Initialization

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Ens"];
<< "../../Projects/WKO4/FreeLie.m";
$IterationLimit = ∞; $SeriesShowDegree = 1;
Arb = Arbitrator → (Replace[#, _ :↦ RandomInteger[{-3, 3}], 1] &);
{BDegr, EDegr} = {5, 5};
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⎵, ⏜}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.
```

Braidors Extend

```
In[•]:= Clear[B, Bs];
        Bs[3, 2] = 1/2; Bs[2, 1] = 0;
        B = DKS[3, Bs];
        BInfo = SeriesSolve[B, (B^σ[1,2,3] ** B^σ[13,2,4] ** B^σ[1,3,4] ≡ B^σ[12,3,4] ** B^σ[1,2,4] ** B^σ[14,2,3]) ∧
            ((1/2) DKS[t[1, 2]]^σ[12,3] ≡ (1/2) B ** DKS[t[1, 3]] ** B^σ[1,3,2]), Arb];
        B@{6} // Timing
        ArbAll = Length[Last[#]] & /@ Read[BInfo]
```

SeriesSolve: In degree 3 arbitrarily setting {Bs[3, 1, 2, 2] → 1}.

SeriesSolve: In degree 5 arbitrarily setting {Bs[3, 1, 1, 1, 2, 2] → 0}.

Out[•]= $\left\{3.46875, \mathrm{DKS}\left[\frac{\overline{t_{23}}}{2}, \frac{1}{12}\overline{t_{13}\,t_{23}}, \overline{t_{13}\,t_{23}\,t_{23}}, -\frac{1441\,\overline{t_{13}\,t_{23}\,t_{23}\,t_{23}}}{5760} + \frac{361}{720}\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,t_{23}} - \frac{1}{720}\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}, \right.\right.$
$-\frac{481\,\overline{t_{13}\,t_{23}\,t_{23}\,t_{23}\,t_{23}}}{7680} + \frac{1}{6}\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,\overline{t_{13}\,t_{23}}} + \frac{721\,\overline{t_{13}\,t_{23}\,\overline{t_{13}\,t_{23}}\,t_{23}}}{8640} + \frac{\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,t_{23}\,t_{23}}}{3840},$
$\frac{13\,441\,\overline{t_{13}\,t_{23}\,t_{23}\,t_{23}\,t_{23}\,t_{23}}}{645\,120} - \frac{160\,273\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,t_{23}\,\overline{t_{13}\,t_{23}}}}{145\,152} - \frac{3\,074\,471\,\overline{t_{13}\,t_{23}\,\overline{t_{13}\,t_{23}}\,t_{23}\,t_{23}}}{2\,903\,040} -$
$\frac{272\,183\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,t_{23}\,t_{23}\,t_{23}}}{483\,840} + \frac{1681\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,\overline{t_{13}\,t_{23}}}}}{20\,160} + \frac{6143\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}\,\overline{t_{13}\,t_{23}}\,t_{23}}}{11\,340} +$
$\left.\left.\frac{90\,733\,\overline{t_{13}\,t_{13}\,\overline{t_{13}\,t_{23}}\,t_{23}\,t_{23}}}{161\,280} - \frac{631\,\overline{t_{13}\,t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}\,t_{23}}}{15\,120} + \frac{\overline{t_{13}\,t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}}}{30\,240}, \ldots\right]\right\}$

Out[•]= {0, 0, 1, 0, 1, 0}

```
In[•]:= B^σ[{},1,2]@{7}
```

SeriesSolve: In degree 7 arbitrarily setting {Bs[3, 1, 1, 1, 1, 1, 2, 2] → 2}.

Out[•]= $\mathrm{DKS}\left[\frac{\overline{t_{12}}}{2}, 0, 0, 0, 0, 0, 0, \ldots\right]$

```
In[•]:= Table[Length@AllLyndonWords[n, {1, 2}], {n, 10}]
```

Out[•]= {2, 1, 2, 3, 6, 9, 18, 30, 56, 99}

Syzygy a)

```
In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
        SyzInfo = SeriesSolve[EE, (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]), Arb];
        EE@{7}; // Timing
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → 2, EEs[3, 2] → 2, EEs[4, 2] → 1, EEs[4, 3] → 2}.

