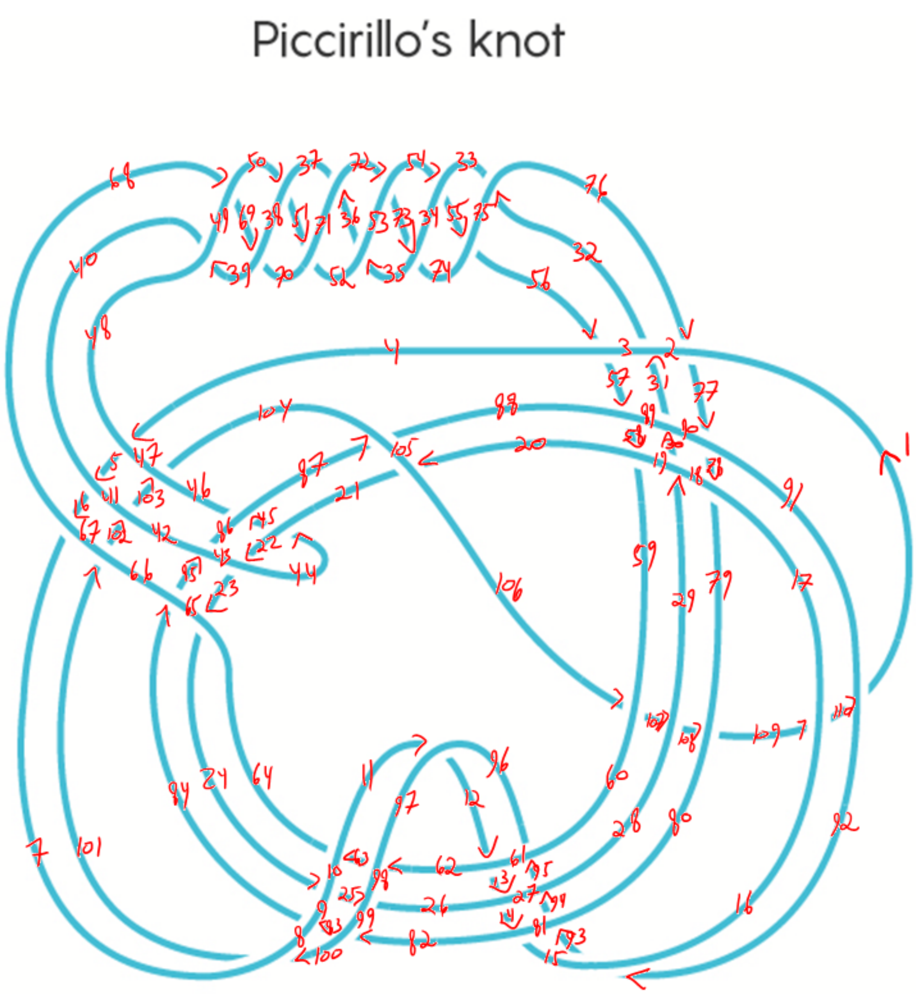
To do.

1. Make a Day 1 Handout, with syllabus and a page 2 with some Quanta magazine content, a fresh drawing of Piccirilo's knot, pictures of some pathologies, and the Reidemeister theorem.

2. Write a super-slim naive-KH program.

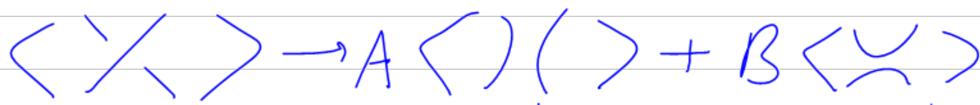


Day 1 Notes

Hour 1. Go over Day1Gallery.html.

3-colourings.

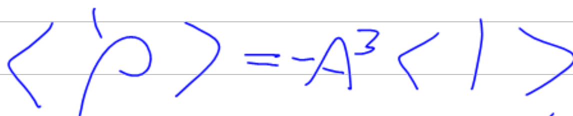
Hour 2. The Kauffman bracket.



