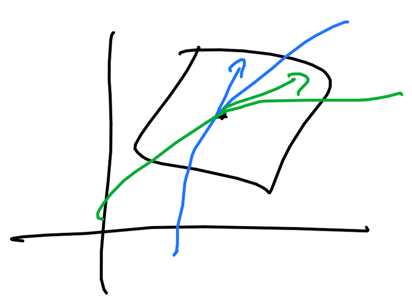
$f(v_1, v_2) = -f(v_2, v_1)$

$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = -\begin{vmatrix} w_1 & v_1 \\ w_2 & v_2 \end{vmatrix} = \det(v_1, w_1)$

$V = T_p M \quad (M = \mathbb{R}^n)$



Plan for today: □ HW 11
□ Problems on Matrices

Q1: isomorphism = invertible homomorphism

1) $f(v_1 + \alpha v_2) = f(v_1) + \alpha f(v_2)$
2) bijective ie. f full rank $(\dim V_1 = \dim V_2)$
(Df full rank)

Q1 says: $\boxed{(V^*)^* \underset{\text{iso.}}{=} V}$

(Infinite dimensions: Hahn-Banach Theorem)

$B \subset B^* \subset B^{**}$

Lemma: $f \in V^*$, $f \neq 0 \Leftrightarrow \exists v \in V$ st. $f(v) \neq 0$.
If $v \in V$, $v \neq 0$ then $\exists f \in V^*, f \neq 0$, st. $f(v) \neq 0$.

$\boxed{L^2[0,\pi]} \subset \{f : [0,2\pi] \to \mathbb{R},\ f(0) = f(2\pi)\}$ ⟶ reflexive $B \simeq B^*$

$\int |f|^2 < \infty \}$ ↗ Hilbert space

$\{\sin nx, \cos nx\}$

Proof: sketch $V = \langle e_1, \ldots, e_n \rangle$ where $f_i = e_i^*$
$V^* = \langle f_1, \ldots, f_n \rangle$ ie. $f_i(e_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\forall f \in V^*$: $f = f(e_1) f_1 + \ldots + f(e_n) f_n$

$\iota: V \to (V^*)^*$. $\quad$ ($\iota(e_i)$ dual dual basis)

$\iota(v)(\phi) = \phi(v) \in \mathbb{R}$.
↑
V^*

Q2: Idea: linear independence, span

$\phi_x(p) = 0$ for $x = -1, 0, 1$?

Use ι from Q1 to get $\iota(\phi_x)$ basis for V^{**}
≀
basis for V.

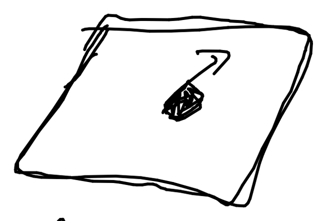
$\iota(p)(\phi_x) = \phi_x(p) = p(x)$

V^{**}
$V^* \to V$
?

Find $\{p_{-1}, p_0, p_1\}$ st. $\phi_x(p_y) = \delta_{xy}$.

Q3: B bilinear on k-linear functionals on V.

$T^k(V) = \{f : V^k \to \mathbb{R},\ f \text{ multilinear}\}$



$T^n(T_p M)$

$B: T^2 \to \mathbb{R}$ bilinear, WTS
$B_1(T_1 + \alpha T_2, J) = B(T_1, J) + \alpha B(T_2, J)$
& " " second component

$B(T,T) > 0$ WTS $B(T,T)$ = sum of squares of reals.
$B(T_2, T_1) = B(T_2, T_1)$

---

1. Orthogonal group of $n \times n$ matrices $O(n)$
ie. A $n \times n$, real : $A^t A = A A^t = I$.
is "compact".

Solution: $\text{Mat}_n = \{n \times n \text{ matrices}\} \simeq \mathbb{R}^{n^2}$.

$\Phi: \text{Mat}_n \simeq \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \simeq \text{Mat}_n$

$[A_{ij}] \longmapsto [(A^t A)_{ij}] = \sum_{k=1}^{n} A_{ik} A_{kj}$

"$\Phi : A \mapsto A^t A$"

$O(n) = \Phi^{-1}(I)$ | Φ clearly conts (quadratic)

$\{I\}$ closed $\Rightarrow$ $O(n)$ closed.

$A \in O(n)$: $\|A\|^2 = \sum_{ij} A_{ij}^2 = \sum_i \sum_j A_{ij}^2 = n < \infty$

$(A^t A)_{jj} = 1$

"$O(n)$ closed & bounded $\Rightarrow$ compact.
subset of $\mathbb{R}^{n^2}$"

∎