Term Test 3

(Author's name here)

March 8, 2022

Q1 20

Solve all 5 problems. Write your solutions only where indicated, or write explicitly, "continued on page X".
Neatness counts! Language counts!

Problem 1. In this question, we say that a function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves one coordinate if there is some $k \in \underline{n}$ such that $g_k(x_1, \ldots, x_n) = x_k$, where g_k is the kth component function of g. Prove that if $n \geq 2$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f'(0)$ is invertible, then on a neighborhood of 0 we can write $f = g_1 \circ g_2$ where g_1 and g_2 are continuously differentiable functions $\mathbb{R}^n \to \mathbb{R}^n$ and each preserves one coordinate.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. You may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space left here for solutions.

Your solution of Problem 1.

We will assume that for at least one i, $\frac{\partial f_i}{\partial x_i} \neq 0$.

Then, let us pick one $i \in \{1, \ldots, n\}$ such that $\frac{\partial f_i}{\partial x_i} \neq 0$.

Then, let us define $g_2 : \mathbb{R}^n \to \mathbb{R}^n$ by:

$g_2(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) := (x_1, \ldots, x_{i-1}, f_i(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n)$.

Then, we can find $\det g_2'(x_1, \ldots, x_n)$ as follows:

$$\det g_2'(x) = \det \begin{pmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \cdots & \frac{\partial x_1}{\partial x_i} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_2}{\partial x_i} & \cdots & \frac{\partial x_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{\partial f_i(x)}{\partial x_1} & \frac{\partial f_i(x)}{\partial x_2} & \cdots & \frac{\partial f_i(x)}{\partial x_i} & \cdots & \frac{\partial f_i(x)}{\partial x_n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{\partial x_n}{\partial x_1} & \frac{\partial x_n}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_i} & \cdots & \frac{\partial x_n}{\partial x_n} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \frac{\partial f_i(x)}{\partial x_1} & \frac{\partial f_i(x)}{\partial x_2} & \cdots & \frac{\partial f_i(x)}{\partial x_i} & \cdots & \frac{\partial f_i(x)}{\partial x_n} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

(Continued on page 4)

Your solution of Problem 1, continued.

For all $1 \leq j, k \leq n$, let $a_{j,k}$ denote the entry in the j^{th} row and k^{th} column of $g_2'(x)$. Then, $\det g_2'(x)$, by definition, equa[ls] $\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{i,\sigma(i)} \cdots a_{n,\sigma(n)}$. Note that, for all $1 \leq j \leq n$ such that $j \neq i$, the only nonzero element in the j^{th} row of $g_2'(x)$ is $a_{j,j} = 1$, so we need $\sigma(j) = j$ for a term in the summation to be nonzero. Since we need this for all $j \neq i$, the only nonzero term in the summation occurs when σ is the identity permutation, so:

$$\begin{aligned} \det g_2'(x) &= 1 \cdot a_{1,1} \cdot a_{2,2} \cdot \cdots \cdot a_{i,i} \cdot \cdots \cdot a_{n,n} \\ &= 1 \cdot 1 \cdot 1 \cdot \cdots \cdot \frac{\partial f_i(x)}{\partial x_i} \cdot \cdots \cdot 1 \\ &= \frac{\partial f_i(x)}{\partial x_i} \end{aligned}$$

Explicitly, this proves that $\det g_2'(0) \neq 0$.

In particular, $\det g_2'(0) = \left.\frac{\partial f_i}{\partial x_i}\right|_{x=0}$, which we assumed to be nonzero. Moreover, g_2' is continuously differentiable since f_i is continuously differentiable. Thus, by the Inverse Function Theorem we obtain open neighbourhoods A, B around 0 and $g_2(0)$, respectively, such that $g_2 : A \to B$ has a continuously differentiable inverse $g_2^{-1} : B \to A$. Then, let us define $g_1 : B \to \mathbb{R}^n$ by $g_1 := f|_A \circ g_2^{-1}$ so that $f|_A = f|_A \circ g_2^{-1} \circ g_2 = g_1 \circ g_2$, as desired. Moreover, we have the diagram $(x_1, \ldots, x_i, \ldots, x_n) \xmapsto{g_2} (x_1, \ldots, f_i(x), \ldots, x_n) \xmapsto{g_1} (f_1(x), \ldots, f_i(x), \ldots, f_n(x))$, which shows that g_1 preserves the i^{th} coordinate and g_2 preserves all other coordinates, as required. Finally, g_1 is continuously differentiable as a composition of continuously differentiable functions, as required.

