MAT237

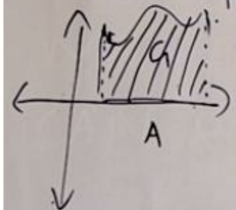
TT2. #1. $f: A \to \mathbb{R}$ cts. bdd. nonnegative.

$A \subset \mathbb{R}^n$ rectangle. $x \in A$

$G \subset \mathbb{R}^{n+1} = \{(x,y) : 0 \le y \le f(x)\}$.

Then $\int_A f = v(G)$.

Proof. - G is Jordan msrable



- f is bdd. so $|f| < M$ for some M

can bound G in the set

$A \times [0,M]$.

$\partial G = A \times \{0\}$

$\cup \{(x,y) : x \in \partial A, 0 \le y \le f(x)\}$ B

$\cup \{(x,y) : x \in A, y = f(x)\}$. G_0

$B \subset \partial A \times [0,M]$. msr 0.

G_0 is the graph of a cts fn.

$v(G_0) = \int_{\mathbb{R}^{n+1}} \chi_{G_0} = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_{G_0}(x,y)\,dy\,dx$

$= \int_{\mathbb{R}^n} 0\,dx$ $\chi_{G_0}(x,y) = 0$ unless $f(x) = y$.

$= 0$.

$\Rightarrow \partial G$ is msr 0.

$\int_{\mathbb{R}^{n+1}} \chi_G = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_G\,dy\,dx$ $\chi_G(x,y) = \begin{cases} 1 & 0 \le y \le f(x) \\ 0 & \text{o/w} \end{cases}$

$= \int_{\mathbb{R}^n} \int_0^{f(x)} 1\,dy\,dx$

$= \int_{\mathbb{R}^n} f(x)\,dx = \int_A f$. □

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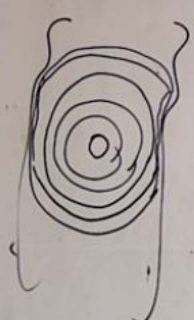
TT2. #2. $U \subset \mathbb{R}^n$ open. Then can find compact sets $\{C_k\}_{k=1}^{\infty}$ st. $C_k \subset \text{int}\, C_{k+1}$. and $U = \bigcup_{k=1}^{\infty} C_k$.



Proof. Take $C_k = \overline{B(0,k)} \cap \{x \in A : d(x, A^c) \ge 1/k\}$.

defined since A^c closed

3. $f: \mathbb{R}_{\ge 0} \to \mathbb{R}$ cts. supp f cpt.

$g: \mathbb{R}^3 \to \mathbb{R}$. $g(x) = f(|x|)$.

Then $\int_{\mathbb{R}^3} g = 4\pi \int_0^{\infty} r^2 f(r)\,dr$.

Proof. CoV using spherical coords.

$h(r,\phi,\theta) = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)$

$|\det Dh| = r^2\cos\phi$.

$(g \circ h)(x) = f(|h(r,\phi,\theta)|) = f(r)$.

supp f cpt. so supp $f \subset (0,R)$ for some R.

supp $g \subset \overline{B(0,R)}$.

$A = [0,R] \times [-\pi/2, \pi/2] \times [0, 2\pi]$.

then $h(A) \supset$ supp g.

$\int_{\mathbb{R}^3} g = \int_{h(A)} g = \int_A (g \circ h)|\det Dh| = \int_A f(r)\, r^2 \cos\phi$

$= 4\pi \int_0^R r^2 f(r)\,dr$.

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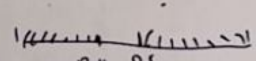
4. $S = \{x \in [0,1] :$ decimal expansion of x does not contain a $4\}$.

has msr 0.

$0.5 = 0.\underline{4}99\cdots$

Hint: Consider the length of the set $C_k := \{x \in [0,1] :$ first k digits in decimal expansion of x does not contain any 4s$\}$

C_1 

$C_1 = [0,1] \setminus [0.4, 0.5]$.

$C_2 = [0, 0.1] \setminus (0.14, 0.15] \cup \cdots \cup [0.3, 0.4] \setminus (0.34, 0.35]$
$\cup [0.5, 0.6] \setminus (0.54, 0.55] \cup \cdots$

$C_{k+1} = C_k \setminus \bigcup_{a_i \ne 4 \text{ for all } i} (0.a_1a_2\cdots a_k4, 0.a_1a_2\cdots a_k5]$

9^k disjoint intervals. each has length $\frac{1}{10^{k+1}}$.

C_k finite union of intervals $\Rightarrow$ msrable.

$v(C_{k+1}) = v(C_k) - \frac{9^k}{10^{k+1}}$

$v(C_1) = \frac{9}{10}$.

$v(C_k) = \frac{9}{10} - \sum_{i=1}^{k-1} \frac{9^i}{10^{i+1}} = \frac{9^k}{10^k}$

$S = \bigcap_{k \ge 1}^{\infty} C_k$ $v(S) = \lim_{k\to\infty} v(C_k) = \lim_{k\to\infty} \left(\frac{9}{10}\right)^k = 0$.