Term Test 2

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January 18, 2022

1. Let any $\varepsilon > 0$ be given. Then, pick $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. Next, for all integers $1 \le i \le k$, define the closed rectangle $C_i := [\frac{i-1}{k}, \frac{i}{k}] \times [\frac{i-1}{k}, \frac{i}{k}]$.
Then, for all $(x,x) \in \{(t,t) : t \in \mathbb{Q} \cap [0,1]\}$, we have $x \in [0,1]$, so there exists some integer $1 \le i \le k$ such that $\frac{i-1}{k} \le x \le \frac{i}{k}$. (This is because this inequality is equivalent to $i-1 \le kx \le i$, and kx must be between two consecutive integers.) From $\frac{i-1}{k} \le x \le \frac{i}{k}$, we get $(x,x) \in [\frac{i-1}{k}, \frac{i}{k}] \times [\frac{i-1}{k}, \frac{i}{k}] = C_i$.
Thus, for all $(x,x) \in \{(t,t) : t \in \mathbb{Q} \cap [0,1]\}$, we found $1 \le i \le k$ such that $(x,x) \in C_i$, so the rectangles $C_1, \dots, C_k$ cover $\{(t,t) : t \in \mathbb{Q} \cap [0,1]\}$, as required.
Next, the rectangles have a total volume of:
$\sum_{i=1}^{k} C_i = \sum_{i=1}^{k} (\frac{i}{k} - \frac{i-1}{k})^2 = \sum_{i=1}^{k} (\frac{1}{k})^2 = \frac{1}{k} < \varepsilon.$
Thus, for all $\varepsilon > 0$, we found finitely many rectangles of total volume less than ε which cover $\{(t,t) : t \in \mathbb{Q} \cap [0,1]\}$, so this set has content 0, as required.

2. Let U be any open set in $\mathbb{R}^n$, and let U^c be its complement. Then, for all $i \in \mathbb{N}$, define $C_k \subseteq \mathbb{R}^n$ by: ~~$C_k := \{x \in \mathbb{R}^n \mid |x| <$~~

$C_k := \{x \in \mathbb{R}^n \mid |x| \leq k, \text{ and } |x-y| \geq \frac{1}{k} \text{ for all } y \in U^c\}$.

First, C_k is bounded since $|x| \leq k$ for all $x \in C_k$.

Next, C_k is the intersection of the following sets:

$C_k = \{x \in \mathbb{R}^n \mid |x| \leq k\} \cap \bigcap_{y \in U^c} \{x \in \mathbb{R}^n \mid |x-y| \geq \frac{1}{k}\}$.

The set $\{x \in \mathbb{R}^n \mid |x| \leq k\}$ is closed as a closed ball. For all $y \in U^c$, $\{x \in \mathbb{R}^n \mid |x-y| \geq \frac{1}{k}\}$ is closed as the complement of the open ball $B_{\frac{1}{k}}(y)$.

Thus, each C_k is closed as an intersection of closed sets. Since C_k is closed and bounded, C_k is compact for all $k \in \mathbb{N}$.

Next, for all $y \in U^c$, and all $k \in \mathbb{N}$ we have $|y-y| = 0 < \frac{1}{k}$, so $y \notin C_k$. In other words, we have for all k that $U^c \subseteq C_k^c$, so $C_k \subseteq U$.

Thus, $\bigcup_{k=1}^{\infty} C_k \subseteq U$.

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2. (Continued)

Next, for all $x \in U$, since U is open, there exists some radius $r>0$ such that $B_r(x) \subseteq U$. In other words, $|x-y| \geq r$ for all $y \in U^c$. Then, there exists $k \in \mathbb{N}$ such that $\frac{1}{k} \leq r$ i.e., $k \geq \frac{1}{r}$ and such that $k \geq |x|$. This gives us $|x| \leq k$, and $|y-x| \geq r \geq \frac{1}{k}$ for all $y \in U^c$, so $x \in C_k$. Thus, for all $x \in U$, there exists $k \in \mathbb{N}$ such that $x \in C_k$, so $U \subseteq \bigcup_{k=1}^{\infty} C_k$. Overall, we proved $U \subseteq \bigcup_{k=1}^{\infty} C_k$ and $\bigcup_{k=1}^{\infty} C_k \subseteq U$, so $U = \bigcup_{k=1}^{\infty} C_k$, which is a union of countably many compact sets, as required.