Q2 20

Solve all 5 problems. Write your solutions only where indicated, or write explicitly, "continued on page X".
Neatness counts! Language counts!

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Problem 2. Let $\phi\colon \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$ be given by $\phi(x,y) = (e^x \cos y, e^x \sin y)$. Compute $\phi^*(du \wedge dv)$ and $\phi_*\xi$, where ξ is the tangent vector to $\mathbb{R}^2_{x,y}$ given by $\xi = \left(\begin{pmatrix}0\\ \pi/2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$.

Your solution of Problem 2.

If $\phi(x,y)=(e^x\cos y, e^x \sin y)$, then $u=e^x\cos y$, $v=e^x\sin y$,

$$\begin{aligned}
\phi^*(du\wedge dv) &= d(e^x\cos y)\wedge d(e^x\sin y)\\
&= \left(dx\wedge\frac{\partial(e^x\cos y)}{\partial x} + dy\wedge\frac{\partial(e^x\cos y)}{\partial y}\right)\\
&\quad\wedge\left(dx\wedge\frac{\partial(e^x\sin y)}{\partial x} + dy\wedge\frac{\partial(e^x\sin y)}{\partial y}\right)\\
&= (e^x\cos y\,dx - e^x\sin y\,dy)\wedge(e^x\sin y\,dx + e^x\cos y\,dy)\\
&= (e^x\cos y\cdot e^x\sin y)\,dx\wedge dx + (e^x\cos y\cdot e^x\cos y)\,dx\wedge dy\\
&\quad -(e^x\sin y\cdot e^x\sin y)\,dy\wedge dx - (e^x\sin y\cdot e^x\cos y)\,dy\wedge dy\\
&= 0 + e^{2x}\cos^2 y\,dx\wedge dy + e^{2x}\sin^2 y\,dx\wedge dy + 0\\
&= e^{2x}(\cos^2 y + \sin^2 y)\,dx\wedge dy\\
&= \boxed{e^{2x}\,dx\wedge dy}
\end{aligned}$$

good 10

$$\begin{aligned}
\phi_*\xi &= \phi_*\left(\begin{pmatrix}0\\ \frac{\pi}{2}\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)\\
&= \left(\phi\begin{pmatrix}0\\ \frac{\pi}{2}\end{pmatrix}, \phi'\begin{pmatrix}0\\ \frac{\pi}{2}\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)
\end{aligned}$$

(Continued on Pg. 6)

$$\phi'\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_1}{\partial y}\\ \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_2}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial(e^x\cos y)}{\partial x} & \frac{\partial(e^x\cos y)}{\partial y}\\ \frac{\partial(e^x\sin y)}{\partial x} & \frac{\partial(e^x\sin y)}{\partial y}\end{pmatrix}$$

$$=\begin{pmatrix}e^x\cos y & -e^x\sin y\\ e^x\sin y & e^x\cos y\end{pmatrix}$$

$$\phi'\begin{pmatrix}0\\\frac{\pi}{2}\end{pmatrix}=\begin{pmatrix}e^0\cos\frac{\pi}{2} & -e^0\sin\frac{\pi}{2}\\ e^0\sin\frac{\pi}{2} & e^0\cos\frac{\pi}{2}\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Finally, $\phi_* \xi = \left(\phi\begin{pmatrix}0\\\frac{\pi}{2}\end{pmatrix}, \phi'\begin{pmatrix}0\\\frac{\pi}{2}\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)$

$$=\left(\begin{pmatrix}e^0\cos\frac{\pi}{2}\\ e^0\sin\frac{\pi}{2}\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$\boxed{\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right)}$$

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Q3 20

olve all 5 problems. Write your solutions only where indicated, or rite explicitly, "continued on page X".
Jeatness counts! Language counts!