SeriesSolve: In degree 2 arbitrarily setting {EEs[4, 2, 3] → 1}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → −3, EEs[4, 2, 2, 3] → −3, EEs[4, 2, 3, 3] → 2}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
Out[•]= {19.2969, Null}
```

```
In[•]:= ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
Out[•]= {4, 1, 3, 5, 11, 17, 35}
```

```
In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
        SyzInfo = SeriesSolve[EE,
            (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧ (EE^σ[{},2,3,4] ≡ DKS[0]), Arb];
        EE@{7}; // Timing
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → 0}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → 3}.

SeriesSolve: In degree 4 arbitrarily setting {EEs[4, 1, 1, 2, 3] → −2, EEs[4, 1, 2, 3, 3] → 1}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
Out[•]= {19.1094, Null}
```

```
In[•]:= ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
Out[•]= {1, 0, 1, 2, 5, 8, 17}
```

```
In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
        SyzInfo = SeriesSolve[EE, (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧
            (EE^σ[{},2,3,4] ≡ DKS[0]) ∧ (EE^σ[1,{},3,4] ≡ DKS[0]), Arb];
        EE@{7}; // Timing
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → −2}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → −3}.

SeriesSolve: In degree 4 arbitrarily setting {EEs[4, 1, 1, 2, 3] → −1, EEs[4, 1, 2, 3, 3] → −2}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
Out[•]= {19.0938, Null}
```

```
In[•]:= ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
Out[•]= {1, 0, 1, 2, 5, 8, 17}
```

```
In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE,
    (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧ (EE^σ[{},2,3,4] ≡ DKS[0]) ∧
     (EE^σ[1,{},3,4] ≡ DKS[0]) ∧ (EE^σ[1,2,{},4] ≡ DKS[0]) ∧ (EE^σ[1,2,3,{}] ≡ DKS[0]), Arb];
EE@{7}; // Timing
```

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → 3}.

SeriesSolve: In degree 4 arbitrarily setting {EEs[4, 1, 1, 2, 3] → −1, EEs[4, 1, 2, 3, 3] → 0}.

SeriesSolve: In degree 5 arbitrarily setting

{EEs[4, 1, 1, 1, 2, 3] → 3, EEs[4, 1, 1, 2, 3, 3] → 3, EEs[4, 1, 1, 3, 2, 3] → −1, EEs[4, 1, 2, 3, 3, 3] → 0, EEs[4, 1, 3, 2, 3, 3] → 1}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
Out[•]= {19.6563, Null}
```

```
In[•]:= ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
Out[•]= {0, 0, 1, 2, 5, 8, 17}
```

```
In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧
     (EE^σ[1,{},3,4] ≡ DKS[0]) ∧ (EE^σ[1,2,{},4] ≡ DKS[0]) ∧ (EE^σ[1,2,3,{}] ≡ DKS[0]), Arb];
EE@{7}; // Timing
```

SeriesSolve: In degree 2 arbitrarily setting {EEs[4, 2, 3] → 0}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → 2, EEs[4, 2, 2, 3] → −3, EEs[4, 2, 3, 3] → −2}.

SeriesSolve: In degree 4 arbitrarily setting

{EEs[4, 1, 1, 2, 3] → −3, EEs[4, 1, 2, 3, 3] → −3, EEs[4, 2, 2, 2, 3] → 3, EEs[4, 2, 2, 3, 3] → 1, EEs[4, 2, 3, 3, 3] → −1}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
Out[•]= {19.3438, Null}
```

```
In[•]:= ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
Out[•]= {0, 1, 3, 5, 11, 17, 35}
```

```
In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[1,3,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]), Arb];
EE@{8}; // Timing
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[3, 2] → 3, EEs[4, 2] → −3, EEs[4, 3] → −1}.