~~3. For all $(x,y) \in \mathbb{R}^2$, let us define $f(x,y) = \frac{1}{1+x^2+y^2}$.~~
~~Then, we wish to find~~

3. Let us define $f: \mathbb{R}^2_{(x,y)} \to \mathbb{R}$ by $f(x,y) = \frac{1}{1+x^2+y^2}$.
Let us also define the set $B \subseteq \mathbb{R}^2_{(x,y)}$ by
$B := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq R^2\}$. Then, we want to find $\int_B f$.

First, let us define $g: \mathbb{R}^2_{(r,\theta)} \to \mathbb{R}^2_{(x,y)}$ by
$g(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$. Then, we will select bounds on r, θ such that $g(r,\theta) \in B$.
First, we need $x^2+y^2 \leq R^2$, so:

$$(r\cos\theta)^2 + (r\sin\theta)^2 \leq R^2$$
$$r^2(\cos^2\theta + \sin^2\theta) \leq R^2$$
$$r^2 \leq R^2$$
$$r \leq R$$

Thus, let us pick the bounds $0 < r < R$. Since θ is an angle, let us also pick the bounds $0 < \theta < 2\pi$.
Then, let us define the rectangle $A \subseteq \mathbb{R}^2_{(r,\theta)}$ by
$A := (0,R) \times (0, 2\pi)$. Under the bounds picked above, $g(A)$ is approximately B, and $g|_A$ is 1-1.

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3. (Continued)

Next, let us compute $\det g'(r,\theta)$ for all $(r,\theta)\in A$:

$$\det g' = \det\begin{pmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{pmatrix}$$

$$= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= \cos\theta\cdot r\cos\theta - (-r\sin\theta)\cdot\sin\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Since $r>0$ for all $(r,\theta)\in A$, we get $\det g'\neq 0$, as required. Thus, we can apply Change of Variables:

$$\int_B f = \int_{g(A)} f$$

$$= \int_A (f\circ g)\,|\det g'|$$

$$= \int_A f(r\cos\theta, r\sin\theta)\cdot r$$

$$= \int_A \frac{1}{1+(r\cos\theta)^2+(r\sin\theta)^2}\cdot r$$

$$= \int_A \frac{r}{1+r^2}$$

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2. (Continued)

$$\int_B f = \int_A \frac{r}{1+r^2}$$
$$= \int_0^{2\pi}\left(\int_0^R \frac{r}{1+r^2}\,dr\right)d\theta \quad \text{(Fubini)}$$

We will use the MAT157 u-substitution $u = 1+r^2$, with $\frac{du}{dr} = 2r$, so $du = 2r\,dr$:

$$\int_B f = \int_0^{2\pi}\left(\int_0^R \frac{1}{2(1+r^2)}\cdot 2r\,dr\right)d\theta$$
$$= \int_0^{2\pi}\left(\int_{r=0}^{r=R} \frac{1}{2u}\,du\right)d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}\ln|u|\Big|_{r=0}^{r=R}\,d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}\ln|1+r^2|\Big|_{r=0}^{r=R}\,d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}\left(\ln(1+R^2) - \ln(1)\right)d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}\ln(1+R^2)\,d\theta$$
$$= \frac{1}{2}\ln(1+R^2)\,\theta\Big|_0^{2\pi}$$
$$= \boxed{\pi\ln(1+R^2)}$$

4. Let $\mathcal{U} := \{B_1(x)\}_{x \in \mathbb{R}^n}$ be the open cover of $\mathbb{R}^n$ using open balls of radius 1 centred at every point in $\mathbb{R}^n$. Also, let $\Phi = \{\varphi_i\}_{i \in \mathbb{R}^n}$ be a partition of unity of $\mathbb{R}^n$ subordinate to $\mathcal{U}$. Then, to show that f is integrable on $\mathbb{R}^n$, it suffices to show that f is $(\mathcal{U}, \Phi)$-integrable, meaning that $\sum_{i=1}^{\infty} \int \varphi_i |f|$ converges.