0-smoothing 1-smoothing

$$\langle D \bigcirc \rangle = d\langle D \rangle$$

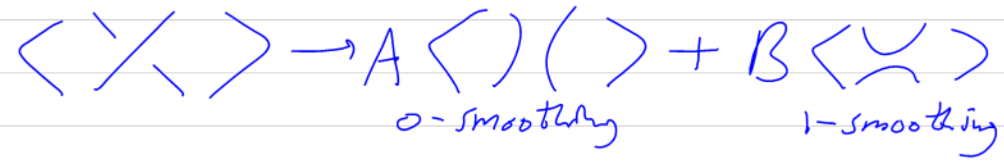
$$\longrightarrow B = A^{-1},\ d = -A^2 - A^{-2}$$



$$J(K) := (-A^3)^{-w(K)} \frac{\langle K \rangle}{d} \ /.\ A \to q^{-1/4}$$

Hour 3. Implement!

```
KB[pd_PD] := Module[{p, t1, t2, t3, t4, B, d},
  SetAttributes[p, Orderless];
  t1 = pd /. X[i_, j_, k_, l_] → A * p[i, j] * p[k, l] + B * p[i, l] p[j, k];
  t2 = Expand[t1 /. PD → Times];
  t3 = t2 //. {p[i_, j_] p[j_, k_] → p[i, k]};
  t4 = t3 /. {p[i_, i_] → d, p[i_, j_]^2 → d};
  Expand[t4 /. {B → 1/A, d → -A^2 - 1/A^2}]
 ]
```

On board: 

$\langle O^k \rangle = d^k$

$B = A^{-1}$, $d = -A^2 - A^{-2}$

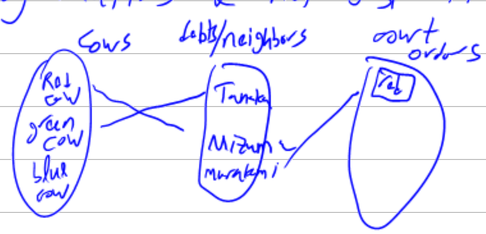
$$J(K) := (-A^3)^{-w(K)} \frac{\langle K \rangle}{d} \;/.\; A \to q^{-1/4}$$

How 1.
1. Divide an conquer
2. Rank $TL_{2n} = \frac{1}{n+1}\binom{2n}{n} < 2^{2n}$
3. The scanner method.

```
EFKB[pd_PD] := Module[{p, t1, t2, t3, t4, B, d, KB, todo, front, x, v},
  SetAttributes[p, Orderless];
  KB = 1;
  todo = List @@ pd;
  front = {};
  v[x_X] := Length[front ⋂ (List @@ x)];
  While[Length[todo] > 0,
   x = RandomChoice[MaximalBy[todo, v]];
   todo = DeleteCases[todo, x];
   t1 = KB (x /. X[i_, j_, k_, l_] → A * p[i, j] * p[k, l] + B * p[i, l] * p[j, k]);
   t2 = Expand[t1];
   t3 = t2 //. {p[i_, j_] × p[j_, k_] → p[i, k]};
   t4 = t3 /. {p[i_, i_] → d, p[i_, j_]^2 → d};
   KB = Expand[t4 /. {B → 1 / A, d → -A^2 - 1 / A^2}];
   front = Complement[front ⋃ (List @@ x), front ⋂ (List @@ x)];
  ];
  KB
 ]
```

How 2. "Dream categorification": All numbers are replaced by sets
All equalities by bijections

Advantage $7 = \ldots\ldots = 7$ could be interesting (loosely, that how 4D geometry enters the game)
The journey matters & not just the destination!



In practice: vector spaces
direct sums
tensor products

complexes
homology
Euler characteristic.

Then graded everything

Day 3 Notes

on board: 1. Class photo?
2. Homework submission?
3. The program could still be improved, 1,000x at least!
4. It's all wrong! And it gets worse.

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J : \times \mapsto q)(- q^2 \asymp, \quad J : \times \mapsto -q^{-2} \asymp + q^{-1})($

Sketch.

Vector spaces, sums, tensor products, dimensions.
Complexes.
Homology and Euler characteristic.

X : finite set. $X^* : \{f : X \to \mathbb{Q}\}$ $\quad \dim X^* = |X|$

$(X \cup Y)^* = X \oplus Y = \left\{ (f,g) : \begin{matrix} f \in X^* \\ g \in Y^* \end{matrix} \right\}$ $\quad a(f_1,g_1)+b(f_2,g_2) := (af_1+bf_2, ag_1+bg_2)$

Now generalize: $V \oplus W := \left\{ (v,w) : \begin{matrix} v \in V \\ w \in W \end{matrix} \right\}$ ...

