Math 273 Course Description
Knot Theory as an Excuse

- **Time and place:** MWF 11AM, Science Center 507. **Instructor:** Dror Bar-Natan, Science Center 426G, 5-8797, dror@math.

- **Goal:** Use knot theory as an excuse to learning amusing mathematics and to having fun.

- **Intended for:** Math graduate students and everybody else. **Prerequisites:** Not being too far behind everybody else on the enclosed prerequisites quiz.

- **Texts:** My own papers (will be distributed), additional photocopied handouts, Kauffman’s “Knots and Physics”, and maybe more.

- **Course plan, main strand (about 75% of classes):** Explain the following paragraph:

  Vassiliev invariants are extremely simple to define, seem to be very powerful, and known to be at least as powerful as the standard knot polynomials. There are two natural and contradicting conjectures about Vassiliev invariants —
  1. that they all come (in an appropriate sense) from Lie algebras.
  2. that they separate knots.

  Besides, being so easy to define and so closely related to other knot invariants, they lead to new insight about these other invariants.

  First, we will study the general theory — definitions, the relation with the Jones-like polynomials, the relation with weight systems and chord diagrams, the relation with Lie algebras and the Hopf algebra structure. Then, we will say more on the “Lie algebras” conjecture: the diagrammatic PBW theorem, the map into surfaces and the relation with the classical groups. Then we will say some about the “separation” conjecture — proving it for braids and for string links up to homotopy. Finally, we will see how Vassiliev invariants are useful in proving theorems about knot invariants — the Melvin-Morton-Rozansky conjecture and the power of HOMFLY over braids.

  Along the way we will need to use many results and ideas from knot theory and from several other disciplines (and part of my motivation in giving the course is to finally understand these results and ideas myself). We will cover some of these results, many of them in lectures given by the students. These include: consistency of the classical knot polynomials, Hopf algebras, some representation theory, connections and holonomies, braids and free groups, the topological theory of the Alexander polynomial, etc.

  We will postpone the discussion of two related, necessary, deep, and beautiful subjects, quantum groups and quasi-Hopf algebras, to the second semester. In the second semester we will study quasi-Hopf algebras following Drinfel’d’s original papers on the subject, discussing their relation to the existence of a canonical universal Vassiliev invariant. We will also say some things about quantum groups, but I haven’t decided yet on the source. It may be the relatively low-level but readable exposition in Kauffman, or some deeper and less comprehensible other source.

  We may or may not say something about the relation with perturbative Chern-Simons theory.

  We may or may not say something about related three-manifold invariants.
- **Course plan, secondary strand** (about 25% of classes): Play with elementary, isolated, and fun topics in and around knot theory, such as the 15 or so such topics in the second part ("miscellany") of Kauffman’s book, the 20 or so such topics in the older Kauffman book ("On Knots"), alternating links, resistor networks, rational tangles, Kuperberg’s polynomial, non-invertible knots, and more.

I hope that many of these topics will be covered by visitors and/or students.

- **Grading:** Hopefully, almost everybody taking this course does not need a grade. About the rest we will worry later. **Homework:** Sporadic exercises and projects. If you do need a grade for the course, do the homework and impress me with the projects.

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Regular isotopy of Figure Eight and its Mirror Image
The Alexander duality theorem.

The Poincaré–Hilbert theorem.

Calculus theory for covering spaces.

The relation between the fundamental group and the first homology of a space.

The Van-Kampen theorem about the fundamental group of a union.

Connections, curvature, and holonomies.

The structure theorem of co-commutative Hopf algebras.

The definition of a Hopf algebra.

The classification of simple Lie algebras and their irreducible representations.

Finite dimensional representations of $SL(2,\mathbb{C})$.

The Poincaré-Birkhoff-Witt theorem.

The definition of a Lie algebra, structure constants.

The tensor algebra, symmetric algebra, and exterior algebra of a vector space.

The tensor product of two vector spaces.

5 - Feel very confident. Know it and can derive all results.
4 - Feel confident. Can prove most details.
3 - Can state it.
2 - Heard about it.
1 - Huh?

Please rate your level of understanding of the following topics using the scale:

<table>
<thead>
<tr>
<th>WILL you need a gradet?</th>
<th>NAME:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Maybe</td>
<td></td>
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</tbody>
</table>

Problem: Probability of taking the course: 20-30% (optional)
Math 273, Sep 19, 1998

Read out course description

A knot: imbedding $S^1 \to \mathbb{R}^3$ considered up to "ambient isotopy"

Examples: $\emptyset$, Millet, Fig-8, $\natural_1$.

Is there really knotted? Classify knots.

