BIRS RESEARCH IN PAIRS APPLICATION

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1. Overview

Over the last years we (the applicants Dror Bar-natan and Zsuzsanna Dancso) have been working on a research program exploring an intricate connection between topology and quantum algebra: several questions of algebraic interest are equivalent to the problem of finding a "homomorphic expansion" for some class of knotted objects. In our recent series of papers [6, 7] and [8] (nearly complete), we study the relationship between *welded* knotted objects and the Kashiwara-Vergne (KV) problem in Lie theory, leading to a topological proof of the KV conjecture and a topological context for its relationship to Drinfel'd associators.

In classical knot theory, spaces of knotted objects[1] are equipped with a filtration called the Vassiliev or "finite type" filtration. We do not define this filtration here but instead comment that it is an analogue of filtering a group ring by powers of the augmentation ideal, that is, the ideal generated by differences of group elements. In fact, for knot-theoretic spaces with a rich enough structure (say "knotted trivalent graphs" or "parenthesized tangles") it can be interpreted as exactly that, replacing "group ring" by "monoid ring" or a similar structure.

When one encounters a filtered space one often prefers to study the associated graded space instead, which by virtue of being graded submits more easily to inductive study. The (degree completed) associated graded space of a given class of knotted objects is a space $\mathcal{A}$, which usually has a combinatorial (diagrammatic) description in terms of *chord diagrams*. This is a complicated infinite dimensional vector space, but it is in some ways much easier to understand than the knotted objects themselves. Furthermore, many operations defined on the knotted objects (e.g. connected sum or cabling) induce corresponding operations on the space of chord diagrams.

A *expansion* (aka *universal finite type invariant*) is a map Z from a space of knotted objects to its associated graded space satisfying a certain universality condition. We say that an expansion is *homomorphic* if it intertwines knot operations with their induced diagrammatic counterparts. In the case of knots there is a well-known homomorphic expansion: the *Kontsevich integral*. In the richer setting of parenthesized tangles or knotted trivalent graphs, where there are more operations, homomorphic expansions are in one-to-one correspondence with *Drinfel'd associators*. A Drinfel'd associator is a powerful tool in quantum algebra which can be interpreted as an element of the space of "chord diagrams on three strands".

In [6, 7], and [8], we study homomorphic expansions for *welded* knotted trivalent tangles. Topologically, welded knots can be thought of as knotted hollow tubes ($S^1 \times \mathbb{R}$) embedded in $\mathbb{R}^4$, and which bound a solid tube ($D^2 \times \mathbb{R}$) with only *ribbon* type singularities. As a Reidemeister theory, welded knots are a quotient space of *virtual knots*, which

[1]By "space of" we mean allowing formal linear combinations; "knotted objects" can mean knots, links, tangles, knotted graphs, etc.

in particular means that they are represented by knot diagrams involving two kinds of crossings: classical and "virtual", factored out by a set of Reidemeister relations.[2]

The associated graded space of welded knotted objects is a space of *arrow diagrams*: similar to the classical space of chord diagrams, but with directed chords and similar, yet different relations. Most importantly, homomorphic expansions for welded trivalent tangles turn out to be in bijection with solutions to the Kashiwara–Vergne problem of Lie theory.

The Kashiwara–Vergne problem, originally posed in 1978 is a property of the Baker–Campbell–Hausdorff series and has many implications in Lie theory and harmonic analysis. The conjecture was first proven by Alekseev and Meinrenken in 2006. In [5] Alekseev and Torossian construct a correspondence between solutions of the Kashiwara-Vergne problem and Drinfel'd associators. In [7] we reinterpret this work as a study of welded trivalent tangles. Note that there is a topologically defined map –known as "Satoh's tubing map"– from classical to welded knotted objects, which helps explain the relationship between associators (which appear in classical knot theory), and the Kashiwara-Vergne conjecture (which arises from the welded version).

In [8] we use these results to give a topological proof for the KV conjecture: using Satoh's tubing map we give a topological construction of the homomorphic expansion for welded trivalent tangles from the Kontsevich integral for knotted trivalent graphs. This in particular results in an explicit formula for KV-solutions in terms of Drinfel'd associators, which agrees with Alekseev, Enriquez and Torossian's formula [1].

2. Objectives

Recently, a new series of papers by Alekseev, Kawazumi, Kuno, and Naef [4, 3, 2] has appeared, giving a new and different topological interpretation to the KV problem in terms of curves on surfaces. Namely, instead of knotted tubes, they consider homotopy classes of free loops on a surface Σ, with the Goldman bracket and Turaev cobracket as operations. Loosely speaking, the Goldman bracket measures intersections of two loops, and the Turaev cobracket depends on self-intersections of a loop. The space of formal linear combinations of loops (modulo the constant loop) is a Lie bialgebra with these operations, called the Goldman–Turaev Lie bialgebra. In the case where Σ is a twice-punctured disk, [3] re-interprets the Kashiwara-Vergne problem as the problem of finding a homomorphic expansion for the Goldman–Turaev Lie bialgebra.

The objectives of this research in pairs can be summarised as follows:

(1) We have a conjectural interpretation of the result of [3] as an expansion of the HOMFLY skein module of links in a punctured disk, interpreted in a "$\mathfrak{gl}_n$-quotient" of the target space. We plan to work this out in detail.
(2) Since the problem of finding a homomorphic expansion for tangled tubes in $\mathbb{R}^4$ and for the Goldman–Turaev Lie bialgebra of loops in a twice-punctured disk are both equivalent to the KV problem, the two topological subjects ought to be related directly, not only via Lie theory. We want to understand how.
(3) We plan to write the sketch of a paper explaining our findings.

References

[1] A. Alekseev, B. Enriquez, and C. Torossian: Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations, *Publications Mathématiques de L'IHÉS*, **112-1** (2010) 143–189, *arXiv:0903.4067*.

[2] It is still an open problem whether this set of Reidemeister relations is complete.

[2] A. Alekseev, F. Naef: Goldman–Turaev formality from the Knizhnik–Zamolodchikov connection *arXiv:1703,05813*

[3] A. Alekseev, N. Kawazumi, Y. Kuno, F. Naef: The Goldman–Turaev Lie bialgebra in genus zero and the Kashiwara–Vergne problem, *arXiv:1703.05813*

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[5] A. Alekseev and C. Torossian: *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Annals of Mathematics, **175** (2012) 415–463, *arXiv:0802.4300*.

[6] D. Bar-Natan, Z. Dancso: Finite type invariants of w-knotted objects I: w-knots and the Alexander polynomial, to appear in *Alg. and Geom. Topology*, *arXiv:1405.1956*.

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[8] D. Dar-natan, Z. Dancso: Finite type invariants of w-knotted objects III: the double tree construction, *in preparation*.