Problem 3. Let V be a vector space, let $\phi\colon V \to V \times V$ be given by $\phi(v) = (v, v)$ and let $\psi\colon V \times V \to V \times V$ be given by $\psi(v, w) = (w, v)$. Let $B\colon V \times V \to \mathbb{R}$ be a bilinear function. Prove that $\phi^* B = 0$ iff $B + \psi^* B = 0$.

Your solution of Problem 3.

"$\Rightarrow$" direction: Suppose $\phi^* B = 0$.

Then, for all $(v_1, v_2) \in V \times V$, we have:

$(\phi^* B)(v_1 + v_2) = 0$

$B(\phi(v_1 + v_2)) = 0$

$B(v_1 + v_2, v_1 + v_2) = 0$

$B(v_1, v_1) + B(v_1, v_2) + B(v_2, v_1) + B(v_2, v_2) = 0$ (Bilinear)

$B(\phi(v_1)) + B(v_1, v_2) + B(\psi(v_1, v_2)) + B(\phi(v_2)) = 0$

$(\phi^* B)(v_1) + B(v_1, v_2) + (\psi^* B)(v_1, v_2) + (\phi^* B)(v_2) = 0$

$0 + (B + \psi^* B)(v_1, v_2) + 0 = 0$

Thus, $(B + \psi^* B)(v_1, v_2) = 0$ for all $(v_1, v_2) \in V \times V$, so $B + \psi^* B = 0$, as required for "$\Rightarrow$" direction.

"$\Leftarrow$" direction: Suppose $B + \psi^* B = 0$.

(Continued on Pg. 8)

Then, for all $v \in V$, we have:

$(B + \psi^* B)(v, v) = 0$

$B(v, v) + (\psi^* B)(v, v) = 0$

$B(v, v) + B(\psi^*(v, v)) = 0$

$B(v, v) + B(v, v) = 0$

$2B(v, v) = 0$

$B(v, v) = 0$

$B(\phi(v)) = 0$

$(\phi^* B)(v) = 0$

Thus, $(\phi^* B)(v) = 0$ for all $v \in V$, so $\phi^* B = 0$, as require for "$\Leftarrow$" direction.

We proved both directions, so $\phi^* B = 0$ if and only if $B + \psi^* B = 0$, as required.

good **20**

Q4 20

...e all 5 problems. Write your solutions only where indicated, or ...e explicitly, "continued on page X".
...atness counts! Language counts!

...oblem 4. Explain in detail how the vector field op...ator curl arises as an instance of the exterior derivative ...erator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

...eminder. $\operatorname{curl}\begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix} = \begin{pmatrix}\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\\ \frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}\\ \frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\end{pmatrix}.$

...our solution of Problem 4.

For any vector field $F = F_1e_1+F_2e_2+F_3e_3$ we can associate forms:
$\omega_1^F \in \Omega^1(\mathbb{R}^3)$ defined by $\omega_1^F = F_1dx_1+F_2dx_2+F_3dx_3$
$\omega_2^F \in \Omega^2(\mathbb{R}^3)$ defined by $\omega_2^F = F_1dx_2\wedge dx_3 + F_2dx_3\wedge dx_1 + F_3dx_1\wedge dx_2$.
Then, we will show $\omega_2^{\operatorname{curl} F} = d(\omega_1^F)$.