The Kauffman bracket (aka the Jones polynomial)

$$\langle \chi \rangle = A \langle \psi \rangle + B \langle \phi \rangle$$

$$\langle \text{fig-8} \rangle = A^{k-1} \quad (\text{e.g., } \langle \text{fig-8} \rangle = A^3 + 3A^2B + 3AB^2 + B^3)$$

Reidemeister:
1. $\chi \leftrightarrow \psi$  \hspace{1cm} $\rightarrow B = A^{-1}, j = -\frac{A^2 - 1}{4A}$
2. $\chi \rightarrow \psi$  \hspace{1cm} $\rightarrow \sqrt{\frac{A^4 + 1}{1}}$
3. $\psi \leftrightarrow 1$  \hspace{1cm} $\rightarrow 0$

Write:

$$\langle \chi \rangle = +1 \quad \langle \psi \rangle = -1$$

$$\Rightarrow \mathcal{J}(k) = (A^3)^{\omega(k)} \langle \phi \rangle$$

is a knot invariant. Evaluated at $q^{\frac{1}{4}}, \omega(k)$, we get Jones.
Distributed Handout
Matteo Mainetti
Math 273, Sep 21 1994
623-2699

\[ 9^{-1} J(X) - 9 J(X^2) = \]
\[ = 9^{-1} q^{\frac{3}{2}} \langle X \rangle + 99^{3/4} \langle \rangle \]
\[ = 9^{3/4} (q^{-1/2} \langle \rangle + q^{1/2} \langle \rangle) \langle \rangle - 9^{-1/4} (q^{-1/2} \langle \rangle \langle \rangle \times q^{1/2} \langle \rangle) \]
\[ = (9^{1/2} - 9^{-1/2}) \langle \rangle \]

Review:
\[ X \twoheadrightarrow X \]
\[ \langle X \rangle = A \langle X \rangle + B \langle \rangle \]
\[ \langle \emptyset \rangle = \delta_{k,1} \]

R1,2 \rightarrow B = A^{-1}, \delta = -A^{-2} - \frac{1}{A^2}
\[ \langle \emptyset \rangle = (-A^3) \langle 1 \rangle \]

Do the trefoil example!
Do the Jones relation
Discuss Conway, HOMFLY, SU(N)-HOMFLY.
If time - Define Vassiliev.
The Taxonomy of Knot (Link) Polynomials

The Alexander polynomial \( \left( \frac{\partial}{\partial t} \text{GL}(N) \right)_{t=0} \)

\[ A(x) - A(x^{-1}) = (t^{1/2} - t^{-1/2})A(x) \quad A(0^+) = \begin{cases} 1 & k=1 \\ 0 & k > 0 \end{cases} \]

The Conway polynomial

\[ C(x) - C(x^{-1}) = 2C(x) \quad C(0^+) = \begin{cases} 1 & k=1 \\ 0 & k > 1 \end{cases} \]

The Jones Polynomial \((\text{SU}(2))\)

\[ 9J(x) - 9^{-1}J(x^{-1}) = (q^{1/2} - q^{-1/2})J(x) \quad J(0^+) = (q^{1/2} + q^{-1/2}) \]

Framed version: \( J^F(x) - J^F(x^{-1}) = (q^{1/2} - q^{-1/2})J^F(x) \quad J^F(0^+) = \text{same} \)

The HOMFLY (Lymph-Tofu) Polynomial \((\text{SU}(N))\)

\[ q^{N/2}H(x) - q^{-N/2}H(x^{-1}) = (q^{1/2} - q^{-1/2})H(x) \quad H(0^+) = \left( \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} \right) \]

Framed version: \( H^F(x) - H^F(x^{-1}) = (q^{1/2} - q^{-1/2})H^F(x) \quad H^F(0^+) = \text{same} \)

The Kauffman Polynomial \((\text{SO}(N))\)

\[ q^{N/2}F(x) - q^{-N/2}F(x^{-1}) = (q^{1/2} - q^{-1/2})(F(x) - F(x^{-1})) \]

\[ F(0^+) = \left( 1 + \frac{q^{1/2} - q^{-1/2}}{q^{1/2} - q^{-1/2}} \right)^k \quad F^F(0^+) = \text{same} \]

Framed version: \( F^F(x) - F^F(x^{-1}) = (q^{1/2} - q^{-1/2})(F^F(x) - F^F(x^{-1})) \)
Math 273, Sep 23 1994

Grades: At least give a talk or two

Ftp: finger dror@math.harvard.edu (dror/273)
within Harvard: ndror/ftp/273

Taxonomical remarks:
1. Links
2. Parametrization
   (usually \( q^{\frac{1}{2}} \rightarrow 1 \))
3. Up: P: l, framed knots (links)
   (regualr isotopy)
   2. blackboard framing, R.E.
   3. linking numbers (\( \frac{1}{2} \) crossings) (symm fn.
   4. Writhe = l.n w/ frame

Vassiliev invariants:
1. lcf
2. big, hope
3. Conway is, HOMFLY is
4. W.S.
Math 273, Sep 26 1994

Distribute paper.
Remind Vassiliev invariants.

Define WIS. Show how to get a WIS.
Compute for Conway & HOMFLY & Kauffman

\[ \text{Kaufman:} \quad \langle \cdots \rangle = -n^{-1} \sqrt{1 + \frac{1}{n-1}} \]

From ind & HT
Define WIS.
Thm 1

(Thm 1' for framed links)

List algebras & tensors.
Math 273, Sep 28, 1994

Reminder for gradables:

Pavel's proof of singularity flips = CD's.

