y}, 2, 3];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[12,3]}$ , { $\alpha$ , bas}]
```

```
Out[*]=
```

```
EDK[1, 2, AW[x, x]]  $\rightarrow$  EDK[1, 3, AW[x, x]] + EDK[2, 3, AW[x, x]]
EDK[1, 2, AW[x, y]]  $\rightarrow$  EDK[1, 3, AW[x, y]] + EDK[2, 3, AW[x, y]]
EDK[1, 2, AW[y, x]]  $\rightarrow$  EDK[1, 3, AW[y, x]] + EDK[2, 3, AW[y, x]]
EDK[1, 2, AW[y, y]]  $\rightarrow$  EDK[1, 3, AW[y, y]] + EDK[2, 3, AW[y, y]]
EDK[1,  $x \overline{xy}$ ]  $\rightarrow$  EDK[1,  $x \overline{xy}$ ] + EDK[2,  $x \overline{xy}$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]
EDK[1,  $\overline{xy}y$ ]  $\rightarrow$  EDK[1,  $\overline{xy}y$ ] + EDK[2,  $\overline{xy}y$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]
EDK[2,  $x \overline{xy}$ ]  $\rightarrow$  EDK[3,  $x \overline{xy}$ ]
EDK[2,  $\overline{xy}y$ ]  $\rightarrow$  EDK[3,  $\overline{xy}y$ ]
```



```
In[ ]:= bas = EDKBasis[{x, y}, 2, 4];
Column@Table[α → ασ[12,3], {α, bas}]
```

Out[]:=

$$\begin{aligned}
 & \text{EDK}[1, 2, \text{AW}[x, x, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, x, x]] + \text{EDK}[2, 3, \text{AW}[x, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, x, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, x, y]] + \text{EDK}[2, 3, \text{AW}[x, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, y, x]] + \text{EDK}[2, 3, \text{AW}[x, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, y, y]] + \text{EDK}[2, 3, \text{AW}[x, y, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, x, x]] + \text{EDK}[2, 3, \text{AW}[y, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, x, y]] + \text{EDK}[2, 3, \text{AW}[y, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, y, x]] + \text{EDK}[2, 3, \text{AW}[y, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, y, y]] + \text{EDK}[2, 3, \text{AW}[y, y, y]] \\
 & \text{EDK}\left[1, \overline{x \ x \ xy}\right] \rightarrow \text{EDK}\left[1, \overline{x \ x \ xy}\right] + \text{EDK}\left[2, \overline{x \ x \ xy}\right] + \text{EDK}[1, 2, -2 \text{AW}[x, x, y] + 2 \text{AW}[y, x, x]] \\
 & \text{EDK}\left[1, \overline{x \ xy \ y}\right] \rightarrow \text{EDK}\left[1, \overline{x \ xy \ y}\right] + \text{EDK}\left[2, \overline{x \ xy \ y}\right] + \\
 & \quad \text{EDK}[1, 2, -\text{AW}[x, x, y] - \text{AW}[x, y, y] + \text{AW}[y, x, x] + \text{AW}[y, y, x]] \\
 & \text{EDK}\left[1, \overline{xy \ y \ y}\right] \rightarrow \text{EDK}\left[1, \overline{xy \ y \ y}\right] + \text{EDK}\left[2, \overline{xy \ y \ y}\right] + \text{EDK}[1, 2, -2 \text{AW}[x, y, y] + 2 \text{AW}[y, y, x]] \\
 & \text{EDK}\left[2, \overline{x \ x \ xy}\right] \rightarrow \text{EDK}\left[3, \overline{x \ x \ xy}\right] \\
 & \text{EDK}\left[2, \overline{x \ xy \ y}\right] \rightarrow \text{EDK}\left[3, \overline{x \ xy \ y}\right] \\
 & \text{EDK}\left[2, \overline{xy \ y \ y}\right] \rightarrow \text{EDK}\left[3, \overline{xy \ y \ y}\right]
 \end{aligned}$$

```
In[ ]:= bas = EDKBasis[{x, y}, 2, 4];
Column@Table[α → ασ[1,23], {α, bas}]
```

Out[]:=

$$\begin{aligned}
 & \text{EDK}[1, 2, \text{AW}[x, x, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, x, x]] + \text{EDK}[1, 3, \text{AW}[x, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, x, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, x, y]] + \text{EDK}[1, 3, \text{AW}[x, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, y, x]] + \text{EDK}[1, 3, \text{AW}[x, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, y, y]] + \text{EDK}[1, 3, \text{AW}[x, y, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, x, x]] + \text{EDK}[1, 3, \text{AW}[y, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, x, y]] + \text{EDK}[1, 3, \text{AW}[y, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, y, x]] + \text{EDK}[1, 3, \text{AW}[y, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, y, y]] + \text{EDK}[1, 3, \text{AW}[y, y, y]] \\
 & \text{EDK}\left[1, \overline{x \ x \ xy}\right] \rightarrow \text{EDK}\left[1, \overline{x \ x \ xy}\right] \\
 & \text{EDK}\left[1, \overline{x \ xy \ y}\right] \rightarrow \text{EDK}\left[1, \overline{x \ xy \ y}\right] \\
 & \text{EDK}\left[1, \overline{xy \ y \ y}\right] \rightarrow \text{EDK}\left[1, \overline{xy \ y \ y}\right] \\
 & \text{EDK}\left[2, \overline{x \ x \ xy}\right] \rightarrow \text{EDK}\left[2, \overline{x \ x \ xy}\right] + \text{EDK}\left[3, \overline{x \ x \ xy}\right] + \text{EDK}[2, 3, -2 \text{AW}[x, x, y] + 2 \text{AW}[y, x, x]] \\
 & \text{EDK}\left[2, \overline{x \ xy \ y}\right] \rightarrow \text{EDK}\left[2, \overline{x \ xy \ y}\right] + \text{EDK}\left[3, \overline{x \ xy \ y}\right] + \\
 & \quad \text{EDK}[2, 3, -\text{AW}[x, x, y] - \text{AW}[x, y, y] + \text{AW}[y, x, x] + \text{AW}[y, y, x]] \\
 & \text{EDK}\left[2, \overline{xy \ y \ y}\right] \rightarrow \text{EDK}\left[2, \overline{xy \ y \ y}\right] + \text{EDK}\left[3, \overline{xy \ y \ y}\right] + \text{EDK}[2, 3, -2 \text{AW}[x, y, y] + 2 \text{AW}[y, y, x]]
 \end{aligned}$$