$\dim(V \oplus W) = \dim V + \dim W$

Now repeat for $(X \times Y)^*$

An aside on singular homology.
Definition using polytops.
Examples: spheres, surfaces.
Maps and functoriality

Graded vector spaces and graded complexes.

The definition of Khovanov homology (with a handout, but not following it)

The 0 -> B -> A -> A/B -> 0 theorem.

Day 4 Notes.

On board:

Theorem. For a knot diagram D, there is a complex K(D) whose homology KH(D) is invariant and whose Euler characteristic is J(D).

Skeleton of the KH(Trefoil) diagram.

Class photo during break!

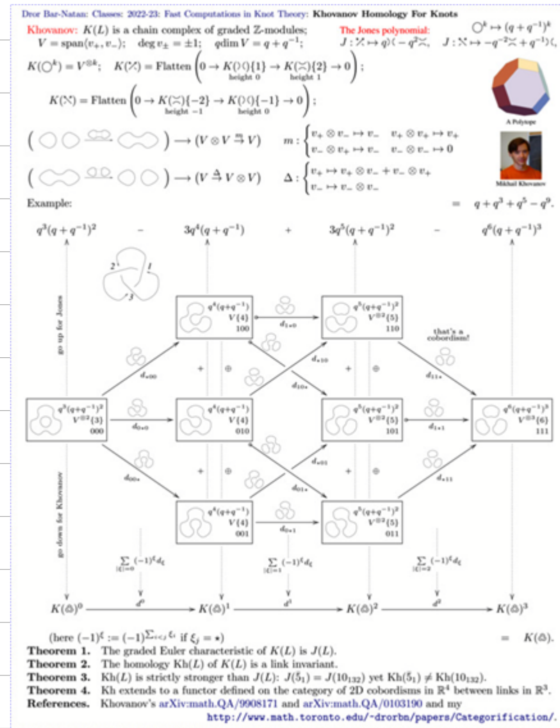
HW2 will be online by midnight! (Comment on HW1).

Piccirillo's knot has only 55 crossings!

The definition of Khovanov homology (with the handout, but not following it)

The 0 -> B -> A -> A/B -> 0 theorem (prove only the part assuming B is acyclic).

Proof of invariance under R2:


Dror Bar-Natan: Classes: 2022-23: Fast Computations in Knot Theory: **Khovanov Homology For Knots**

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules; $V = \mathrm{span}(v_+, v_-)$; $\deg v_\pm = \pm 1$; $q\dim V = q + q^{-1}$;

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$K(\bigcirc^k) = V^{\otimes k}$

$m: v_+\otimes v_- \mapsto v_-$, $v_+\otimes v_+ \mapsto v_+$, $v_-\otimes v_+ \mapsto v_-$, $v_-\otimes v_- \mapsto 0$

$\Delta: v_+ \mapsto v_+\otimes v_- + v_-\otimes v_+$, $v_- \mapsto v_-\otimes v_-$

Example: $= q + q^3 + q^5 - q^9$.

$q^3(q+q^{-1})^2 \quad - \quad 3q^4(q+q^{-1}) \quad + \quad 3q^5(q+q^{-1})^2 \quad - \quad q^6(q+q^{-1})^3$

(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$)

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\mathrm{Kh}(L)$ of $K(L)$ is a link invariant.

Theorem 3. $\mathrm{Kh}(L)$ is strictly stronger than $J(L)$: $J(5_1) = J(10_{132})$ yet $\mathrm{Kh}(5_1) \neq \mathrm{Kh}(10_{132})$.

Theorem 4. Kh extends to a functor defined on the category of 2D cobordisms in $\mathbb{R}^4$ between links in $\mathbb{R}^3$.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my http://www.math.toronto.edu/~drorbn/papers/Categorification/.

