Pensieve header: The naive Kh Program - aborted attempt to remove np/nm.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\23-FastComputations"];
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
pd = PD[Knot[5, 2]]
```

```
PD[X[1, 4, 2, 5], X[3, 8, 4, 9], X[5, 10, 6, 1], X[9, 6, 10, 7], X[7, 2, 8, 3]]
```

pdf

```
Sign[X[i_, j_, k_, l_]] /; j - l == 1 ∨ l - j > 1 ^:= 1;
Sign[X[i_, j_, k_, l_]] /; l - j == 1 ∨ j - l > 1 ^:= -1;
w[pd_] := Plus @@ (Sign /@ pd);
```

```
Sign /@ pd
```

```
PD[-1, -1, -1, -1, -1]
```

```
w[pd]
```

```
-5
```

pdf

```
SetAttributes[p, Orderless]
```

pdf

```
m_ ⋄ n_ := Min[m, n];
S[pd_PD, a_List] := Times @@ ({List @@ pd, a}^T /. {
        {x : X[i_, j_, k_, l_], 0} :> q^(3/2 Sign[x] - 1/2) t^(1/2 Sign[x] - 1/2) p[i, j]_(i⋄j) p[k, l]_(k⋄l),
        {x : X[i_, j_, k_, l_], 1} :> q^(3/2 Sign[x] + 1/2) t^(1/2 Sign[x] + 1/2) p[i, l]_(i⋄l) p[j, k]_(j⋄k),
        {x_X, ⋆} :> x
      }) //. p[i_, j_]_m_ p[j_, k_]_n_ :> p[i, k]_(m⋄n) //.
   {X[i_, j_, k_, l_] p[i_, j_]_m_ p[k_, l_]_n_ :> (c_m c_n → c_(m⋄n)),
    X[i_, j_, k_, l_] p[i_, l_]_m_ p[j_, k_]_n_ :> (c_(m⋄n) → c_m c_n)} //. p[__]_m_ :> c_m
```

```
{
 S[PD[Mirror[Knot[3, 1]]], {0, 0, 0}],
 S[PD[Mirror[Knot[3, 1]]], {0, 1, 0}],
 S[PD[Mirror[Knot[3, 1]]], {0, ⋆, 0}]
}
```

$$\left\{q^3\, c_1\, c_2,\ q^4\, t\, c_1,\ q^2\, (c_1\, c_2 \to q\, c_1)\right\}$$

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```
V[pd_PD, a_] := List @@ Expand[S[pd, a] /. c_x_ :→ (q vp_x + q^-1 vm_x)]
V[pd_PD, a_, deg_] := Select[V[pd, a], Exponent[#, q] == deg &]
```

```
V[pd, {0, 0, 0, 0, 0}]
```

$$\left\{\frac{vm_1\, vm_2\, vm_3}{q^{13}\, t^5}, \frac{vm_2\, vm_3\, vp_1}{q^{11}\, t^5}, \frac{vm_1\, vm_3\, vp_2}{q^{11}\, t^5}, \frac{vm_3\, vp_1\, vp_2}{q^9\, t^5}, \frac{vm_1\, vm_2\, vp_3}{q^{11}\, t^5}, \frac{vm_2\, vp_1\, vp_3}{q^9\, t^5}, \frac{vm_1\, vp_2\, vp_3}{q^9\, t^5}, \frac{vp_1\, vp_2\, vp_3}{q^7\, t^5}\right\}$$

```
V[pd, {0, 0, 0, 0, 0}, -9]
```

$$\left\{\frac{vm_3\, vp_1\, vp_2}{q^9\, t^5}, \frac{vm_2\, vp_1\, vp_3}{q^9\, t^5}, \frac{vm_1\, vp_2\, vp_3}{q^9\, t^5}\right\}$$

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```
d[pd_PD, a_] := S[pd, a] /. {
   (c_x_ c_y_ → c_z_) * _. :→ {vp_x vp_y → t vp_z, vp_x vm_y → t vm_z, vm_x vp_y → t vm_z, vm_x vm_y → 0},
   (c_z_ → c_x_ c_y_) * _. :→ {vp_z → t vp_x vm_y + t vm_x vp_y, vm_z → t vm_x vm_y}}
```