```

In[*]:= bas = EDKBasis[{x, y}, 2, 2];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[3,12]}$ , { $\alpha$ , bas}]

Out[*]=
EDK[1, 2, AW[x]]  $\rightarrow \sigma[\{3\}, \{1, 2\}]$  [EDK[1, 2, AW[x]]]
EDK[1, 2, AW[y]]  $\rightarrow \sigma[\{3\}, \{1, 2\}]$  [EDK[1, 2, AW[y]]]
EDK[1,  $\overline{xy}$ ]  $\rightarrow$  EDK[3,  $\overline{xy}$ ]
EDK[2,  $\overline{xy}$ ]  $\rightarrow$  EDK[1,  $\overline{xy}$ ] + EDK[2,  $\overline{xy}$ ]

In[*]:= bas = EDKBasis[{x, y}, 2, 3];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[3,12]}$ , { $\alpha$ , bas}]

Out[*]=
EDK[1, 2, AW[x, x]]  $\rightarrow \sigma[\{3\}, \{1, 2\}]$  [EDK[1, 2, AW[x, x]]]
EDK[1, 2, AW[x, y]]  $\rightarrow \sigma[\{3\}, \{1, 2\}]$  [EDK[1, 2, AW[x, y]]]
EDK[1, 2, AW[y, x]]  $\rightarrow \sigma[\{3\}, \{1, 2\}]$  [EDK[1, 2, AW[y, x]]]
EDK[1, 2, AW[y, y]]  $\rightarrow \sigma[\{3\}, \{1, 2\}]$  [EDK[1, 2, AW[y, y]]]
EDK[1,  $\overline{x \overline{xy}}$ ]  $\rightarrow$  EDK[3,  $\overline{x \overline{xy}}$ ]
EDK[1,  $\overline{xyy}$ ]  $\rightarrow$  EDK[3,  $\overline{xyy}$ ]
EDK[2,  $\overline{x \overline{xy}}$ ]  $\rightarrow$  EDK[1,  $\overline{x \overline{xy}}$ ] + EDK[2,  $\overline{x \overline{xy}}$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]
EDK[2,  $\overline{xyy}$ ]  $\rightarrow$  EDK[1,  $\overline{xyy}$ ] + EDK[2,  $\overline{xyy}$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]

In[*]:= bas1 = EDKBasis[{x, y}, 2, 4];
bas2 = EDKBasis[{x, y}, 2, 4];
s =  $\sigma[12, 3]$ ;
Column@
DeleteCases[_  $\rightarrow$  0] [Flatten@Table[ $\{\alpha, \beta\} \rightarrow \mathbf{b}[\alpha, \beta]^s - \mathbf{b}[\alpha^s, \beta^s]$ , { $\alpha$ , bas1}, { $\beta$ , bas2}]]

Out[*]=

In[*]:= bas1 = EDKBasis[{x, y}, 2, 4];
bas2 = EDKBasis[{x, y}, 2, 4];
s =  $\sigma[1, 23]$ ;
Column@
DeleteCases[_  $\rightarrow$  0] [Flatten@Table[ $\{\alpha, \beta\} \rightarrow \mathbf{b}[\alpha, \beta]^s - \mathbf{b}[\alpha^s, \beta^s]$ , { $\alpha$ , bas1}, { $\beta$ , bas2}]]

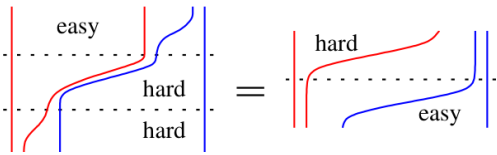
Out[*]=

In[*]:= bas1 = EDKBasis[{x, y}, 2, 4];
bas2 = EDKBasis[{x, y}, 2, 4];
s =  $\sigma[23, 1]$ ;
Column@
DeleteCases[_  $\rightarrow$  0] [Flatten@Table[ $\{\alpha, \beta\} \rightarrow \mathbf{b}[\alpha, \beta]^s - \mathbf{b}[\alpha^s, \beta^s]$ , { $\alpha$ , bas1}, { $\beta$ , bas2}]]

Out[*]=

```

The Full Pole-Strand “Operad”.



Goal: Implement the pentagon,

$$\Phi \star \star \Phi^{\sigma[y \rightarrow y+1, 2]} \star \star \Phi^{\sigma[x \rightarrow y, y \rightarrow 1, 2]} = \Phi^{\sigma[x \rightarrow x+y, y \rightarrow 1, 2]} \star \star \Phi^{\sigma[12]}.$$