SeriesSolve: In degree 2 arbitrarily setting {EEs[4, 2, 3] → 1}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 2, 2, 3] → −1, EEs[4, 2, 3, 3] → 1}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
Out[•]= {76.2656, Null}
```

```
In[•]:= Length[Last[#]] & /@ Read[SyzInfo]
Out[•]= {3, 1, 2, 3, 6, 9, 18, 30}

In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[1,3,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧ (EE^σ[{},2,3,4] ≡ DKS[0]), Arb];
EE@{8}; // Timing
Out[•]= {84.6719, Null}

In[•]:= Length[Last[#]] & /@ Read[SyzInfo]
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[2,1,3,4] - EE ≡ DKS[0]), Arb];
EE@{8}; // Timing
SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → -1, EEs[3, 1] → -2, EEs[4, 1] → 0, EEs[4, 3] → 0}.
SeriesSolve: In degree 2 arbitrarily setting {EEs[4, 1, 3] → 3}.
SeriesSolve: In degree 3 arbitrarily setting
{EEs[3, 1, 1, 2] → -1, EEs[4, 1, 1, 2] → -3, EEs[4, 1, 1, 3] → 3, EEs[4, 1, 2, 3] → -2, EEs[4, 1, 3, 3] → 3}.
General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.
Out[•]= {5.4375, Null}

In[•]:= Length[Last[#]] & /@ Read[SyzInfo]
Out[•]= {4, 1, 5, 9, 27, 60, 165, 414}

In[•]:= Clear[EE, EEs]; EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[13,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]), Arb];
EE@{8}; // Timing
SeriesSolve: In degree 1 arbitrarily setting {EEs[3, 1] → -1, EEs[4, 1] → 0, EEs[4, 3] → 3}.
SeriesSolve: In degree 2 arbitrarily setting {EEs[4, 1, 3] → -3}.
SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 1, 3] → 0, EEs[4, 1, 3, 3] → -2}.
General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.
Out[•]= {276.281, Null}

In[•]:= Length[Last[#]] & /@ Read[SyzInfo]
Out[•]= {3, 1, 2, 3, 6, 9, 18, 30}
```

Syzygy b)

```
Clear[EE, EEs];
EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[1,2,3,4] ≡ EE^σ[1,2,4,3]), Arb];

Block[{$IterationLimit = ∞},
 EE@{EDegr}; // Timing
]
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → −28, EEs[3, 1] → 88, EEs[3, 2] → 21, EEs[4, 3] → 89}.

SeriesSolve: In degree 2 arbitrarily setting {EEs[3, 1, 2] → −75}.

SeriesSolve: In degree 3 arbitrarily setting

{EEs[3, 1, 1, 2] → 58, EEs[3, 1, 2, 2] → 46, EEs[4, 1, 1, 3] → 28, EEs[4, 1, 2, 3] → 69, EEs[4, 2, 2, 3] → −73}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
{1.0205, Null}
```

```
ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
{4, 1, 5, 9, 27}
```

Syzygy c)

```
Clear[EE, EEs];
EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE,
   ħ^-1 (b[DKS[t[2, 3]], EE^σ[1,2,4,5] + EE^σ[12,3,4,5]] ≡ -b[DKS[t[4, 5]], EE^σ[1,4,2,3] + EE^σ[14,5,2,3]]),
   Arb];

Block[{$IterationLimit = ∞},
 EE@{EDegr}; // Timing
]
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → −93, EEs[3, 2] → 15, EEs[4, 2] → −26, EEs[4, 3] → 34}.

