Word Crunching

I

wrote

a poem

on a page

but then each line grew

to the word sum of the previous two

until I began to worry about all these words coming with such frequency

because as you can see, it can be easy to run out of space when a poem gets all Fibonacci sequency

Brian Bilston

Please do the poll!

	1	2	3	4	5	6	7
I did							
I wanna see							

If you can, Video on!

(1) Let A, B, C, D be $n \times n$ matrices such that $AC = CA$. Prove that

$$\underbrace{\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{M} = \det(AD - CB)\ .$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Assuming $DC = CD$

$$\Rightarrow \det M = \det(A - BD^{-1}C)\det D$$
$$= \det(AD - BD^{-1}\underline{CD})$$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC) \text{ if } AC = CA,\ A \text{ is invertible}$$

$$\det\begin{pmatrix} A + \lambda I & B \\ C & D \end{pmatrix} = \det((A + \lambda I)D - BC)$$

For all but finitely many λ.

$$\begin{pmatrix} \cancel{A} & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{I} & 0 \\ & \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \xrightarrow{\cdot\begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix}} \begin{pmatrix} I & B \\ CA^{-1} & D \end{pmatrix}$$

$$\xrightarrow[\qquad]{\cdot \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}} \begin{pmatrix} I & 0 \\ CA^{-1} & -CA^{-1}B+D \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det (D - CA^{-1}B)$$

(2) Let A, B be $n \times n$ matrices (with real entries) that commute (i.e. $AB = BA$.) Prove that if $\det(A+B) \geq 0$, then $\det(A^k + B^k) \geq 0$ for all $k \geq 1$.

Jibran: $A^2 + B^2 = (A+iB)(A-iB)$

$$= (A+iB)\overline{(A+iB)}$$

$$\det(A^2+B^2) = \det(A+iB)\,\overline{\det(A+iB)}$$

$$A^3 + B^3 = (A+B)(\cancel{A^2 - AB + B^2})$$

$$(A+B)(A+wB)(A+\bar{w}B)$$

n odd

$$x^n + y^n = (x+y)\prod(x+w_k y)$$

nth root on 1 $\neq 1$

w_k be the roots of $1 + x^n = 0$

$$1^n + x^n = \prod(x - w_k)$$

$$1 + \left(\frac{b}{a}\right)^n = \prod\left(\frac{b}{a} - w_k\right)$$

$$a^n + b^n = \prod(b - w_k a)$$

$$A^n + B^n = \prod(B - w_k A)$$

$$\det(A^n + B^n) = \prod \det(B - w_k A)$$

(3) Let A be an $n \times n$ matrix such that there exists a positive integer k for which

$$kA^{k+1} = (k+1)A^k .$$

Prove that the matrix $\underbrace{A - I_n}$ is invertible and find its inverse.

$$A^{k+1} = \cancel{\frac{k+1}{k}} A^k \qquad \lambda$$

$$(x-1)^{-1} \qquad (1-x)^{-1}$$

$$A^{k+2} = \lambda A^{k+1} = \lambda^2 A^k$$

$$(I-A)^{-1} = \sum_{p\geq 0} A^p$$

$$A^{k+p} = \lambda^p A^k$$

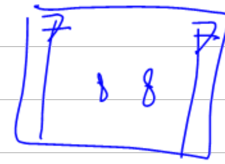
$$= (I + \dots + A^{k-1}) + A^k \sum_{p \geq 0} \lambda^p$$

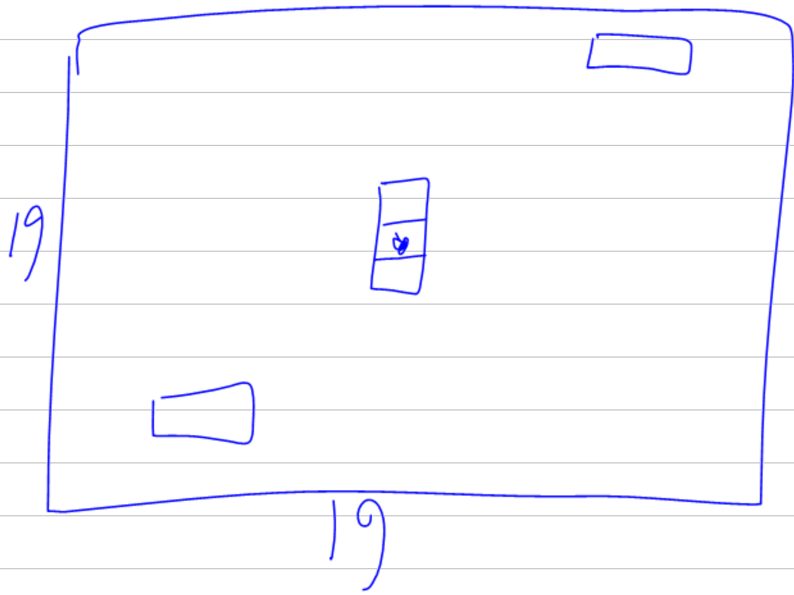
$$= (I + \dots + A^{k-1}) + \frac{1}{1-\lambda} A^k$$

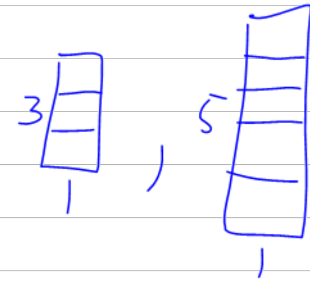
(4) We have n coins of unknown masses and a balance. We are allowed to place some of the coins on one side of the balance and an equal number of coins on the other side. After thus distributing the coins, the balance gives a comparison of the total mass of each side, either by indicating that the two masses are equal or by indicating that a particular side is the more massive of the two. Show that at least $n-1$ such comparisons are required to determine whether all of the coins are of equal mass.

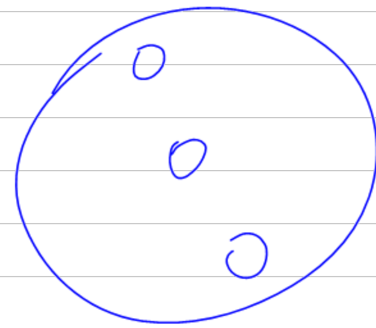
$$v = \begin{pmatrix} m_1 \\ \vdots \\ \vdots \\ m_n \end{pmatrix} \overset{?}{\in} \operatorname{span}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(5) (A2) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?









(6) (A2) Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i,j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{7} \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8}
\end{pmatrix}$$

$$\begin{matrix}
1 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6}-\frac{1}{5} & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7}-\frac{1}{6} & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}-\frac{1}{7}
\end{matrix}$$

$$\prod_{k=1}^{n-1}\left(\frac{1}{k+1}-\frac{1}{k}\right) = \prod_{k=1}^{n-1}\frac{-1}{k(k+1)} = \frac{(-1)^{n-1}}{(n-1)!\,n!}$$

(7) (A3) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

$$\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}$$

$$\boxed{\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}}$$

$$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos\beta - \sin\alpha \sin\beta$$

$$\cos\alpha \cos\beta = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$$

$$\alpha = \frac{x+y}{2} \quad \beta = \frac{x-y}{2}$$

$$\cos\frac{x+y}{2} \cos\frac{x-y}{2} = \frac{\cos x + \cos y}{2}$$