2021/22 MAT257 Term Test 3 Information and Rejected Questions

- The test will take place on Tuesday March 8, 5-7PM, at EX320. It will be a "closed book" exam: no books and no notes of any kind will be allowed, no cell-phones, no calculators, no devices of any kind that can display text. So only stationary will be allowed, as well as minimal hydration and snacks, and stuffed animals for joy and comfort. Don't forget to bring your UofT ID!
- Our TA Jessica Liu will hold extra pre-test office hoursin her usual zoom room. (password vchat), on Monday at 4-5:30PM and on Tuesday at 1-2:30PM.
- I will hold my regular office hours, plus an additional hour and a half, on Tuesday at 9:30-12 at Bahen 6178 and simultaneously at `http://drorbn.net/vchat`.
- Material: Everything up to and including Friday's material, chains and boundaries of chains, with greater emphasis on the material that was not included in Term Test 2 (meaning, starting with the proof of the COV formula, and then k-tensors and all that followed). The questions will be a mix of direct class material, questions from homework, and "fresh" questions. This is more similar to TT1 than to TT2 which was "all fresh".
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- To prepare: Do last years' 2021-257-TT3 and the TT3 "rejects" available below. But more important: make sure that you understand every single bit of class material so far!

The following questions were a part of a question pool for the 2020-21 MAT257 Term Test 3, but at the end, they were not included.

1. Prove that the Change of Variables (COV) theorem holds even without the assumption on the invertibility of g'.
2. It is common to identify $\mathbb{R}^3$ with the space of column vectors of length 3, and to identify $(\mathbb{R}^3)^*$ with the space of row vectors of length 3. With this in mind, find the dual basis to the basis $v_1 = \begin{pmatrix}1\\3\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\4\\0\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\0\\5\end{pmatrix}$ of $\mathbb{R}^3$.
3. Let V be a vector space, let $\phi\colon V \to V \times V$ be given by $\phi(v) = (v, v)$ and let $\psi\colon V \times V \to V \times V$ be given by $\psi(v, w) = (w, v)$. Let $B\colon V \times V \to \mathbb{R}$ be a bilinear function. Prove that $\phi^* B = 0$ iff $B + \psi^* B = 0$.
4. Prove that in S_k, for $k > 1$, there is an equal number of odd and even permutations.
5. Let $\sigma \in S_n$ be the permutation given by $\sigma i = i + 1$ for $i < n$ and $\sigma n = 1$. What is $\text{sign}(\sigma)$?
6. Let $\phi\colon \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$ be given by $\phi(x, y) = (x^2 - y^2, 2xy)$. Compute $\phi^*(du \wedge dv)$ and $\phi_*\xi$, where ξ is the tangent vector to $\mathbb{R}^2_{x,y}$ given by $\xi = ((0, 1), (1, 0))$.
7. Let $\xi = (p, v)$ be a tangent vector to $\mathbb{R}^n$. Prove that there exists a path $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ such that for every differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ we have that $D_\xi f = (f \circ \gamma)'(0)$.
8. Let $\omega = \dfrac{xdy - ydx}{x^2 + y^2} \in \Omega^1\left(\mathbb{R}^2_{x,y} \setminus \{0\}\right)$, and let $f\colon Q = (0, \infty)_r \times [0, 2\pi]_\theta \to \mathbb{R}^2$ be given by $f(r, \theta) = (r\cos\theta, r\sin\theta)$.
   (a) Compute $f^*(\omega)$.
   (b) Show that ω is closed.

(c) Show that $f^*(\omega)$ is exact on Q.

(d) Show that ω is not exact on $\mathbb{R}^2_{x,y} \setminus \{0\}$.

9. Explain in detail how the vector-field operator grad arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

10. Explain in detail how the vector-field operator div arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

11. (Not in this years' material!) Let $f\colon [0,1] \to \mathbb{R}$ be a differentiable function, and let $c\colon [0,1]^2_{u,v} \to \mathbb{R}^2_{x,y}$ be the 2-cube given by $c(u,v) = (u, f(u)v)$. Use Stokes' theorem and the form $\omega = -y dx$ to show that $\int_c dx \wedge dy = \int_0^1 f(x) dx$. Can you interpret this result geometrically?

12. It is common to identify $\mathbb{R}^n$ with the space of column vectors of length n and to identify $(\mathbb{R}^n)^*$ with the space of row vectors of length n. Suppose $\phi \in (\mathbb{R}^m)^*$ is a row vector, and suppose $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation presented relative to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ by the matrix $A \in M_{m\times n}(\mathbb{R})$. Compute the row vector $L^*\phi$ (the pullback of ϕ via L).

Please watch this page for changes — I may add to it later. Last modified: Wednesday 2nd March, 2022, 13:37