$$LS = \omega_2^{\operatorname{curl} F}$$
$$= \omega_2\left[\left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\right)e_1+\left(\frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}\right)e_2+\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)e_3\right]$$
$$= \left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\right)dx_2\wedge dx_3+\left(\frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}\right)dx_3\wedge dx_1+\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)dx_1\wedge dx_2$$

$$RS = d(\omega_1^F)$$
$$= d(F_1dx_1+F_2dx_2+F_3dx_3)$$
$$= dx_1\wedge \frac{\partial}{\partial x_1}(F_1dx_1+F_2dx_2+F_3dx_3)$$
$$+dx_2\wedge\frac{\partial}{\partial x_2}(F_1dx_1+F_2dx_2+F_3dx_3)$$
$$+dx_3\wedge\frac{\partial}{\partial x_3}(F_1dx_1+F_2dx_2+F_3dx_3)$$

good 20

(Continued on page 10)

$$d(\omega_1^F) = \left(\cancelto{0}{\frac{\partial F_1}{\partial x_1} dx_1 \wedge dx_1} + \frac{\partial F_2}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial F_3}{\partial x_1} dx_1 \wedge dx_3\right)$$
$$+ \left(\frac{\partial F_1}{\partial x_2} dx_2 \wedge dx_1 + \cancelto{0}{\frac{\partial F_2}{\partial x_2} dx_2 \wedge dx_2} + \frac{\partial F_3}{\partial x_2} dx_2 \wedge dx_3\right)$$
$$+ \left(\frac{\partial F_1}{\partial x_3} dx_3 \wedge dx_1 + \frac{\partial F_2}{\partial x_3} dx_3 \wedge dx_2 + \cancelto{0}{\frac{\partial F_3}{\partial x_3} dx_3 \wedge dx_3}\right)$$
$$= \left(\frac{\partial F_2}{\partial x_1} dx_1 \wedge dx_2 - \frac{\partial F_3}{\partial x_1} dx_3 \wedge dx_1\right)$$
$$+ \left(-\frac{\partial F_1}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial F_3}{\partial x_2} dx_2 \wedge dx_3\right)$$
$$+ \left(\frac{\partial F_1}{\partial x_3} dx_3 \wedge dx_1 - \frac{\partial F_2}{\partial x_3} dx_2 \wedge dx_3\right)$$
$$= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1 \wedge dx_2$$
$$= LS$$

Thus, $LS = RS$, so $\omega_2^{\text{curl} F} = d(\omega_1^F)$. In other words, to find curl F, one can compute $d(\omega_1^F)$, then the e_1-coefficient of curl F equals the $(dx_2 \wedge dx_3)$-coefficient of $d(\omega_1^F)$, the e_2-coefficient of curl F equals the $(dx_3 \wedge dx_1)$-coefficient of $d(\omega_1^F)$ (or negative of the $(dx_1 \wedge dx_3)$-coefficient), and the e_3-coefficient of curl F equals the $(dx_1 \wedge dx_2)$-coefficient of $d(\omega_1^F)$. This shows how curl arises from $d: \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$, as required.

Q5 20

roblem 5. If $L\colon V \to W$ is an invertible linear transrmation between oriented vector spaces (vector spaces quipped with an orientation), we say that L is *orientaon preserving* if it pushes the orientation of V forward to the orientation of W (or equivalently, if it pulls the rientation of W back to the orientation of V). Otherwise, L is called *orientation reversing*. Decide for each of he cases below, if L_i is orientation preserving or reversing. In this question $\mathbb{R}^n$ always comes equipped with its tandard orientation $(e_1, e_2, \dots, e_n)$.

1. $L_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ via $(x, y) \mapsto (-x, y)$.
2. $L_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ via $(x, y) \mapsto (y, x)$.
3. $L_3\colon \mathbb{R}^2 \to \mathbb{R}^2$, the counterclockwise rotation by $2\pi/7$.
4. $L_4\colon \mathbb{R}^2 \to \mathbb{R}^2$, the clockwise rotation by $2\pi/7$.
5. $L_5\colon \mathbb{R}^2 \to \mathbb{R}^2$, the complex conjugation map $z \mapsto \bar{z}$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ via $(x, y) \leftrightarrow x + iy$.
6. $L_6\colon \mathbb{R}^3 \to \mathbb{R}^3$ via $(x, y, z) \mapsto (y, z, x)$.
7. $L_7\colon \mathbb{R}^n \to \mathbb{R}^n$ via $v \mapsto -v$.
8. $L_8\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ via $(u, v) \mapsto (v, u)$, where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

This is because L_i pushes basis $(e_1, \dots, e_n)$ to basis $(M_i e_1, \dots, M_i e_n)$, and these basis have same orientation if and only if $\det M_i > 0$.