Remind me who can post a better one.

From ind & YH

Define w's.

Thm 1

(Thm 1' for framed links)

Lie algebras and tensors

SU, IHX, AS

\( gl(n), su(n) \)
Math 273, Sep 30 1994

Reminder for gradables.

Remind Thm 1,
“actuality-table”

Thm 1’

Tensors

Lie algebras & Tensors.

STU THX, 45.
Math 273, Oct 3 1994

Remind Lie algebra as a tensor
Rps as tensors.

$\mathfrak{sl}(n)$ all the way.
8.5 Complete evaluation for the classical algebras

By the remark at the end of the previous section, to calculate $C_g$ for the classical algebras (in their defining representations) it is enough to consider the four complex classical algebras.

The first step is to use relation $STU$ repeatedly, with each usage reducing the number of $G^3$ vertices by one, until we are left with a diagram $D$ that has no $G^3$ vertices. The basic building block of such diagrams is the tensor

$$T^{\alpha \gamma}_{\beta \delta} = \begin{array}{c|c|c|c}
\alpha & \delta \\
 \hline 
 b & a \\
\beta & \gamma 
\end{array}.$$ 

This tensor will be evaluated explicitly for each of the complex classical algebras, and the results will turn out to have representations in terms of diagrams that have no propagators in them. Using this repeatedly, we are left with disjoint unions of circles which again are easy to evaluate explicitly.

I will show in detail the computations for $so(N, C)$, and just state the results for $gl(N, C)$, $sl(N, C)$, and $sp(N, C)$.

8.5.1 The algebra $so(N, C)$.

A convenient choice of generators for $so(N, C)$ are the $N \times N$ matrices $M_{ij}$ ($i < j$), given by

$$(M_{ij})_{\alpha \beta} = \delta_{\alpha i} \delta_{j \beta} - \delta_{\alpha j} \delta_{i \beta}.$$ 

That is, the $ij$ entry of $M_{ij}$ is $+1$, the $ji$ entry of $M_{ij}$ is $-1$, and all other entries of $M_{ij}$ are zero. The invariant bilinear form that we pick on $so(N, C)$ is the matrix trace in the defining representation, and so

$$t^{(ij)(kl)} \overset{\text{def}}{=} tr(M_{ij} M_{kl}) = -2 \delta_{ik} \delta_{jl}.$$ 

Inverting the $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$ matrix $t^{(ij)(kl)}$ we get

$$t^{(ij)(kl)} = -\frac{1}{2} \delta^{ik} \delta^{jl},$$ (8.14)

and so

$$T^{\alpha \gamma}_{\beta \delta} = \sum_{i < j < k < l} t^{(ij)(kl)} (M_{ij})_{\alpha \beta} (M_{ij})_{\gamma \delta}.$$ (8.15)

Using (8.14) and some algebraic manipulations we can simplify (8.15), and then represent it by a diagram:

$$(8.15) = \frac{1}{2} (\delta_{\alpha \beta} \delta_{\gamma \delta} - \delta_{\alpha \gamma} \delta_{\beta \delta}) = \frac{1}{2} \left( \begin{array}{c|c|c|c}
\alpha & \delta \\
 \hline 
 b & a \\
\beta & \gamma 
\end{array} \right).$$ (8.16)
The last thing to note is that
\[ C_{so(N,C)}(k \text{ disjoint circles}) = N^k. \]

**Example** For \( so(N, C) \) in its defining representation we can calculate \( d, r, \) and \( g \) using: (suppressing the '\( C_{so(N,C)} \) symbols)

\[
d = \bigcirc = N, \\
dr = \bigcirc \bigcirc = \frac{1}{2} \left( \bigodot + \bigodot \right) = \frac{N(N-1)}{2}, \\
dr \left( r - \frac{1}{2} g \right) = \bigotimes = \frac{1}{4} \bigotimes - \frac{1}{2} \bigotimes + \frac{1}{4} \bigotimes = \frac{N(N-1)}{4}.
\]

### 8.5.2 The algebra \( gl(N, C) \).

Similar considerations lead to the even simpler rule

\[
\begin{array}{c}
\alpha \\
(ij)
\end{array}
\bigg|_{(kl)}
\begin{array}{c}
\beta \\
\gamma
\end{array}
\delta = \begin{array}{c}
\alpha \\
\delta
\end{array}
\bigg|_{\gamma} \begin{array}{c}
\beta \\
\gamma
\end{array}
\delta,
\]

while retaining
\[ C_{gl(N,C)}(k \text{ disjoint circles}) = N^k. \]

**Example** For \( gl(N, C) \) in its defining representation,
\[ \bigotimes = \bigcirc - \bigotimes = \bigotimes - \bigotimes = N(N^2 - 1). \]

