Term Test 1

(Author's name here) November 2, 2021

Summaries and Notes on Intuition

This solution set consists of handwritten solutions that I wrote during the test. Unfortunately, my solutions are quite messy because they were written under time pressure. Hence, before I present the handwritten solutions, I will summarize the key ideas for each solution, and I will also discuss some methods to approach each problem and motivate each solution. (Note: This section was not submitted for grading.)

1. For this question, we had to use all scale fidelity to prove that f is surjective, using the same method that was done in lecture (while proving the Inverse Function Theorem). The key idea is to use a recursive sequence $\{x_n\}_{n\in\mathbb{N}}$ to find better approximations of x over time. Since all scale fidelity says that $f(x_n)-f(x_{n-1})$ is close to x_n-x_{n-1} , and since we want $f(x_n)$ to be close to y, it is reasonable to pick x_n such that $y-f(x_{n-1})=x_n-x_{n-1}$. This gives us our recursive definition $x_n=x_{n-1}+(y-f(x_{n-1}))$.

Now, we want our sequence $\{x_n\}_{n\in\mathbb{N}}$ to better approximate x over time, so these terms should also get closer to each other over time. This motivates us to show that $|x_n-x_{n-1}|$ decreases quickly over time. The naive approach of rewriting a single recursion $x_n=x_{n-1}+(y-f(x_{n-1}))$ as $x_n-x_{n-1}=y-f(x_{n-1})$ does not work because we do not know, a priori, how well $f(x_{n-1})$ actually approximates y. Instead, we can subtract consecutive recursive relations. The left-hand side is precisely x_n-x_{n-1} , and the right-hand side is $(x_{n-1}-x_{n-2})-(f(x_{n-1})-f(x_{n-2}))$. Directly applying all scale fidelity, the right-hand side can be bounded to have a magnitude of at most $\frac{1}{7}|x_{n-1}-x_{n-2}|$. Thus, $|x_n-x_{n-1}|\leq \frac{1}{7}|x_{n-1}-x_{n-2}|$, so the distances between consecutive terms experiences exponential decay. This allows us to prove that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy (with some technicalities involving geometric series), so it converges to some limit x.

Next, recall the "naive step" that we tried above: $x_n - x_{n-1} = y - f(x_{n-1})$. Now that we know that $|x_n - x_{n-1}|$ experiences exponential decay, this step becomes helpful because we discover that $|y - f(x_{n-1})|$ also experiences exponential decay. As a result, $f(x_n)$ must approach y as $x \to \infty$. Finally, since f is continuous, it follows that:

$$f(x) = f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) = y,$$

as required.

2. This question asks us to prove Spivak's Theorem 2-8. We can prove it using the "axis crawl" solution presented during lecture. In other words, for all a+h near a, we want to travel from a to a+h in n steps, where we travel along the x_k -direction during the k^{th} step. We formalize this idea by defining the points $b_k := (a_1 + h_1, \ldots, a_k + h_k, a_{k+1}, \ldots, a_n)$ for all indices $0 \le k \le n$; the k^{th} step travels along the x_k -direction from b_{k-1} to b_k .

Next, we are interested in how much f changes at each step. At the $k^{\rm th}$ step, since this step of the axis crawl only travels along x_k -direction, we can apply the MAT157 Mean Value Theorem for the $k^{\rm th}$ partial derivative. This gives us:

$$f(b_k) - f(b_{k-1}) = h_k \frac{\partial f(c_k)}{\partial x_k}, \tag{*}$$

where c_k is between b_k and b_{k-1} . Since f has continuous partial derivatives, the term $\frac{\partial f(c_k)}{\partial x_k}$ above should approach $\frac{\partial f(a)}{\partial x_k}$ as $h \to 0$.

