

Pensieve header: The naive Kh Program.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\23-FastComputations"];
Once[<< KnotTheory`]
```

```
pd = PD[Knot[5, 2]]
```

```
PD[X[1, 4, 2, 5], X[3, 8, 4, 9], X[5, 10, 6, 1], X[9, 6, 10, 7], X[7, 2, 8, 3]]
```

pdf

```
np[pd_PD] := Count[pd, X[i_, j_, k_, l_] /; j - l == 1 ∨ l - j > 1];
nm[pd_PD] := Count[pd, X[i_, j_, k_, l_] /; l - j == 1 ∨ j - l > 1];
```

```
{np[pd], nm[pd]}
```

```
{0, 5}
```

pdf

```
SetAttributes[p, Orderless]
```

pdf

```
S[pd_PD, s_String] := S[pd, Characters[s] /. {"0" → 0, "1" → 1}]
```

pdf

```
m_ ◊ n_ := Min[m, n];
S[pd_PD, a_List] := Times@@({List@@pd, a}^T /. {
  {X[i_, j_, k_, l_], 0} => p[i, j]_{i◊j} p[k, l]_{k◊l},
  {X[i_, j_, k_, l_], 1} => p[i, l]_{i◊l} p[j, k]_{j◊k},
  {x_X, "*" } => x}) // . p[i_, j_]_m p[j_, k_]_n => p[i, k]_{m◊n} // .
{X[i_, j_, k_, l_] p[i_, j_]_m p[k_, l_]_n => (c_m c_n → c_{m◊n}),
 X[i_, j_, k_, l_] p[i_, l_]_m p[j_, k_]_n => (c_{m◊n} → c_m c_n)} // . p[___]_m' => c_m
```

```
{
  S[PD[Mirror[Knot[3, 1]]], {0, 0, 0}],
  S[PD[Mirror[Knot[3, 1]]], {0, 1, 0}],
  S[PD[Mirror[Knot[3, 1]]], {0, "*", 0}]
}
```

```
{c_1 c_2, c_1, c_1 c_2 → c_1}
```

```
{
  S[PD[Mirror[Knot[5, 2]]], "00010",
  S[PD[Mirror[Knot[5, 2]]], "00110",
  S[PD[Mirror[Knot[5, 2]]], "00*10"]
}
```

```
{c_1 c_2 c_3, c_1 c_2, (c_1 c_3 → c_1) c_2}
```

pdf

```
Deg[P_] := Count[P, vp_, {0, 1}] - Count[P, vm_, {0, 1}]
```

```
{Deg[vp6 vp7], Deg[vp6 vm7]}
```

```
{2, 0}
```

pdf

```
V[pd_PD, a_] := List@@Expand[S[pd, a] /. c_x_ -> (vp_x + vm_x)]
```

```
V[pd_PD, a_, deg_] := Select[V[pd, a], (deg == Deg[#] + (Plus@@a)) &]
```

```
V[pd, {0, 0, 0, 0, 0}]
```

```
{vm1 vm2 vm3, vm2 vm3 vp1, vm1 vm3 vp2, vm3 vp1 vp2, vm1 vm2 vp3, vm2 vp1 vp3, vm1 vp2 vp3, vp1 vp2 vp3}
```

pdf

```
d[pd_PD, a_] := S[pd, a] /. {
```

```
(c_x_ c_y_ -> c_z_) * _ -> {vp_x vp_y -> vp_z, vp_x vm_y -> vm_z, vm_x vp_y -> vm_z, vm_x vm_y -> 0},
```

```
(c_z_ -> c_x_ c_y_) * _ -> {vp_z -> vp_x vm_y + vm_x vp_y, vm_z -> vm_x vm_y}}
```

```
d[pd, #] & /@Permutations[{0, 0, 0, 1, "*"}]
```

```
{ {vp1 vp3 -> vp1, vm3 vp1 -> vm1, vm1 vp3 -> vm1, vm1 vm3 -> 0},
```

```
{vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0},
```

```
{vp1 vp3 -> vp1, vm3 vp1 -> vm1, vm1 vp3 -> vm1, vm1 vm3 -> 0},
```

```
{vp1 -> vm6 vp1 + vm1 vp6, vm1 -> vm1 vm6}, {vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0},
```

```
{vp1 -> vm6 vp1 + vm1 vp6, vm1 -> vm1 vm6}, {vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0},
```

```
{vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0},
```

```
{vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0},
```

```
{vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0},
```

```
{vp1 vp3 -> vp1, vm3 vp1 -> vm1, vm1 vp3 -> vm1, vm1 vm3 -> 0},
```

```
{vp1 vp3 -> vp1, vm3 vp1 -> vm1, vm1 vp3 -> vm1, vm1 vm3 -> 0},
```

```
{vp1 vp3 -> vp1, vm3 vp1 -> vm1, vm1 vp3 -> vm1, vm1 vm3 -> 0}, {vp1 -> vm2 vp1 + vm1 vp2, vm1 -> vm1 vm2},
```

```
{vp1 -> vm2 vp1 + vm1 vp2, vm1 -> vm1 vm2}, {vp1 vp3 -> vp1, vm3 vp1 -> vm1, vm1 vp3 -> vm1, vm1 vm3 -> 0},
```

```
{vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0}, {vp1 -> vm2 vp1 + vm1 vp2, vm1 -> vm1 vm2},
```

```
{vp1 -> vm2 vp1 + vm1 vp2, vm1 -> vm1 vm2}, {vp1 vp2 -> vp1, vm2 vp1 -> vm1, vm1 vp2 -> vm1, vm1 vm2 -> 0}}
```