For all $k \in \mathbb{N}$, consider the finite sum $\sum_{i=1}^{k} \int \varphi_i |f|$. First, since Φ is subordinate to $\mathcal{U}$, we have for all $1 \leq i \leq k$ that $\operatorname{supp} \varphi_i \subseteq B_1(x_i)$ for some $x_i \in \mathbb{R}^n$. Since each ball $B_1(x_i)$ is bounded, each support $\operatorname{supp} \varphi_i$ is also bounded, so the finite union $\bigcup_{i=1}^{k} \operatorname{supp} \varphi_i$ is also bounded. Then, we can pick a rectangle $R \subseteq \mathbb{R}^n$ such that $\bigcup_{i=1}^{k} \operatorname{supp} \varphi_i \subseteq R$. In other words, for all $1 \leq i \leq k$, we have that φ_i is 0 outside R.

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4. (Continued)

Thus, when we consider $\sum_{i=1}^{k}\int \varphi_i|f|$ we only need to integrate on R, so:

$$\sum_{i=1}^{k}\int \varphi_i|f| = \sum_{i=1}^{k}\int_R \varphi_i|f|$$

$$= \int_R \sum_{i=1}^{k} \varphi_i|f|$$ (since $\int_R f_1 + \int_R f_2 = \int_R (f_1+f_2)$, and we have a finite sum)

$$\leq \int_R \sum_{i=1}^{\infty} \varphi_i |f|$$

$$= \int_R |f|$$ (Since Φ has "sum = 1" property: $\sum_{i=1}^{\infty} \varphi_i(x) = 1$)

$$= \int_R f$$ ($f = |f|$ since f is nonnegative)

$$\leq M$$ (Given)

Thus, $\sum_{i=1}^{k}\int \varphi_i|f| \leq M$ for all $k \in \mathbb{N}$, so the infinite series $\sum_{i=1}^{\infty}\int \varphi_i|f|$ converges (and is at most M). We conclude that f ... NT-integrable, as ...

just because this is bounded doesn't mean it converges yet. you also need monotone

5. First, h is a linear map represented by the matrix $M_h = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Since $\det M_h = (0)(1)-(1)(1) = -1$, we find that h is bijective. Moreover, we can use a bookkeeping matrix to find M_h^{-1}:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \text{(Swap the rows)}$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \text{(Subtract Row 2 from Row 1)}$$

Thus, $M_h^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. This matrix represents the inverse map h^{-1}, so $h^{-1}(x,y) = (-x+y, x)$.

Next, let B be any Jordan-measurable set in $\mathbb{R}^2$. Then, we need to show that $h^{-1}(B)$ is also Jordan-measurable.

First, B is bounded, so B is contained in some rectangle $[a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$.

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5. (Continued)

Then, for all $(x,y) \in B \subseteq (a_1, b_1) \times (a_2, b_2)$, we have $h_1^{-1}(x,y) = -x+y < -a_1+b_2$, as well as $h_1^{-1}(x,y) > -b_1+a_2$. Thus, $h_1^{-1}(x,y) \in (-a_1+b_2, -b_1+a_2)$. Similarly, $h_2^{-1}(x,y) = x \in (a_1, b_1)$. Overall, this shows that $h^{-1}(B)$ is bounded.

Next, consider any (x',y') inside $\partial(h^{-1}(B))$, the boundary of $h^{-1}(B)$. Then, we will show that $(x',y') \in h^{-1}(\partial B)$. Suppose this is not the case. Then, since h^{-1} is bijective, we must have $(x',y') \in h^{-1}(\text{int } B)$ (interior) or $(x',y') \in h^{-1}(\text{ext } B)$ (exterior). If $(x',y') \in h^{-1}(\text{int } B)$, then $h^{-1}(\text{int } B)$ is open since $\text{int } \partial B$ is open and h is continuous, so $h^{-1}(\text{int } B)$ is an open neighbourhood around (x',y') contained in $h^{-1}(B)$. Similarly, if $(x',y') \in h^{-1}(\text{ext } B)$, then $h^{-1}(\text{ext } B)$ is an open neighbourhood around (x',y') contained outside $h^{-1}(B)$. Either way, $(x',y') \notin \partial(h^{-1}(B))$, a contradiction. Thus, by contradiction, $(x',y') \in h^{-1}(\partial B)$ for all $(x',y') \in \partial(h^{-1}(B))$, so $\partial(h^{-1}(B)) \subseteq h^{-1}(\partial B)$.