Next, we shall guess that the differential of f at a is the linear map:

$$L(h) := \sum_{i=1}^{n} \frac{\partial f(a)}{\partial x_i} h_i,$$

as suggested by Spivak's Theorem 2-7. We can verify this by verifying the definition of the differential:

$$\lim_{h\to 0} \frac{\left|f(a+h) - f(a) - L(h)\right|}{|h|} = 0.$$

Our axis crawl allows us to split the numerator into:

$$|(f(b_n) - f(b_{n-1})) + (f(b_{n-1}) - f(b_{n-2})) + \dots + (f(b_1) - f(b_0)) - L(h)|.$$

Since L is linear, we can split L(h) into its n directions to split the numerator further:

$$|(f(b_n) - f(b_{n-1}) - L(0, \dots, h_n)) + \dots + (f(b_1) - f(b_0) - L(h_1, \dots, 0))|.$$

The rest of the solution is computational. We apply (*), then the observation $\lim_{h\to 0} \frac{\partial f(c_k)}{\partial c_k} = \frac{\partial f(a)}{\partial x_k}$ gives us the final push to prove that L is the differential of f at a.

As a final note, my handwritten solution contains several distracting justifications that "we are close enough to a to do this and that". It should be safe to ignore such remarks while reading the solution.

3. This question is Question 5 from Assignment 1. The solution is available in the Assignment 1 Solution Set on the class website, and my handwritten solution here is very similar, so I will simply give a quick review.

First, for all $x \notin [0,1] - A$, there are two ways for a point x to be outside [0,1] - A: Either x is outside [0,1], or x is inside A. If x is outside [0,1], then x is far away from the set $A \subseteq [0,1]$, so x is in the exterior of A. If x is inside A, then it is inside one of the open intervals that comprise A, so x is in the interior of A. Either way, x is not in the boundary of A.

Next, for all $x \in [0,1]-A$, every open interval around x contains the point x outside A. Since the rationals are dense, we can argue that every open interval around x also contains a rational number in [0,1]. Such a number must be in A, so every open interval contains points inside and outside A. Thus, x is in the boundary of A.

These steps, when combined, prove that [0,1]-A is the boundary of A, as required.

4. This question is new. To prove that f is integrable, we could find partitions whose lower sums and upper sums approach each other so that the lower integral and upper integral of f are equal. When in doubt, it is a good idea to pick the partition using uniformly spaced cutpoints $\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ to obtain a simple partition. This partition has subrectangles of the form:

$$S_{i,j} = \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

Note that all subrectangles have the same volume of $\frac{1}{n^2}$. Since f is also relatively simple, we obtain the following cases for $S_{i,j}$:

i) $S_{i,j}$ could be completely above the diagonal y=x, so f(x,y)=0 for all $(x,y)\in S_{i,j}$. Then, $m_{S_{i,j}}(f)=M_{S_{i,j}}(f)=0$. There turn out to be $\frac{(n-1)n}{2}$ such rectangles. (This is Case 1 in the handwritten solution.)

- ii) $S_{i,j}$ could touch the diagonal y=x, so f(x,y) takes values of both 0 and 1 as (x,y) ranges across $S_{i,j}$. Then, $m_{S_{i,j}}(f)=0$ and $M_{S_{i,j}}(f)=1$. There turn out to be n+(n-1)=2n-1 such rectangles, so the proportion of all n^2 rectangles covered by this case is $\frac{2n-1}{n^2}=\frac{2}{n}-\frac{1}{n^2}$. This case is the only source of discrepancy between the lower and upper sums of f, so it is good news that it contains few rectangles. (This is Cases 2 and 3 in the handwritten solution.)
- iii) $S_{i,j}$ could be completely below the diagonal y=x, so f(x,y)=1 for all $(x,y)\in S_{i,j}$. Then, $m_{S_{i,j}}(f)=M_{S_{i,j}}(f)=1$. There turn out to be $\frac{(n-2)(n-1)}{2}$ such rectangles. (This is Case 4 in the handwritten solution.)