### 8.5.3 The algebra \( sl(N, C) \).

The rule here is the so-called "Fierz identity",

\[
\begin{array}{c}
\alpha \\
(ij)
\end{array}
\bigg|_{(kl)}
\begin{array}{c}
\beta \\
\gamma
\end{array}
\delta = \begin{array}{c}
\alpha \\
\delta
\end{array}
\bigg|_{\gamma} \begin{array}{c}
\beta \\
\gamma
\end{array} \left( - \frac{1}{N} \right)\alpha \\
\delta.
\]

with the usual
\[ C_{sl(N,C)}(k \text{ disjoint circles}) = N^k. \]

**Example** For \( sl(N, C) \) in its defining representation we can calculate \( d, r, \) and \( g \) using:

\[
d = \bigcirc = N, \\
dr = \bigcirc \bigcirc = \bigodot - \bigodot = N^2 - 1, \\
dr \left( r - \frac{1}{2} g \right) = \bigotimes = \bigotimes = \frac{2}{N} \bigodot + \frac{1}{N^2} \bigcirc = \frac{1 - N^2}{N}.
\]

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8.5.4 The algebra $sp(N, C)$.

This is the most complicated case. Let $D$ be a diagram with no $G^3$ vertices. The computation of $C_{sp(N, C)}(D)$ now proceeds in two steps:

1. Mark each Wilson loop segment in $D$ with either the symbol $P$ or the symbol $Q$, in such a way that the number of $P$'s entering each subdiagram of $D$ of the form $\cdots$ is equal to the number of $P$'s leaving it. (Remember that the Wilson loops are directed).

2. Simplify $D$ using the following rules:

$$\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \\
\downarrow
\end{array}
\end{array}
\end{array} & = 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Q \\
\downarrow
\end{array}
\end{array}
\end{array}
\end{array} = \frac{1}{2} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} \right), \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Q \\
\downarrow
\end{array}
\end{array}
\end{array} & = 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \\
\downarrow
\end{array}
\end{array}
\end{array}
\end{array} = -\frac{1}{2} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} \right), \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \\
\downarrow
\end{array}
\end{array}
\end{array} & = 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Q \\
\downarrow
\end{array}
\end{array}
\end{array}
\end{array} = \frac{1}{2} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} \right).
\end{align*}$$

3. Similarly to the usual,

$$C_{sp(N, C)}(k \text{ disjoint marked circles}) = N^k.$$ 

(Notice that this time $\dim R = 2N \neq N$).

Example For $sp(N, C)$ in its defining representation we can calculate $d$, $r$, and $g$ using:

$$\begin{align*}
d & = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
P \\
\vdots
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} = 2N, \\
dr & = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
P \\
\vdots
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
P \\
\vdots
\end{array}
\end{array} \\
& = 2 \frac{1}{2} \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \right) = 2N \left( N + \frac{1}{2} \right), \\
dr \left( r - \frac{1}{2} g \right) & = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}
\end{array} = 2 \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \\
\vdots
\end{array}
\end{array}
\end{array} + 4 \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \\
\vdots
\end{array}
\end{array}
\end{array} \\
& = \frac{1}{2} \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \right) - \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \right) = -\frac{1}{2} N(1 + 2N).
\end{align*}$$

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Distribute handout

\[ \Phi^c : A^c \to A^t \]

and

\[ \Phi^k : A^k \to A^c \quad (k = \# \text{ of internal, induct on } k) \]

IHx, AS.

\[ \Phi^l : A^l \to A^t \quad \text{is iso!} \]

The product on \( A^l, A^t \) is a connected sum of knots.

The product of Vass. is Vass; 1-topr algebras.
Egor Pak (October 7, 1994)

The Chromatic Polynomial

Definition for a graph $P(G,n)$

$P(G_{\emptyset}, k) = n^k$

$P(G_{\text{complete}}, k) = n(n-1) \ldots (n-k+1)$

$P(G_{\text{erdos-reed}}, k) = n(n-1)^{k-1}$

Def (G-e) G/e
definition restriction

Thm $P(G-e,x) = P(G,x) - P(G/e)$

Example $P(G-e) = P(G) - P(G/e) = x^2 - x$

$P(\Delta) = P(\bigtriangleup) - P(\bigtriangledown) = x(x-1)^2 - x(x-1) = x(x-1)(x-2)$

Thm (Whitney) $P(G,x) = \sum (-1)^i \alpha_i x^i$, $n \neq \chi(G)$

$\alpha_i = \# \{S \subseteq E : S \text{ doesn't contain a broken cycle} \}$
Def assume the edges of $G$ are ordered, i.e.

$$E = \{e_1, e_2, \ldots \}$$

$$C \subseteq E$$

is called "a broken cycle" if it is a cycle with the maximal edge removed.

Idea of proof of Whitney: use chromatic relations for the maximal edge.

Tutte polynomial

$$T(G) = \#E - \#C + \#L$$

where $C$ is the number of connected components of $G$.

$$T(G, x, y) = \sum_{C \subseteq E} (x-1)^{\#(G-C)} (y-1)^{\#(G-\{e\})}$$

"The Tutte polynomial".