SeriesSolve: In degree 2 arbitrarily setting {EEs[4, 2, 3] → 60}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → −92, EEs[4, 1, 3, 2] → −63, EEs[4, 2, 3, 3] → −46}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
{15.6825, Null}
```

```
ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
{4, 1, 3, 4, 8}
```

Syzygy ab)

```
Clear[EE, EEs]
EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧
    (EE^σ[1,2,3,4] ≡ EE^σ[1,2,4,3]), Arb];

Block[{$IterationLimit = ∞},
 EE@{EDegr}; // Timing
]
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → 20, EEs[3, 2] → −72, EEs[4, 3] → 25}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → 81, EEs[4, 2, 2, 3] → −36}.

SeriesSolve: In degree 4 arbitrarily setting {EEs[4, 1, 2, 3, 3] → −90, EEs[4, 2, 2, 3, 3] → −32}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
{2.57811, Null}
```

```
ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
{3, 0, 2, 2, 6}
```

Syzygy ac)

```
Clear[EE, EEs]
EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE,
    (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧ ħ^-1 (b[DKS[t[2, 3]],
        EE^σ[1,2,4,5] + EE^σ[12,3,4,5]] ≡ -b[DKS[t[4, 5]], EE^σ[1,4,2,3] + EE^σ[14,5,2,3]]), Arb];

Block[{$IterationLimit = ∞},
 EE@{EDegr}; // Timing
]
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → 15, EEs[3, 2] → −56, EEs[4, 2] → 15, EEs[4, 3] → 3}.

SeriesSolve: In degree 2 arbitrarily setting {EEs[4, 2, 3] → 86}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → −68, EEs[4, 2, 3, 3] → 84}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
{16.3726, Null}
```

```
ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
{4, 1, 2, 3, 6}
```

Syzygy bc)

```
Clear[EE, EEs];
EE = DKS[4, EEs];
SyzInfo =
  SeriesSolve[EE, (EE^σ[1,2,3,4] ≡ EE^σ[1,2,4,3]) ∧ ħ^-1 (b[DKS[t[2, 3]], EE^σ[1,2,4,5] + EE^σ[12,3,4,5]] ≡
       -b[DKS[t[4, 5]], EE^σ[1,4,2,3] + EE^σ[14,5,2,3]]), Arb];

Block[{$IterationLimit = ∞},
 EE@{EDegr}; // Timing
]
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → 85, EEs[3, 2] → –91, EEs[4, 3] → –68}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → 44}.

SeriesSolve: In degree 4 arbitrarily setting {EEs[4, 1, 2, 3, 3] → –87}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
{15.1559, Null}
```

```
ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
{3, 0, 1, 1, 3}
```

All Syzygies

```
Clear[EE, EEs];
EE = DKS[4, EEs];
SyzInfo = SeriesSolve[EE, (EE^σ[1,3,4,5] + EE^σ[13,2,4,5] - EE^σ[1,2,4,5] - EE^σ[12,3,4,5] ≡ DKS[0]) ∧
     (EE^σ[1,2,3,4] ≡ EE^σ[1,2,4,3]) ∧ ħ^-1 (b[DKS[t[2, 3]], EE^σ[1,2,4,5] + EE^σ[12,3,4,5]] ≡
        -b[DKS[t[4, 5]], EE^σ[1,4,2,3] + EE^σ[14,5,2,3]]), Arb];

Block[{$IterationLimit = ∞},
 EE@{EDegr}; // Timing
]
```

SeriesSolve: In degree 1 arbitrarily setting {EEs[2, 1] → 95, EEs[3, 2] → 26, EEs[4, 3] → –44}.

SeriesSolve: In degree 3 arbitrarily setting {EEs[4, 1, 2, 3] → 60}.

SeriesSolve: In degree 4 arbitrarily setting {EEs[4, 1, 2, 3, 3] → 99}.

General: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation.

```
{15.3187, Null}
```

```
ArbSyz = Length[Last[#]] & /@ Read[SyzInfo]
```

```
{3, 0, 1, 1, 3}
```