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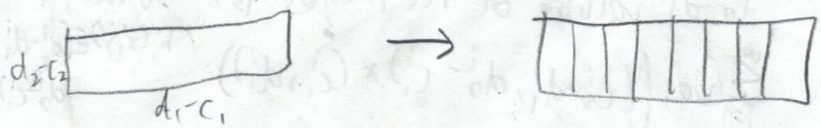
5. (Continued)

Then, to show that $\partial(h^{-1}(B))$ has measure 0, it suffices to show that $h^{-1}(\partial B)$ has measure 0.

Let any $\varepsilon > 0$ be given. Then, since ∂B has measure 0, cover ∂B with countably many open rectangles of total volume less than $\frac{\varepsilon}{2}$. Also, we can require that these rectangles have a shorter x-length than y-length; in other words, if a rectangle is of the form $(c_1, d_1) \times (c_2, d_2)$, we can require $d_1 - c_1 < d_2 - c_2$. If a rectangle does not satisfy this, we can simply split it:



Let us express this open cover of ∂B as $\bigcup_{i=1}^{\infty} ((c_1^i, d_1^i) \times (c_2^i, d_2^i))$, which also gives us

$h^{-1}(\partial B) \supseteq \bigcup_{i=1}^{\infty} h^{-1}((c_1^i, d_1^i) \times (c_2^i, d_2^i))$. (*)

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5. (Continued)

Then, we claim that the rectangles:

$\bigcup_{i=1}^{\infty}((c_2^i - d_1^i, d_2^i - c_1^i) \times (c_1^i, d_1^i))$

cover $h^{-1}(\partial B)$. Indeed, it suffices to show

$(c_2^i - d_1^i, d_2^i - c_1^i) \times (c_1^i, d_1^i) \supseteq h^{-1}((c_1^i, d_1^i) \times (c_2^i, d_2^i))$

because of $(*)$. This is true since, for all

$(x', y') \in (c_1^i, d_1^i) \times (c_2^i, d_2^i)$, we have:

$h_1^{-1}(x', y') = y' - x' < d_2^i - c_1^i$, $h_1^{-1}(x', y') > c_2^i - d_1^i$,

$h_2^{-1}(x', y') = x' \in (c_1^i, d_1^i)$, as required.

So we get: $h_1^{-1}(x', y') \in (c_2^i - d_1^i, d_2^i - c_1^i)$

Total volume of rectangles is:

$$\sum_{i=1}^{\infty} vol((c_2^i - d_1^i, d_2^i - c_1^i) \times (c_1^i, d_1^i))$$

$$= \sum_{i=1}^{\infty} ((d_2^i - c_2^i) + (d_1^i - c_1^i))(d_1^i - c_1^i)$$

$$\leq \sum_{i=1}^{\infty} ((d_1^i - c_1^i) + (d_2^i - c_2^i))(d_1^i - c_1^i)$$

$$= \sum_{i=1}^{\infty} 2 vol((c_1^i, d_1^i) \times (c_2^i, d_2^i))$$

$$< 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

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5. (Continued)

Thus, we covered $h^{-1}(\partial B)$ with countably many rectangles with volume $< \varepsilon$, so $h^{-1}(\partial B)$ is measure 0, so $\partial(h^{-1}(B)) \subseteq h^{-1}(\partial B)$ is measure 0. $h^{-1}(B)$ is also bounded, so $h^{-1}(B)$ is Jordan-measurable.

Next, h^{-1} is 1-1, and

$\det(h^{-1})' = \det M_h^{-1} = \det \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$, so we can apply change of variable:

$$\mathrm{vol}(h^{-1}(B)) = \int_{R} \chi_{h^{-1}(B)}$$

R = Rectangle covering B

$$= \int_R (\chi_{h^{-1}(B)} \circ h^{-1}) \, |\det h^{-1}|$$

$$= \int_R (\chi_B) \cdot |-1|$$ (Since $x \in B$ if and only if $h^{-1}(x) \in h^{-1}(B)$)

$$= \int_R \chi_B$$

$$= \mathrm{vol}(B),$$

so $\mathrm{vol}(h^{-1}(B)) = \mathrm{vol}(B)$, as required.