Thm (Tutte)

$$T(G, x, y) = \begin{cases} 0 & \text{if } G = \emptyset \\ \sum_{C \subseteq E} y^{\#C} T(G/C, x, y) + T(G/e, x, y) & \text{if } e \text{ is a loop} \\ \sum_{C \subseteq E} T(G/e, x, y) & \text{if } e \text{ is a bridge} \\ 0 & \text{if } \#E = 0 \end{cases}$$
Cor

\[ g'(6, 1, 2, 0) = 2 \cdot \rho(6, 2) \cdot \zeta^{-1} \]

Examples

\[ g(0) = x \quad g(9) = y \]

\[ g(\infty) = x + y \]

\[ g(\triangle) = x^2 + x + y \]

Thm. If \( G \) is planar, \( G^* \) its dual.

\[ g(6, x, y) = g(6^*, y, x) \]

Obs

\[ g(6, 1, 2, 2) = 2 \cdot E \]

\[ g(6, 1, 1, 1) = \# \text{ of connected subgraphs} \]

\[ (\text{Stanley}) \quad (-1)^n \cdot \rho(6, -1) = \# \text{ acyclic orientations} \]
Hopf algebra = an algebra whose dual is also an algebra, in a compatible way.

Examples:
1. $G$ finite group, $\mathbb{Z}G = \sum_{g \in G} g$
2. Generalizations
4. Vassiliev invariants.

Claim
$$ (V_1(V_2)) - (V_1 V_2(-)) = (V_1(+)) - V_1(-) V_2(+) + V_1(-) V_2(-) $$
$$ = V_1(\text{dbl}) V_2(+) + V_1(-) V_2(\text{dbl}) $$

More Formally: $A$ is an algebra if $A \otimes A \rightarrow A$ s.t.

$$ A \otimes A \rightarrow A \otimes A \quad \text{unit: } \eta : Q \rightarrow A \text{ s.t. Commutativity} $$

$$ \Rightarrow A \otimes A \rightarrow A $$

Co-algebra $M : A^* \otimes A^* \rightarrow A^*$, or simply $D : A \rightarrow A \otimes A$ s.t.

Count

Hopf alg: $A \otimes A \xrightarrow{m} A$ $\Delta$ is an algebra morphism $m$ is a co-algebra morphism

Thm $A$ is a commutative & co-commutative Hopf algebra
Factoring $\emptyset$ out

Operators involved:

$\emptyset$: $\emptyset m\mathcal{A} \to \emptyset m+1\mathcal{A}$ - multiplication by $\emptyset$

$F$: $\emptyset m\mathcal{A} \to \emptyset m-1\mathcal{A}$ - Forget one (any) chord

$[F, \emptyset] = F\emptyset - \emptyset F = I$

$S$: $\emptyset m\mathcal{A} \to \emptyset m\mathcal{A}$ - $S\emptyset = \emptyset - \emptyset F\emptyset + \frac{1}{2} \emptyset^2 F^2\emptyset - \frac{1}{3!} \emptyset^3 F^3\emptyset + \ldots$

$R$: $\emptyset m\mathcal{A} \to \emptyset m+1\mathcal{A}$ - $R = \emptyset - \frac{1}{2} \emptyset^2 F + \frac{1}{3!} \emptyset^3 F^3 - \ldots$

$RF = I - S$

Claim: If $W = W_1 \cdot W'$, then $W(\emptyset) = W'(\emptyset F\emptyset)$

- If trivial

Claim: If $W = W_1 \cdot W'$, then $W'(\emptyset) = W(\emptyset R\emptyset)$

- If $W(R\emptyset) = W'(F R\emptyset) = W'(D)$ because $FR\emptyset = 0$ (See below)

Claim: $FR = I$, $FS = 0$ (trivial if $F$ is interpreted as $\emptyset F\emptyset$)

- If easy

Claim: If $W \circ S = 0$, then $W = W_1 \cdot W'$ for some $W'$

Define $W' = W \circ R$, $D = R \emptyset F\emptyset + SD$

$(W_1 \cdot W')(D) = (W_1 \cdot W')(R \emptyset F\emptyset + SD) = (W_1(F R\emptyset F \emptyset SD)) = W_1(F R\emptyset F SD) = W_1(F R\emptyset D) = W(\emptyset D) = W(\emptyset SD)$

9/11/94
Math 273, October 17 1994

A is generated by \( \Omega \), \( \delta \) \( \in \Omega \in A : \delta \in \set{\delta : \delta = 1 \otimes 0 + 0 \otimes 1} \)

1. \( \Omega \) in terms of knots.