Combining these cases, the rest of the solution computes the upper and lower sums:

$$U(f,P) = \sum_{S \in P} \text{vol}(S) M_S(f) = \frac{1}{n^2} \sum_{S \in P} M_S(f), \qquad L(f,P) = \sum_{S \in P} \text{vol}(S) m_S(f) = \frac{1}{n^2} \sum_{S \in P} m_S(f).$$

Then, it finds that both the upper and lower sums converge to $\frac{1}{2}$ as $n \to \infty$, giving us the chain of inequalities:

$$\frac{1}{2} \le L(f) \le U(f) \le \frac{1}{2}.$$

We conclude that $L(f)=U(f)=\frac{1}{2}$, so f is integrable (with $\int_{[0,1]\times[0,1]}f=\frac{1}{2}$)), as required.

Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page X". Neatness counts! Language counts!

Problem 1. A continuous function $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfies

 $|(x_1-x_2)-(f(x_1)-f(x_2))| \le \frac{1}{2}|x_1-x_2|$ We will call this

for every x_1 and x_2 in \mathbb{R}^n . Prove that for every $y \in \mathbb{R}^n$ there is an $x \in \mathbb{R}^n$ such that f(x) = y.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. You may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space left here for solutions.

Your solution of Problem 1.

Let any $y \in \mathbb{R}^n$ be given. Then, define the sequence $\{x_n\}_{n\geq 0}$ by the following recursive formula: $\{x_n\}_{n\geq 0} = \{0, x_n\}_{n\geq 0} = \{x_n\}_{n\geq 0}$

Next, for all n > 2, we have the following tho equations, by definition:

$$x_n = x_{n-1} + (y - f(x_{n-1}))$$
 (1)
 $x_{n-1} = x_{n-2} + (y - f(x_{n-2}))$ (2)

$$| x_{n-1} - (x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2})) | so:$$

$$| x_{n-1} - (x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2})) |$$

$$| x_{n-1} - (x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2})) |$$

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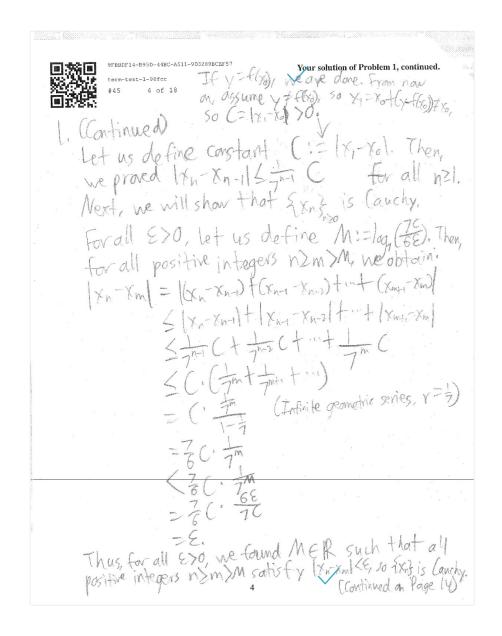
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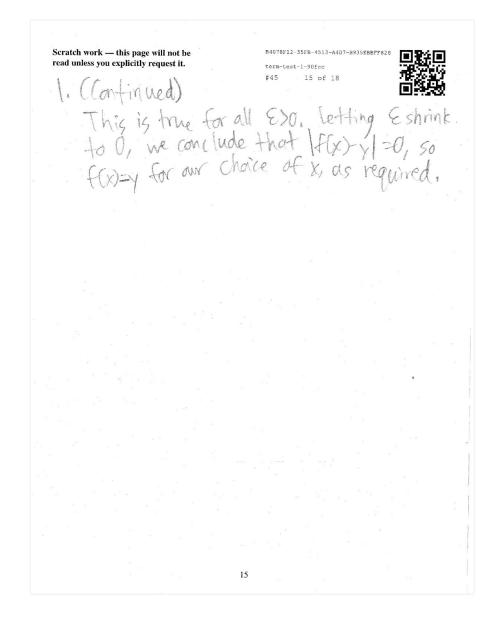
$$| x$$



Scratch work — this page will not be read unless you explicitly request it. 1. (Continued)
Therefore, the Cauchy sequence Exnt has a limit; call this limit x.
Next, for all n 21, we have x=xn+ty-f(x, x), So: [y-f(xn-)]=(xn-xn-1) \lefter fn-C.