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. First, we notice that the given set $\{(t,t) : t \in \mathbb{Q} \cap [0,1]\}$ is a subset of the diagonal joining the points $(0,0)$ and $(1,1)$. This diagonal is a 1-dimensional set in the 2-dimensional space $\mathbb{R}^2$, which motivates us to prove that the diagonal has content 0. To do this, we must cover the diagonal with finitely many rectangles with arbitrarily small total volume. However, since it is a diagonal and it is not horizontal or vertical, we cannot simply cover it with one very thin rectangle. Instead, the solution is to cover it with tiny squares. Since the area of a square decreases at a quadratic rate as its side length decreases, this allows us to decrease the total volume by using a larger number of smaller squares. Then, we can make the total volume arbitrarily small, as required.

2. Recall that, when we proved Partitions of Unity for open sets U in class, we proved it by expressing U as a union of a sequence of strongly nested compact sets. We defined these compact sets by:
$$C_k := \{x \in \mathbb{R}^n : |x| \leq k \text{ and } \mathrm{dist}(x, U^c) \geq \frac{1}{k}\}.$$
Intuitively, the condition "$\mathrm{dist}(x, U^c) \geq \frac{1}{k}$" makes the compact sets stay inside U while covering points that are arbitrarily close to the boundary of U, so the union eventually covers the entire set U. Moreover, the condition "$|x| \leq k$" is needed so that C_k is bounded.
For the test question, we no longer need the compact sets to be strongly nested, but the same construction still works.

3. Similarly to Assignment 10 Question 2, this question heavily involves the quantity $x^2 + y^2$, which motivates us to apply Change of Variables with a polar coordinate transformation. After some computations, we reach a step whre we must evaluate $\int_0^R \frac{1}{1+r^2} r dr$. To do this, we apply the standard u-substitution $u = 1 + r^2$, with $du = 2rdr$. After evaluating this integral, we are done.

4. As the question suggests, we begin with a partition of unity $\Phi = \{\varphi_i\}_{i \in \mathbb{N}}$ subordinate to some open cover $\mathcal{U}$ of $\mathbb{R}^n$. Then, to show that f is NT-integrable, we need to show that $\sum_{i=1}^{\infty} \int \varphi_i |f|$ converges. Since we are given that $\int_R f \leq M$ for all rectangles R in $\mathbb{R}^n$ no matter how large R is, we suspect that $\sum_{i=1}^{\infty} \int \varphi_i |f| \leq M$. Since the given information only applies for bounded R, we need to focus our attention on finite sums $\sum_{i=1}^{k} \int \varphi_i |f|$. Here, it is important that the open sets inside $\mathcal{U}$ are bounded: That way, the support of each φ_i is also bounded, and we only need to integrate along a bounded set to evaluate $\sum_{i=1}^{k} \int \varphi_i |f|$. This lets us obtain $\sum_{i=1}^{k} \int \varphi_i |f| \leq M$, so the finite sums are bounded above by M. Finally, since each term $\int \varphi_i |f|$ is nonnegative, we know that the finite sums $\sum_{i=1}^{k} \int \varphi_i |f|$ are nondecreasing as k increases, so the infinite series $\sum_{i=1}^{\infty} \int \varphi_i |f|$ converges.
(Note: On the test, I made a minor error where I forgot to mention that the finite sums are nondecreasing. This is an important condition for the infinite series to converge.)

5. First, we prove that $g^{-1}(B)$ is Jordan-measurable, which requires us to prove that the boundary of $g^{-1}(B)$ has measure 0. Intuitively, g^{-1} maps the boundary of B onto the boundary of $g^{-1}(B)$. Then, since the boundary of B has measure 0, we could cover that boundary with rectangles with arbitrarily small total volume, then apply g^{-1} to those rectangles to cover the boundary of $g^{-1}(B)$. The problem is that the rectangles get mapped to parallelograms, not new rectangles. To solve

this, we find bounds for these small parallelograms so that we can cover them with rectangles that are still small enough.
Once we prove that $g^{-1}(B)$ is also Jordan-measurable, we finish by applying Change of Variables. This application of Change of Variables is motivated by the fact that we need to "integrate over $g^{-1}(B)$" to evaluate its volume.