2. \( \Omega \) in terms of Lie algebras

Primitives & generators

\[ A = S \Omega(A) \quad \Omega(A) = \set{\delta \in A : \delta = 1 \otimes 0 + 0 \otimes 1} \]

Example \( \Omega(A) = \mathbb{F}_2 \) not a great desc.

\[ \mathbb{W} \text{ is a sub-Hopf alg of } A^* \quad \Omega(\mathbb{W}) = \Omega(A^*) \]

\[ = \partial'(A^*) \]

\[ \Rightarrow \exists \text{ proj. } \mathbb{W} \rightarrow \mathbb{W} \]

\[ F, \Omega \text{ desc.} \]
Math 273, October 19, 1994

\[ Q[x_0, x_2, \ldots] = A \xrightarrow{S^2} A^r = A / \langle \Theta \rangle \cong Q[x_2, x_3, \ldots] \]
\[ Q[x_1, x_2, \ldots] = A \xrightarrow{S^2} W \xrightarrow{2} Q[x_2, x_3, \ldots] \]

1. Indeed \( W = Q[x_2, x_3, \ldots] \)
   - \( W \) is a sub-space (proof in)
   - \( \Delta W = W \Theta + E \otimes W \)
   \[ \Rightarrow W \text{ is prime if it vanishes on reducible diagrams.} \]

2. Definition of \( S \):
   - \( \Theta : \mathcal{C}_k A \to \mathcal{C}_{k+1} A \)
   - \( W_1 : \mathcal{C}_k A^* \to \mathcal{C}_{k+1} A^* \)
   - \( F : \mathcal{C}_k A \to \mathcal{C}_{k-1} A \)
   - \( F(0) = \Sigma \) (The root vector)
   - \( [F, \Theta] = 1 \)
   - \( \Theta = \text{mult by } \Theta \)
   - \( F = \frac{\partial}{\partial \Theta} \)

\[ S = \sum_{n=0}^{-1} (-1)^n \Theta^n F^n = e^{-\Theta F} \]
\[ R = \sum_{n=0}^{\infty} \Theta^{n+1} S F^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Theta^{n+1} F^n \]

**Prop 1.** \( S \) is a Hopf morphism:
   - \( S(0, D_2) = S(0_1) S(D_2) \) (Follows of bicrossed)
   - \( S(D, S(0_1)) = (S \otimes S)(0, D) \) \( \Rightarrow \Delta \Theta = (0_1 \otimes 1 + 1_0 \otimes \Theta) \Delta \)
   - \( FS = 0 \)
   - \( FR = I \)

**Example:** \( \Theta = 0 \)
\[ S \Theta = 0 \]
\[ RF = I - S \]
Math 273, October 26 1994

Alexander Postnikov, Hecke Alg & Jones.

Following Jones' paper Ann Math 126 (1987) 335-356

$A^k + B^k + C^k = 0$ no good.

$S_n = \langle \sigma_1, \sigma_{n-1}, \sigma_i^2 = 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$

$B_n$ same, dropping $\sigma_i^2 = 1$.

Markov's theorem.

RPs of $B_n$: 1. Permutations.

$\otimes \otimes : \sigma_i (b_{i+1}) = (b_{i+1} + 1)$

(equiv to adding the relation

$(\sigma_i - 9)(\sigma_i + 1) = 0$)

or $\sigma_i^2 = (9-1)\sigma_i + 9$

$\Rightarrow$ Hecke $= \langle \sigma_1 \ldots \rangle$ $B_n$ relations

$\otimes (\sigma_i - 9)(\sigma_i + 1)$

$\Rightarrow H_n = H(\sigma_i)$

Has the same reps as $S_n$ for generic $\sigma_i$.
Oceana's trace:

\[ H_0 \subset H_1, \ldots, \text{tr}: Y / H_n \to \mathbb{C} \]

s.t.

1. \( \text{tr}(1) = 1 \)

2. \( \text{tr}(ab) = \text{tr}(ba) \)

3. \( \text{tr}(a \sigma_i) = z \cdot \text{tr}a \) for some fixed \( z \).

**Thm (0)**: \( \exists \beta \) such \( \text{tr} \).

**Sketch of PE**: Assume for \( H_{n-1} \).

Basis of \( H_n \) is

\[
\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_n} \]

\( i_1 < i_2 < \ldots < i_n \)

Property: \( \alpha \in H_{n-1} \) or \( b \in \text{coker} \), \( \beta \in H_{n-1} \) generate basis.

\( \Rightarrow \) forces \( \text{ker} \) or \( \text{tr} ? \)

Have to check to relations? \( \Box \).

We have an "almost knot invariant."
Indeed, define

$$X(q, X(A, X, B) = (z - v\lambda)\bar{\beta}^{(n+1)}(\tilde{\beta}) + \gamma$$

with \( \lambda \) defined by \( z = -\frac{1}{1 - \lambda v} \).

\( \pi : Br_{1} \rightarrow H_{2} \) Proj

and

\( r(b) = \text{writhe}(\text{link}(b)) \)

Thm: \( X_{0,1} \) is a R-b\_link invariant.