Tip. The answers for L_7 and for L_8 may depend on n and m.

Your solution of Problem 5.

L_i is orientation preserving if and only if $\det M_i > 0$, where M_i is matrix representing L_i in the standard basis $(e_1, \dots, e_n)$.

1. $M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\det M_1 = (-1)(1) - (0)(0) = -1 < 0$, $\boxed{\text{reversing}}$

2. $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det M_2 = (0)(0) - (1)(1) = -1 < 0$, $\boxed{\text{reversing}}$

(Standard rotation matrix by $\frac{2\pi}{7}$)

3. $M_3 = \begin{pmatrix} \cos\frac{2\pi}{7} & -\sin\frac{2\pi}{7} \\ \sin\frac{2\pi}{7} & \cos\frac{2\pi}{7} \end{pmatrix}$, $\det M_3 = \cos^2\frac{2\pi}{7} + \sin^2\frac{2\pi}{7} = 1 > 0$, $\boxed{\text{Preserving}}$

(Continued on Pg. 12)

(Standard rotation matrix by $-\frac{2\pi}{7}$)

4. $M_4 = \begin{pmatrix} \cos\frac{2\pi}{7} & \sin\frac{2\pi}{7} \\ -\sin\frac{2\pi}{7} & \cos\frac{2\pi}{7} \end{pmatrix}$, $\det M_4 = \cos^2\frac{2\pi}{7} + \sin^2\frac{2\pi}{7} >$

$\boxed{\text{preserving}}$

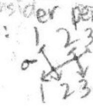
5. $L_5(x,y) = L_5(x+yi) = x-yi = (x,-y)$

$M_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\det M_5 = (1)(-1)-(0)(0) = -1 < 0$, $\boxed{\text{reversing}}$

6. $M_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\det M_6 = (1)(1)(1) + 0 + 0 = 1 \geq 0$,

$\boxed{\text{preserving}}$

Consider permutation $\sigma \in S_3$ define by:



Then, we count 2 crossings, so $\det M_6 = (-1)^\sigma = (-1)^2 = 1$.

7. $M_7 = \underbrace{\begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}}_{n} \Big\} n$, $\det M_7 = (-1)^n$

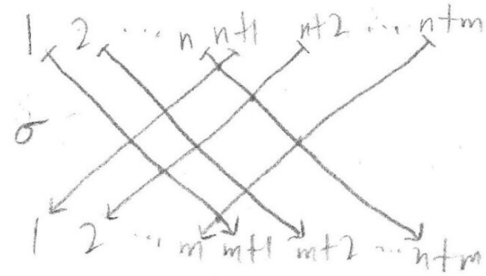
If n is even, $\det M_7 = (-1)^n > 0$, $\boxed{\text{preserving}}$

If n is odd, $\det M_7 = (-1)^n < 0$, $\boxed{\text{reversing}}$

8. M_8 is the block matrix $\begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$, where I_k denotes the identity matrix of size k. Then, consider the permutation $\sigma \in S_{n+m}$ defined by:

(Continued on Pg. 13)

·8. (Continued)



Then, we count nm crossings, since all n arrows starting from $\{1,\dots,n\}$ cross all m arrows starting from $\{n+1,\dots,n+m\}$.

Thus, $\det M_8 = (-1)^{\sigma} = (-1)^{nm}$.

If n and m are odd, then $\det M_8 = -1 < 0$, $\boxed{\text{reversing}}$.

If n or m is even, then $\det M_8 = 1 > 0$, $\boxed{\text{preserving}}$.