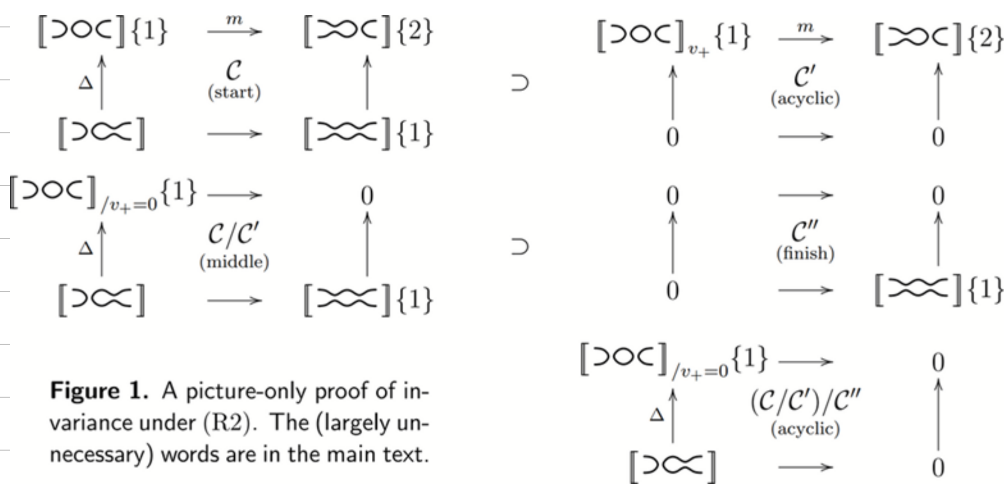


Figure 1. A picture-only proof of invariance under (R2). The (largely unnecessary) words are in the main text.

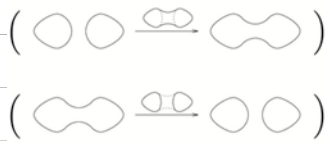
Day 5 Notes. (3 sessions)

Class photo: Send me your names!

$V = \mathrm{span}\langle v_+, v_-\rangle;\quad \deg v_\pm = \pm 1;\quad q\dim V = q + q^{-1};$

$$K(\bigcirc^k) = V^{\otimes k};\quad K(\times) = \mathrm{Flatten}\Big(0 \to \underset{\text{height } 0}{K()()\{1\}} \to \underset{\text{height } 1}{K(\asymp)\{2\}} \to 0\Big);$$

$$K(\times) = \mathrm{Flatten}\Big(0 \to \underset{\text{height } -1}{K(\asymp)\{-2\}} \to \underset{\text{height } 0}{K()()\{-1\}} \to 0\Big);$$



$\longrightarrow (V \otimes V \xrightarrow{m} V)\qquad m: \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$

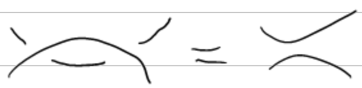
$\longrightarrow (V \xrightarrow{\Delta} V \otimes V)\qquad \Delta: \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$

Thm: The complex

$$0 \to \underset{\text{height } -n_-}{V^{\otimes -}\{n_+ - 2n_-\}} \to \oplus V^{\otimes \cdots}\{..\} \to \ \ldots \ \to \oplus V^{\otimes \cdots}\{..\} \to \underset{\text{height } n_+}{V^{\otimes \cdots}\{2n_+ - n_-\}} \to 0$$

has invariant homology. board line

Pf (partial).



?

Thm If $B^\circ \subset A^\circ$ is an inclusion of complexes,

1. $H(B) = 0 \Rightarrow H(A) = H(A/B)$
2. $H(A/B) = 0 \Rightarrow A(A) = H(B)$

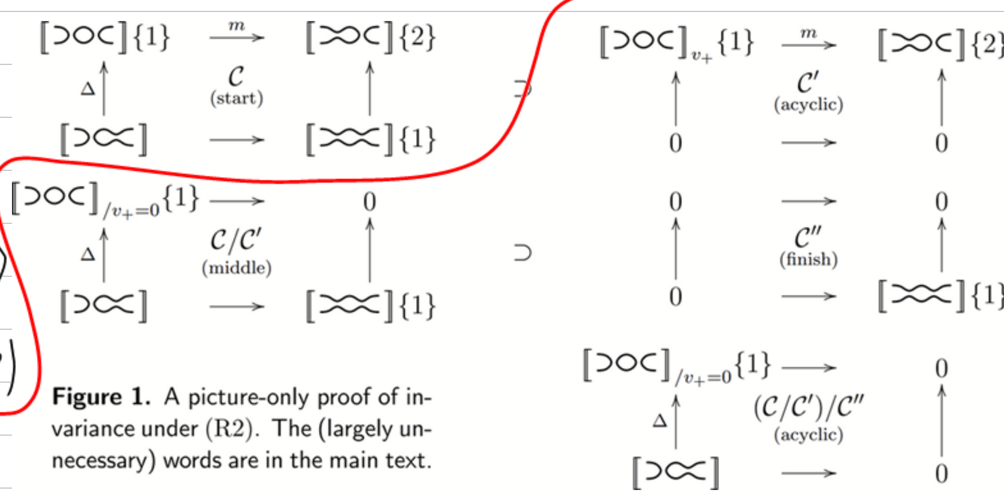


Figure 1. A picture-only proof of invariance under (R2). The (largely unnecessary) words are in the main text.