If \( .. \) D

Thm: \( X_{0,1}(b) = \text{HOMFLY}(\text{link}(b)) \)

where \( \text{HOMFLY} \) is defined by

\[
\begin{align*}
t & = \sqrt{x} \sqrt{y} \\
x & = (\sqrt{x} - \frac{1}{\sqrt{y}}) \\
-1 \rightarrow & -t \rightarrow \\
\text{pf} & \quad \square
\end{align*}
\]
Math 273, October 28 1994

\[ \mathcal{A} \leftarrow \mathcal{A}^t \leftarrow \mathcal{A} \rightarrow \mathcal{D} \rightarrow 2^{-1} \rightarrow \mathcal{M} \]

\[ \text{A} \text{ links} \rightarrow \mathcal{A} \]

\[ \mathcal{A} \leftarrow \mathcal{D} \leftarrow \mathcal{D} \rightarrow \mathcal{M} \]

\[ \mathcal{A} \leftarrow \mathcal{D} \leftarrow \mathcal{D} \rightarrow \mathcal{M} \leftarrow \mathcal{M} \]

Need to prove:

1. \( \mathcal{A} \) descends. (start with boring)

2. \( \mathcal{D} \) descends ("\( \mathcal{D} \) is a relation in \( \mathcal{D} \) is preserved")

3. \( \mathcal{D} \) is 1-1!
Math 273  October 30  1994

A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M
A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M
A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M
A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M
A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M
A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M
A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M
A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M

(interrupt & discuss A & primitives)

\text{grad} \text{ed by some simple function of the genus.}

\text{Claim} \ (\Phi \circ \mu') \in \text{all classical Lie algs}

A \xrightarrow{\phi} \Theta \xrightarrow{\delta} M

\text{Products:}

\text{genus = 1.}

\Phi \text{ is to } C_{1/0,1/2}
\Phi^* \text{ is to } C_{1/0,1/2,2}
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\text{grad} \text{ed by some simple function of the genus.}

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Math 273, 11/4/94

David Goldberg - $SO(3)$ & four colors.

1. The franks trick for $SO(3)$
2. E-calculus for trivalent graphs
3. $O - O - O - X$
4. The definition of $E$
5. The franks trick satisfies the relations of before.
6. The relation between $E$ & $SO(3)$.
7. The relation between $(Z/2Z)^2$, map 4-colors & edge 3-colors.
8. Signs using red & blue filters, (the # of intersections of the views is even)
Math 273  November 7  1994

To show that the $M_{\lambda_1 \cdots \lambda_n}$ span $M^*_\mathbf{R}$ we need to know a little more about reps & characters.

**Rep $\rightarrow$ character (class function)**

**Theorem:** Every class function is a linear comb of characters

\[
\psi^\alpha \in C_c(G) \quad \text{by} \quad \chi^\alpha: \chi_1 \chi_2 \chi_3 \rightarrow \psi^\alpha(R(x_1)R(x_2)R(x_3)) = a^1_\alpha a^2_\alpha a^3_\alpha \chi_1(R(x_1)R(x_2)R(x_3))
\]

\[
\chi_R \in C_c(G) \quad \text{by} \quad \chi(x_1x_2) \rightarrow \psi_R(x_1x_2x_3) = a^1_R a^2_R a^3_R \chi(x_3)
\]

Conclude covering formula for $\psi^\alpha$:

\[
Wg_{\psi_R}(\alpha) = Wg_{\chi_R}(\psi^\alpha / \psi^\alpha(\alpha))
\]

Covering formulas for tensor products
\[ \text{Math 273, November 14, 1994} \]

\[ \text{tr}(M_1 M_2 M_3) = \text{tr}(M_3 M_1 M_2) = \text{tr}\left(\left(\text{tr}(M_3) M_1 M_2\right)^T\right) = \text{tr}(M_1 M_2 M_3) = \left(\text{tr}(M_1 M_2 M_3)\right) \]

\[ \psi^t W_{3,1} = W_{3,1} \longrightarrow \psi^t W_{3,1} \]

---

1. Interpretation of coverings in terms of knots.

2. \( \psi^t \) on \( U, A, \delta \rightarrow \delta \psi^t \) on knots.

3. \( \partial \delta \omega = 0 \rightarrow \) classical Lie algs do not detect orientation.

4. No Lie algs detect orientations.

Kuperberg:

\[ S_T(K) = \text{tr} T \text{ around } K \]

\( S_T^* \) preserves \( \psi^t \): \( V \circ S_T \) is vass.

If \( k = k^{-1} \), \( (V \circ S_T)(k) = (V \circ S_T)(k^{-1}) \) for all \( V \) but it may be that \( (V \circ S_T)(k) \) & \( (V \circ S_T)(k^{-1}) \) are completely different.
1. Refs Appendix B of [citation].

2. Def of $A$ a Hopf algebra over a char-0 $k$.

3. $P(A)$, is a lie algebra (proof).

4. $U(\mathfrak{g})$ by the universal prop.

5. PBW

6. $S(V)$ (a universal def!), co-product on $S(V)$.

7. $U(L)$ is a Hopf algebra.

8. $T(V)$ is a Hopf algebra by th cut co-product, $S(V) \rightarrow T(V)$ is a coalg mp.

9. $U(P(A)) \xrightarrow{\alpha} A$

$\alpha$ is an algebra mp & a co-alg mp.