```
d[pd, #] & /@ Permutations[{0, 0, 0, 1, ★}]
```

$\{\{vp_1\, vp_3 \to vp_1,\ vm_3\, vp_1 \to vm_1,\ vm_1\, vp_3 \to vm_1,\ vm_1\, vm_3 \to 0\},$
$\{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\},$
$\{vp_1\, vp_3 \to vp_1,\ vm_3\, vp_1 \to vm_1,\ vm_1\, vp_3 \to vm_1,\ vm_1\, vm_3 \to 0\},$
$\{vp_1 \to vm_6\, vp_1 + vm_1\, vp_6,\ vm_1 \to vm_1\, vm_6\},\ \{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\},$
$\{vp_1 \to vm_6\, vp_1 + vm_1\, vp_6,\ vm_1 \to vm_1\, vm_6\},\ \{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\},$
$\{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\},$
$\{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\},$
$\{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\},$
$\{vp_1\, vp_3 \to vp_1,\ vm_3\, vp_1 \to vm_1,\ vm_1\, vp_3 \to vm_1,\ vm_1\, vm_3 \to 0\},$
$\{vp_1\, vp_3 \to vp_1,\ vm_3\, vp_1 \to vm_1,\ vm_1\, vp_3 \to vm_1,\ vm_1\, vm_3 \to 0\},$
$\{vp_1\, vp_3 \to vp_1,\ vm_3\, vp_1 \to vm_1,\ vm_1\, vp_3 \to vm_1,\ vm_1\, vm_3 \to 0\},\ \{vp_1 \to vm_2\, vp_1 + vm_1\, vp_2,\ vm_1 \to vm_1\, vm_2\},$
$\{vp_1 \to vm_2\, vp_1 + vm_1\, vp_2,\ vm_1 \to vm_1\, vm_2\},\ \{vp_1\, vp_3 \to vp_1,\ vm_3\, vp_1 \to vm_1,\ vm_1\, vp_3 \to vm_1,\ vm_1\, vm_3 \to 0\},$
$\{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\},\ \{vp_1 \to vm_2\, vp_1 + vm_1\, vp_2,\ vm_1 \to vm_1\, vm_2\},$
$\{vp_1 \to vm_2\, vp_1 + vm_1\, vp_2,\ vm_1 \to vm_1\, vm_2\},\ \{vp_1\, vp_2 \to vp_1,\ vm_2\, vp_1 \to vm_1,\ vm_1\, vp_2 \to vm_1,\ vm_1\, vm_2 \to 0\}\}$

```
{V[pd, {0, 0, 0, 1, 0}] , d[pd, {0, 0, ★, 1, 0}]}
```

$$\left\{\left\{\frac{vm_1\, vm_3}{q^{11}\, t^4}, \frac{vm_3\, vp_1}{q^9\, t^4}, \frac{vm_1\, vp_3}{q^9\, t^4}, \frac{vp_1\, vp_3}{q^7\, t^4}\right\}, \{vp_1 \to q\, vm_6\, vp_1 + q\, vm_1\, vp_6,\ vm_1 \to q\, vm_1\, vm_6\}\right\}$$

pdf

```
CC[pd_PD, r_, deg_] := Select[
  Union @@ ((v @@ #) V[pd, #, deg] & /@ Tuples[{0, 1}, Length@pd]), Exponent[#, t] == r &]
```

```
CC[pd, -4, -9]
```

$$\left\{\frac{vm_2\, vp_1\, v[0,0,0,0,1]}{q^9\, t^4}, \frac{vm_1\, vp_2\, v[0,0,0,0,1]}{q^9\, t^4}, \frac{vm_3\, vp_1\, v[0,0,0,1,0]}{q^9\, t^4}, \frac{vm_1\, vp_3\, v[0,0,0,1,0]}{q^9\, t^4}, \frac{vm_3\, vp_1\, v[0,0,1,0,0]}{q^9\, t^4}, \frac{vm_1\, vp_3\, v[0,0,1,0,0]}{q^9\, t^4}, \frac{vm_2\, vp_1\, v[0,1,0,0,0]}{q^9\, t^4}, \frac{vm_1\, vp_2\, v[0,1,0,0,0]}{q^9\, t^4}, \frac{vm_3\, vp_1\, v[1,0,0,0,0]}{q^9\, t^4}, \frac{vm_1\, vp_3\, v[1,0,0,0,0]}{q^9\, t^4}\right\}$$

pdf