```
{V[pd, "00010"], d[pd, "00*10"]}
```

```
{ {vm1 vm3, vm3 vp1, vm1 vp3, vp1 vp3}, {vp1 -> vm6 vp1 + vm1 vp6, vm1 -> vm1 vm6}}
```

```
Expand[V[pd, "00010"] /. d[pd, "00*10"]]
```

```
{vm1 vm3 vm6, vm3 vm6 vp1 + vm1 vm3 vp6, vm1 vm6 vp3, vm6 vp1 vp3 + vm1 vp3 vp6}
```

pdf

```
KhBracket[pd_PD, r_, deg___] :=
```

```
If[r < 0 || r > Length[pd], {}, Join@@((v@@#) V[pd, #, deg]) & /@
```

```
Permutations[Join[Table[0, Length@pd - r], Table[1, r]]]
```

```
CC[pd_PD, r_, deg_] := KhBracket[pd, r + nm[pd], deg - np[pd] + 2 nm[pd]]
```

KhBracket[pd, 2]

```
{vm1 v[0, 0, 0, 1, 1], vp1 v[0, 0, 0, 1, 1], vm1 v[0, 0, 1, 0, 1],
vp1 v[0, 0, 1, 0, 1], vm1 vm3 vm6 v[0, 0, 1, 1, 0], vm3 vm6 vp1 v[0, 0, 1, 1, 0],
vm1 vm6 vp3 v[0, 0, 1, 1, 0], vm6 vp1 vp3 v[0, 0, 1, 1, 0], vm1 vm3 vp6 v[0, 0, 1, 1, 0],
vm3 vp1 vp6 v[0, 0, 1, 1, 0], vm1 vp3 vp6 v[0, 0, 1, 1, 0], vp1 vp3 vp6 v[0, 0, 1, 1, 0],
vm1 v[0, 1, 0, 0, 1], vp1 v[0, 1, 0, 0, 1], vm1 v[0, 1, 0, 1, 0], vp1 v[0, 1, 0, 1, 0],
vm1 v[0, 1, 1, 0, 0], vp1 v[0, 1, 1, 0, 0], vm1 v[1, 0, 0, 0, 1], vp1 v[1, 0, 0, 0, 1],
vm1 vm2 vm3 v[1, 0, 0, 1, 0], vm2 vm3 vp1 v[1, 0, 0, 1, 0], vm1 vm3 vp2 v[1, 0, 0, 1, 0],
vm3 vp1 vp2 v[1, 0, 0, 1, 0], vm1 vm2 vp3 v[1, 0, 0, 1, 0], vm2 vp1 vp3 v[1, 0, 0, 1, 0],
vm1 vp2 vp3 v[1, 0, 0, 1, 0], vp1 vp2 vp3 v[1, 0, 0, 1, 0], vm1 vm2 vm3 v[1, 0, 1, 0, 0],
vm2 vm3 vp1 v[1, 0, 1, 0, 0], vm1 vm3 vp2 v[1, 0, 1, 0, 0], vm3 vp1 vp2 v[1, 0, 1, 0, 0],
vm1 vm2 vp3 v[1, 0, 1, 0, 0], vm2 vp1 vp3 v[1, 0, 1, 0, 0], vm1 vp2 vp3 v[1, 0, 1, 0, 0],
vp1 vp2 vp3 v[1, 0, 1, 0, 0], vm1 v[1, 1, 0, 0, 0], vp1 v[1, 1, 0, 0, 0]}
```

pdf

```
d[pd_PD][expr_] := Expand[expr] /. s_ * a_v => Expand[sign = 1; Sum[
  If[a[[i]] == 0,
    sign * ReplacePart[a, 1, i] * s /. d[pd, List@@ReplacePart[a, "*", i]], sign *= -1;
    0], {i, Length[a]}
]]
```

t9 = d[pd][v[0, 0, 1, 0, 0] vm[3] vp[1]]

-v[0, 0, 1, 0, 1] vm[3] vp[1] - v[0, 0, 1, 1, 0] vm[3] vp[1] +
v[0, 1, 1, 0, 0] vm[3] vp[1] + v[1, 0, 1, 0, 0] vm[3] vp[1]

d[pd][t9]

0

pdf

```
Rank[pd_PD, r_, deg_] := (*Rank[pd,r,deg]=*) Module[{b0, b1, db0, ds0, s1},
  b0 = CC[pd, r, deg]; b1 = CC[pd, r + 1, deg];
  If[b0 == {} ∨ b1 == {}, 0,
    db0 = d[pd][b0];
    MatrixRank[Table[Coefficient[ds0, s1], {ds0, db0}, {s1, b1}]]
  ]
];
```

pdf

```
Betti[pd_PD, r_, deg_] := Length[CC[pd, r, deg]] - Rank[pd, r, deg] - Rank[pd, r - 1, deg]
```

pdf

```
qBetti[pd_PD, r_] := (degs = Union[Deg /@ KhBracket[pd, r + nm[pd]]] + np[pd] - nm[pd] + r;
  (Betti[pd, r, #] & /@ degs) . qdegs)
```

pdf

```
Kh1[pd_PD] := Expand[Sum[tr qBetti[pd, r], {r, -nm[pd], Length[pd] - nm[pd]}]]
```

Kh1[PD[Knot[3, 1]]]

$$\frac{1}{q^3} + \frac{1}{q} + \frac{1}{q^9 t^3} + \frac{1}{q^5 t^2}$$

$$\frac{1}{q^3} + \frac{1}{q} + \frac{1}{q^9 t^3} + \frac{1}{q^5 t^2}$$

$$\frac{1}{q^3} + \frac{1}{q} + \frac{1}{q^9 t^3} + \frac{1}{q^5 t^2}$$

Timing@Sum[Kh[K][q, t] == Kh1[PD@K], {K, AllKnots[{3, 6]}]}

{6.0625, 7 True}