Since lim \frac{1}{n=0} = 0, this implies that lim (y-f(xn-1)=0, so lim f(xn)=lim f(xn-1)=y.

Now, since f is continuous, Xinx f(xx) xxisk (Continued on Page 15)



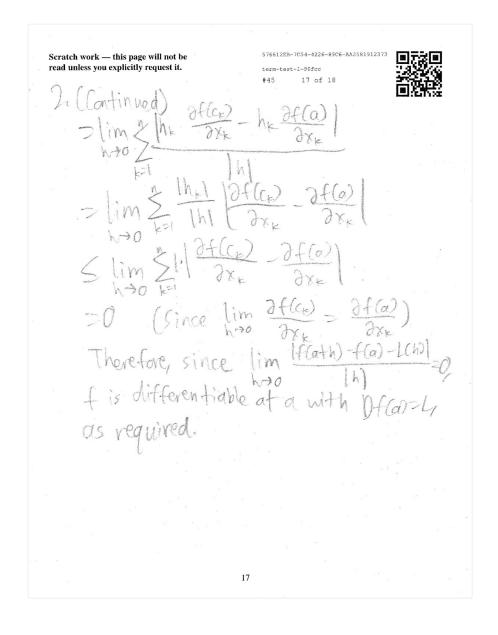
Q2

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Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page X". Neatness counts! Language counts! #45 5 of 18
Problem 2. Show that if a function f is defined near a point $a \in \mathbb{R}^n$ and has continuous partial derivatives near a , then it is differentiable at a . Tip. In math exams, "show" means "prove".
Tip. In math exams, "show" means "prove".
Tip. In math exams, "show" means "prove". Your solution of Problem 2. and has continuous partial derivatives.
I is defined near a, so there exists some radius
r such that torall hEK satistying lhizr,
f(ath) is defined and of are continuous of oth.
Now, let any hER be given such that IhlKr. Then, let us define the following points in Rn:
Then, let us define the tollowing points in in
poi (a, area, on) -a
b, = (a, thi, az,, an)
biolaid all a
bk = (a, th, m, akthe, akt, m, an)
bn = (a, thy m, anthn) = ath.
for all OSKSn, we have:
(bx-9) = (Chu, m, hx, 0, m,0)
= 12/1
< 15h;
1 - 1 h l
 50 f(b) is defined. 5 (Continued on next page)
50 f(bk) is defined. 5 (Continued on next page)

Your solution of Problem 2, continued. (Because by by ane hear a) 2. (Continued)
Now, for all 15k5n, (we have that If
exists between by and bx. (Here, "between
bx and bx reans a point of the form
(ath, and athr, t, axx, and), where t is between
ax and axthr.) Then, by the MATIST Mean Value Theorem, we obtain! f(bx)-f(bxi) = hx 2f(cx) for some Ck between but and bk. Next, since we showed above that lbx-alslhl for all k, we obtain lbx-1-al, lbx-alslhl. Then, since Ck is between bx and bx, we can write Ck as $\lambda b_k + (1-\lambda)b_{k-1}$, where $\lambda \in [0,1]$, As a result: $|C_k-a| = |\lambda b_k + (1-\lambda)b_{k-1} - a|$ (Triangle $\leq \lambda (b_k-a) + (1-\lambda)(b_{k-1}-a)$ (Triangle $\leq \lambda (b_k-a) + (1-\lambda)(b_{k-1}-a)$ (Triangle $\leq \lambda (b_k-a) + (1-\lambda)(b_{k-1}-a)$ (Triangle $\leq \lambda (b_k-a) + (1-\lambda)(b_{k-1}-a)$ (Continued on Page 16)