```
d[pd_PD][expr_] := Expand[expr] /. s_ * a_v :> Expand[sign = 1; Sum[
     If[a[[i]] == 0,
      sign * ReplacePart[a, 1, i] * s /. d[pd, List @@ ReplacePart[a, *, i]], sign *= -1;
      0], {i, Length[a]}
    ]]
```

```
t9 = d[pd][v[0, 0, 1, 0, 0] vm[3] vp[1]]
```

```
-v[0, 0, 1, 0, 1] vm[3] vp[1] - v[0, 0, 1, 1, 0] vm[3] vp[1] +
 v[0, 1, 1, 0, 0] vm[3] vp[1] + v[1, 0, 1, 0, 0] vm[3] vp[1]
```

```
d[pd][t9]
```

```
0
```

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```
Rank[pd_PD, r_, deg_] := (*Rank[pd,r,deg]=*)Module[{b0, b1, db0, ds0, s1},
   b0 = CC[pd, r, deg]; b1 = CC[pd, r + 1, deg];
   If[b0 == {} \/ b1 == {}, 0,
    db0 = d[pd][b0];
    MatrixRank[Table[Coefficient[ds0, s1], {ds0, db0}, {s1, b1}]]
   ]
  ];
```

pdf

```
Betti[pd_PD, r_, deg_] := Length[CC[pd, r, deg]] - Rank[pd, r, deg] - Rank[pd, r - 1, deg]
```

```
{r, (w[pd] - Length[pd]) / 2, (w[pd] + Length[pd]) / 2}
```

```
{r, -5, 0}
```

```
Plus @@ V[pd, Table[0, Length@pd]]
```

$$\frac{vm_1\, vm_2\, vm_3}{q^{13}\, t^5} + \frac{vm_2\, vm_3\, vp_1}{q^{11}\, t^5} + \frac{vm_1\, vm_3\, vp_2}{q^{11}\, t^5} + \frac{vm_3\, vp_1\, vp_2}{q^9\, t^5} + \frac{vm_1\, vm_2\, vp_3}{q^{11}\, t^5} + \frac{vm_2\, vp_1\, vp_3}{q^9\, t^5} + \frac{vm_1\, vp_2\, vp_3}{q^9\, t^5} + \frac{vp_1\, vp_2\, vp_3}{q^7\, t^5}$$

```
{deg, Exponent[Plus @@ V[pd, Table[0, Length@pd]], q, Min],
 Exponent[Plus @@ V[pd, Table[1, Length@pd]], q, Max]}
```

```
{deg, -13, -1}
```

```
Kh1[pd_PD] := Sum[t^r q^deg Betti[pd, r, deg],
  {r, -Length@pd, Length@pd},
  {deg, Exponent[Plus @@ V[pd, Table[0, Length@pd]], q, Min],
   Exponent[Plus @@ V[pd, Table[1, Length@pd]], q, Max], 2}
 ]
```

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```
Kh1[pd_PD] := Sum[t^r q^deg Betti[pd, r, deg],
  {r, -Length@pd, Length@pd},
  {deg, -3 Length@pd, 3 Length@pd, 1}
 ]
```

```
Kh1[PD[Knot[3, 1]]]
```

$$\frac{1}{q^3}+\frac{1}{q}+\frac{1}{q^9\,t^3}+\frac{1}{q^5\,t^2}$$

$$\frac{\mathbf{1}}{\mathbf{q^3}}+\frac{\mathbf{1}}{\mathbf{q}}+\frac{\mathbf{1}}{\mathbf{q^9\,t^3}}+\frac{\mathbf{1}}{\mathbf{q^5\,t^2}}$$

$$\frac{1}{q^3}+\frac{1}{q}+\frac{1}{q^9\,t^3}+\frac{1}{q^5\,t^2}$$

```
Timing@Sum[Kh[K][q, t] == Kh1[PD@K], {K, AllKnots[{3, 6}]}]
```

```
{311.516, 7 True}
```