10. $U(P(A)) \xrightarrow{\alpha} A$

$\beta$ is an alg mp & a co-mp.
11-connected, 1-connected

22-connected \rightarrow connected
Math 273, November 28, 1994

kkz formula:

\[
Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{p \leq m} \frac{(-1)^{\#p}}{2^{\frac{m}{2}}} \sum_{a \leq p} \frac{1}{z_{2a}} \frac{1}{z_{2a+1}}
\]

1. Can move @ time where there's no crit pts.
2. Can (by a needle argument) move critical pts.
3. Can cancel critical points? \( n = 1 \)?

\[
\int_{0}^{1} \frac{d(t-1)}{t} \int_{0}^{1} \frac{dt}{t} = \int_{0}^{1} \frac{dt}{t} \log t = \frac{1}{e}
\]

\( Z(\infty) = Z(0) = 0 + \frac{1}{e} \times \) 

\( \exists \) is a knot invariant

Thm \( \exists(\kappa_0) = 0 + \) higher order terms

(i.e., \( \exists \) is a universal Vassiliev invariant)

Cor Thm 1

Thm \( Z(\infty). \exists \) is a Hopf map (in as much as this makes sense)
Math 273, December 13 1984

Next class: Leonid Kargodsky

\begin{align*}
\text{Pent} & \quad \mathcal{O}^{123} \cdot (\Delta \mathcal{O}) = \mathcal{O}^{111} \mathcal{O} (\Delta \mathcal{O}) \\
\text{Hex} & \quad (\Delta \mathcal{O}) \mathcal{R}^{\pm} = \mathcal{O}^{123} \cdot (\mathcal{R}^{2})^{23} \cdot (\mathcal{O}^{-1})^{32} \cdot (\mathcal{R}^{1})^{3} \cdot \mathcal{O}^{32} \\
& \quad \mathcal{R}^{32} \cdot \mathcal{O} = I \\
\text{Thm (Drinfeld, A. Murakami, . . . )} & \quad \text{If } R \text{ is fixed, all } \mathcal{O}^{'s} \text{ are conjugate.}
\end{align*}

Explain "conjugate" by an anisotropic KZ.

\textbf{PA of thm} \quad \text{Fix } R, \text{ suppose } \mathcal{O} \& \mathcal{O}' \text{ sat.}

\begin{itemize}
\item \text{Pent} \& \text{Hex} \; \text{assume} \; \mathcal{O}' = \mathcal{O} + \mathcal{O}'
\end{itemize}

\text{Set}
\begin{align*}
\mathcal{Y} &= 0 & \mathcal{Y}' &= -\mathcal{Y} \\
\mathcal{Y} - \mathcal{Y}' &= 0 & \mathcal{Y} + \mathcal{Y}' &= 0
\end{align*}

Try \quad F = \mathcal{F} + \mathcal{F}'

\begin{align*}
\mathcal{O}' &= (\mathcal{O} \mathcal{O}) F \cdot \mathcal{F}' - \mathcal{O} \cdot (\mathcal{F}^{-1})' \cdot (\mathcal{O} \mathcal{O}) F \\
\mathcal{Y} &= \mathcal{dF} \\
\mathcal{F} - \mathcal{F}' &= 0
\end{align*}

\text{Prob: is } \bar{H}_{\text{grr}} (A^{\mathcal{O}6}) = \emptyset \; ?
Math 273, Possible topics for 2nd semester:

1. More on Vassiliev invariants:
   a. Vassiliev invariants separate braids.
   b. The Milnor invariants are Vassiliev.
   c. $gl(N)$ invariants separate braids.
   d. Something on Graph Cohomology.
   e. More on Chern-Simons theory.
   f. The Maximal coherence theorem & other omissions.
   g. the Melvin-Morton-Rozansky conjecture.

2. The Alexander polynomial.

3. A joint attempt to read Drinfeld's papers on quasi-Hopf algebras.

Please prioritize (or make other suggestions...)
1. Def of $B_n, P_n$

2. $\emptyset \rightarrow P_n \rightarrow B_n \rightarrow \Sigma_n \rightarrow \infty$

3. $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$:
   - $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$
   - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

4. PF of above, by reduction to $P_n$ in $B_n$:
   a. $P_n = \langle A_{i,j} \mid 1 \leq i < j \leq n, \text{ rels} \rangle$
   b. $\text{rels} = \langle A_{n} A_{1} A_{n}^{-1} = A_{1} A_{n}^{-1} \rangle$

\[3/16/22\]
Leonid Kongorovsky, Ribbon Categories & invariants of Links, 12/19/94

Ribbon Categories = \textit{balanced} \textit{braided} \textit{rigid} monoidal category

\textit{quasi} triangular

\textbf{Example:} Ribbon tangles. (global description, under if a normal is specified)

def by generators.

\textit{colors}, \ast: \sigma \rightarrow \sigma \ (\sigma = \text{set or colors})

\textbf{Def monoidal:} associative, tensor prod., w/ identity

\textbf{Rigid:} dual objects exist, st.

\begin{align*}
& k \rightarrow X \otimes X, \quad X \otimes X \rightarrow k \\
& k \rightarrow X \otimes * X, \quad X \otimes * X \rightarrow k \\
& \text{s.t. all usual axioms.}
\end{align*}

\textbf{branched:}

\textbf{balanced:} \beta_x: X \rightarrow X^*

\textit{quasi}-\textit{triangular Hopf algebras, reps...}