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2. (Continued) We should that Kx-al < lhl. Since f has continuous partial derivatives near of this
implies that: I'm $\frac{\partial f(c_k)}{\partial x_k} = \lim_{c_k \to a} \frac{\partial f(c_k)}{\partial x_k} = \frac{\partial f(a)}{\partial x_k}$
My define the linear map Like
by L(h) = \frac{\partial}{\partial} h_1 + \frac{\partial}{\partial} h_2 + \frac{\partial}{\partial} h_n. Then, we obtain! Lim [f(\partial) - f(\partial) - L(\partial)]
- lim (fCbn)-f(bd)-LCho)
> lim (f(bn)-f(bn-i)-L(0, m, hn))+(f(bn)-f(bn)-L(0, m, hn, 0)) + (f(bi)-f(bd)-L(h, m, 0))
Slim(1fcbn)-fcbn)-lo, m, holf + fcbofcbo-lch, m, o)
Applying Triangle inequality (Continued on page to)



Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page X".
Neatness counts! Language counts!

Problem 3. A set $A \subset [0, 1]$ is a (possibly infinite) union of open intervals and it contains all the rational numbers in [0, 1]. Show that the boundary of A is $[0, 1] \setminus A$.

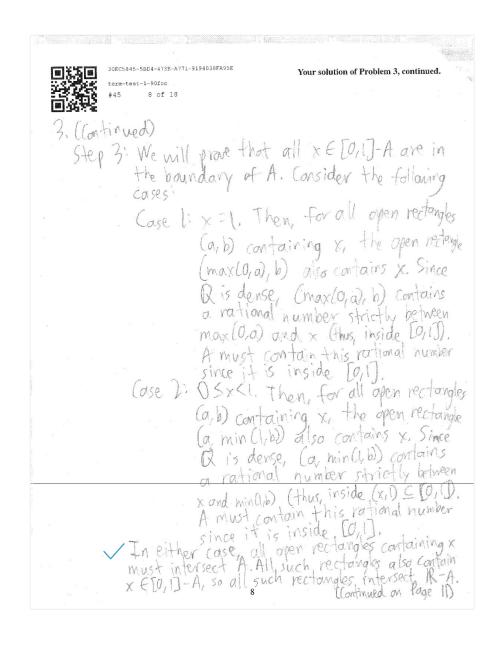
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#45 7 of 18



Your solution of Problem 3. Step I' We will prove that all xEA are not in the boundary of A. Let any XEA be given. Then, since A is a union of open intervals, x is inside some open interval (a, b) SA. In other words, there exists an open rectargle around x contained in A, so x is in the interior of A, not the boundary of A, as required. Step 2: We will prove that all x E (-0,0) U (1,0) are not in the boundary of A.

If x E (-0,0), then x is in the open rectargle (x-1, 0), and this rectangle does not intersect A because $A \subseteq [0,1] \subseteq (\mathbb{R} - (x+0))$. If $x \in (1,\infty)$, then x is in the open rectangle (1,x+1), and this rectangle does not intersect A because A C[0,1] & (R-(1,x+1)). Either may, there exists an open rectangle around x contained in R-A, so x is in the exterior of A, not the baindary, as required (Confinued on next page)



C2AAC4C6-F4AB-4725-AD59-6D42C08128CB Scratch work — this page will not be read unless you explicitly request it. term-test-1-90fcc 3. (Continued) Thus, we proved that all $x \in [0, 0]$ - A are in the boundary of A. O verall, Steps I and 2 shared that all x O verall, Steps I and 2 shared that all x outside [0,1) - A are not in the boundary of A cutside [0,1] or inside A). Moreover, Step 3 showed that all x E[0,1]-A are in the boundary of A. We conclude that [0,1]-A equals the boundary of A. as required.

Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page X". Neatness counts! Language counts!

 $f(x,y) = \begin{cases} 1 & x > y \\ 0 & x \le y \end{cases}.$

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term-test-1-90fcc

Problem 4. Let $f: [0,1] \times [0,1] \to \mathbb{R}$ be given by



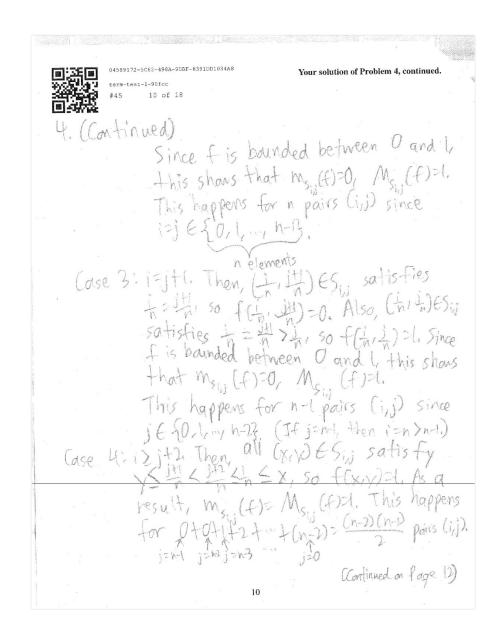
Prove that f is integrable on $[0,1] \times [0,1]$ directly by using partitions (namely, without using theorems about continuity and integrability).

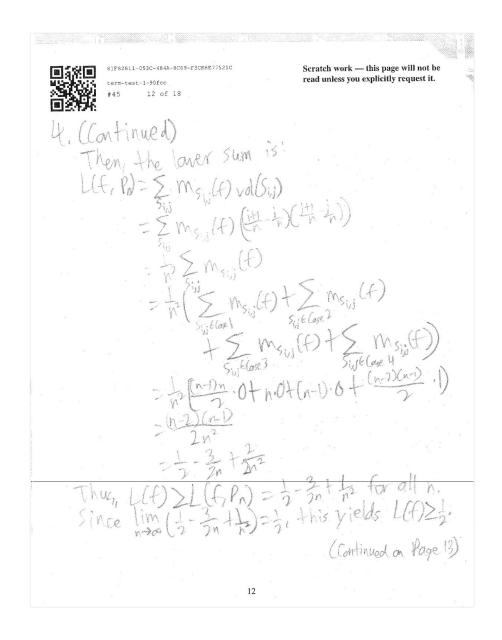
Your solution of Problem 4

For all nEN let us define the partition Pu ms, (f) = Ms, (f) = 0. The number of pairs (i,j) for which this happens is 9+1+2+ (1)= (min) for jen!

(ase 2:1=j. Then, (first) ESi, satisfies in=in, so f(in in)=0. Moreover, (in in) ESi, satisfies in=in, satisfies in=in, so f(in in)=1.

(continued on next page)





Scratch work — this page will not be 6057B685-D7A3-4943-95A5-91A7867B921E read unless you explicitly request it. term-test-1-90fcc #45 13 of 18 U. (Continued) The upper sum is: $U(f_1P_n) = \sum_{s_{ij}} M_{s_{ij}}(f) \, vd(S_{ij})$ = 1/2 E Msin(E) (Since vol(Sin)= p) = fr2 (Subtrace) + Subtrace? Msin (F) + SM SIN(F) + SM SIN(F) 12 (n-Dn otn (+(n-1).1+(n-U(n-2).1) = h + h + (n-1)(n-2) Thus, U(f) & V(f, Pn) = 2+ 2n for all n.

Since ling (3+ 1/2n) = 2' this shows V(f) S3.

Overall, V(f) S3 & L(f). This, combined with

L(f) & V(f) (Heorem in class), process that U(f) = L(f),

so f is integrable by definition, as required.