Prove R2 invariance (review!)

Prove Cancellation thm Part I
(do not review)

You cannot learn the proof of this theorem
You can only learn the technique (diagram chase).

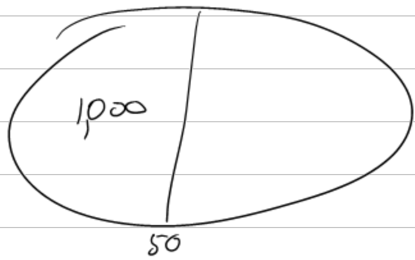
Implement! (as in Kyoto, Sep 2001).

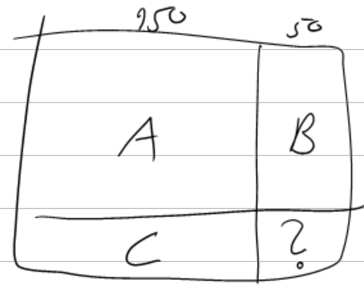
Day 6 Notes. (2 sessions)

Class photo: Send me your names!

<u>Session 2</u> K34 as in cornell-2015.

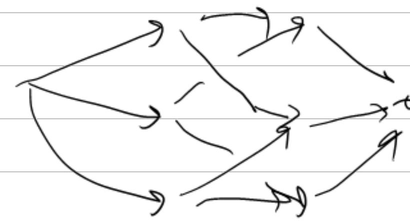
<u>Session 1</u> Pre-compute, given partial information:





Back to KH

Need "maps" between proto-vector spaces.



Need "direct sums"

Need "degrees"

Can be conservative/a coward!

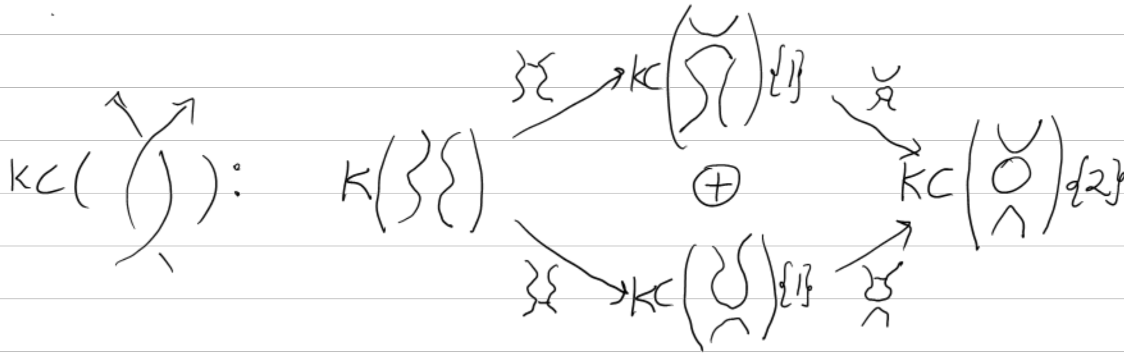
Need $\bigcirc \cong \begin{pmatrix}\emptyset_{+1}\\ \emptyset_{-1}\end{pmatrix}$

Need a sufficiently powerful cancellation rule

Need "Planar Algebra".

Day 6 Notes (2 sessions, take 2).

Session 2: "Knots in 3 and 4 dimensions" as in 2015 in Cornell University.
Class photo: Send me your names!
A half is better than a whole!



board line

1. Need "proto-vector-spaces"

2. Need "proto-maps" between them

3. Need $\bigcirc \cong \langle V_+, V_- \rangle$

4. Need "direct sums"

5. Need "degrees"

6. Need "Planar Algebra".

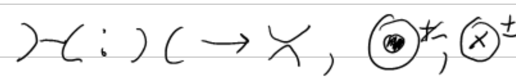
7. Need a sufficiently powerful cancellation rule

must stay on board!

Continue w/ 4–7.

Need "a category"

A word about free groups and the first iso. thm



Claim (probably true) The category with objects { … } and morphisms generated by …, $\odot^{+/-}$, $\otimes^{+/-}$ modulo universal Khovanov relations is