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. This was probably the hardest question on the test. One way to approach this problem is to recall a similar proof we did in lecture: We proved that if $f'(0)$ is invertible, then f can be written as a composition of coordinate swaps and layer preserving maps. A key step in that proof was that we defined $\alpha_k(x) := (x_1, x_2, \dots, x_{n-1}, f_k(x))$, and we wanted to define $\beta_k := f \circ \alpha_k^{-1}$ so that $f = \beta_k \circ \alpha_k$. To do this, we needed to use the Inverse Function Theorem on α_k, which required $\det \alpha_k'(0) \neq 0$. Since $\alpha_k'(x) = \begin{pmatrix} I_{n-1} & 0 \\ * & \frac{\partial f_k(x)}{\partial x_n} \end{pmatrix}$, we obtained $\det \alpha_k'(x) = \frac{\partial f_k(x)}{\partial x_n}$, so we needed $\frac{\partial f_k(x)}{\partial x_n}$ to be nonzero for some k. Since $f'(0)$ is invertible, we knew that some entry in the n^{th} column of $f'(0)$ was nonzero, so we obtained $\frac{\partial f_k(x)}{\partial x_n} \neq 0$ for some k, as required. (Then, we composed α_k and β_k with coordinate swaps to form layer preserving maps.)
   The solution for this test question follows analogously. First, our goal is to obtain the following diagram:

   $$(x_1, \dots, x_i, \dots, x_n) \xmapsto{g_2} (x_1, \dots, f_i(x), \dots, x_n) \xmapsto{g_1} (f_1(x), \dots, f_i(x), \dots, f_n(x))$$

   Then, we define $g_2(x) := (x_1, \dots, f_i(x), \dots, x_n)$. Following the discussion above, we compute $\det g_2'(x) = \frac{\partial f_i(x)}{\partial x_i}$, and assuming that $\frac{\partial f_i(x)}{\partial x_i} \neq 0$ for some i, we proceed to use the Inverse Function Theorem. This allows us to define $g_1 := f \circ g_2^{-1}$, completing the proof.
   If you're curious, a possible counterexample to the original statement (i.e., without $\frac{\partial f_i}{\partial x_i} \neq 0$) is $f(x, y) = (y, x)$. In this example, we have $\det f'(0) = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$, so $f'(0)$ is invertible. However, if we try to write $f = g_1 \circ g_2$, whire g_1 and g_2 each preserve a coordinate, then we can only preserve one coordinate at a time. This means that we must have the following diagram:

   $$(x, y) \xmapsto{g_2} (x, x) \xmapsto{g_1} (y, x)$$

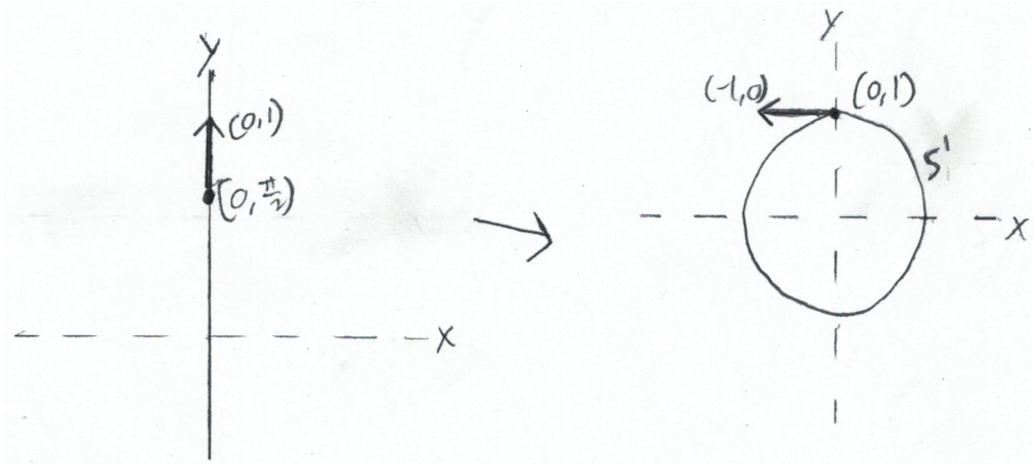
   or the following diagram:

   $$(x, y) \xmapsto{g_2} (y, y) \xmapsto{g_1} (y, x)$$

   Both situations are invalid because g_2 is not invertible: Since $g_2(x, y)$ only contains information about one coordinate, g_1 does not know how to map $g_2(x, y)$ to (y, x). (By the way, this is not a contradiction for the fixed problem statement because $\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = 0$.)

2. (Note: This question was very similar to Question 6 from last year's Test 3 rejects.)
   The question itself was mostly computational, I will proceed by providing a visualization of $\phi_* \xi$. First, $\xi = ((0, \frac{\pi}{2}), (0, 1))$ is a vector on the y-axis that points further in the positive y-direction. Next, let us examine how ϕ "pushes" the y-axis. Plugging in $x = 0$, we obtain $\phi(0, y) = (\cos y, \sin y)$. As a result, ϕ "pushes" the y-axis to the unit circle in $\mathbb{R}^2$, and ϕ "pushes" the positive y-direction to the counterclockwise direction on the circle. Then, it would make sense if ξ gets "pushed" to a vector starting at the point $(\cos \frac{\pi}{2}, \sin \frac{\pi}{2}) = (0, 1)$ and pointing counterclockwise. Indeed, we can compute that $\phi_* \xi = ((0, \frac{\pi}{2}), (0, 1)) = ((0, 1), (-1, 0))$,

where $(-1, 0)$ points counterclockwise, as desired. Here is a diagram of this visualization:



3. (Note: This question also appeared as Question 3 from last year's Test 3 rejects.)
First, let us try to understand what "$\phi^* B = 0$" and "$B + \psi^* B = 0$" actually mean. If we have $\phi^* B = 0$, then it means that $0 = (\phi^* B)(v) = B(\phi(v)) = B(v, v)$ for all $v \in V$. In other words, "$\phi^* B = 0$" is equivalent to "B kills repetitions". Moreover, if $B + \psi^* B = 0$, then it means that $0 = (B + \psi^* B)(u, v) = B(u, v) + B(\psi(u, v)) = B(u, v) + B(v, u)$, so $B(u, v) = -B(v, u)$ for all $u, v \in V$. In other words, "$B + \psi^* B = 0$" is equivalent to "B is alternating". Then, the question is really asking us to prove that B kills repetitions if and only if B is alternating. In fact, we also proved this statement in lecture, so we can re-apply the proof for this test question.

4. (Note: This question also appeared as Question 3 on last year's Test 3, and it is also strongly related to Assignment 15 Question 1).
First, since curl only operates on vector fields in $\mathbb{R}^3$, it makes sense that we should consider d on forms in $\mathbb{R}^3$. Next, since F and $\operatorname{curl} F$ both have three coordinates/components, it makes sense that they correspond to forms in some 3-dimensional space $\Omega^k(\mathbb{R}^3)$. Since we know that $\dim \Omega^0(\mathbb{R}^3) = \dim \Omega^3(\mathbb{R}^3) = 1$ and $\dim \Omega^1(\mathbb{R}^3) = \dim \Omega^2(\mathbb{R}^3) = 3$, this tells us to associate F with 1-forms and $\operatorname{curl} F$ with 2-forms. We associate F with $\omega_1^F = F_1 dx_1 + F_2 dx_2 + F_3 dx_3$ because that is a simple and natural choice. After computing $d\omega_1^F$, we compare that with $\operatorname{curl} F$, and that tells us how to associate $\operatorname{curl} F$ with a corresponding 2-form. After we compile this "scratch work" into a written solution, we are done.

5. This question also appeared as Question 2 on Assignment 13. As with Assignment 13, the key idea was to treat each L_i as a change of basis matrix, then to compute whether its determinant is positive or negative.