M.Sc. Math Workshop — Assignment #8

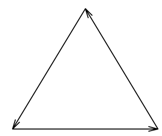
HUJI Spring 1998

Dror Bar-Natan

(37) Prove that if n^c is an integer for every n, then c is an integer.

(38) Prove that in any finite partition of the integers to at least two arithmetic progressions, at least two of the progressions will have the same skip.

(39) Compute the fundamental group and describe the universal covering space of the topological space obtained by identifying the three edges of a triangle as in the figure below:



(40) Let $\gamma_{1,2} : S^1 \to \mathbf{R}^3$ be two non-intersecting smooth paths in $\mathbf{R}^3$. Set

$$l(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} dt_1 dt_2 \, \frac{(\dot\gamma_1(t_1) \times \dot\gamma_2(t_2)) \cdot (\gamma_1(t_1) - \gamma_2(t_2))}{|\gamma_1(t_1) - \gamma_2(t_2)|^3},$$

where $t_{1,2}$ are parameters in S^1, $\dot\gamma$ denotes the derivative of γ with respect to its parameter (i.e., its tangent vector), $\times$ and $\cdot$ denote the vector and scalar products in $\mathbf{R}^3$ (respectively), and $|\ |$ denotes the Euclidean norm.

(a) Prove that l is a link invariant (that is, prove that it is invariant under deformations of $\gamma_{1,2}$ in the spirit of assignment #6). (One of many possibilities is to consider the map $S^1 \times S^1 \to S^2$ given by $(t_1, t_2) \mapsto \frac{\gamma_1(t_1) - \gamma_2(t_2)}{|\gamma_1(t_1) - \gamma_2(t_2)|}$, and to study the pullback of the volume form on S^2 via this map).

(b) Find a simple combinatorial formula for computing l given a generic planar projection of $\gamma_{1,2}$.

(41) Let $f : \mathbf{R} \to \mathbf{C}$ be a unit-norm L^2 function, and let $\tilde f$ be the Fourier transform of f (the Fourier transform is normalized so that L^2 norms are preserved). Let μ be the distribution on $\mathbf{R}$ whose density function is $|f|^2$, let $V = \int x^2 d\mu - \left(\int x d\mu\right)^2$ be the variance of μ, and make similar definitions for $\tilde\mu$ and $\tilde V$ starting from $\tilde f$.

(a) Prove that $V \cdot \tilde V \geq \frac{1}{4}$.

(b) Determine all the functions f for which the above inequality is an equality.

Hint: you may want to recall Q16.