Assignment 19 (Integration on Manifolds)

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1. (a) We will show that Stokes' theorem fails if the manifold is not compact, yet that it holds again if the form has a compact support.

First, consider the 1-dimensional manifold-without-boundary $M := (0,1) \subseteq \mathbb{R}^1_x$, equipped with the standard orientation of $\mathbb{R}^1$, and consider the 0-form $\omega(x) = x$ on M. We can show that M has no boundary by showing that it satisfies definition (M) for manifolds-without-boundary. Indeed, for all $p \in M$, we can pick $U := (0,1) \ni p$, $V := (0,1) \subseteq \mathbb{R}^1$, and the diffeomorphism $h : U \to V$ defined by $h(x) = x$, and we obtain that $h(U \cap M) = h((0,1)) = (0,1) = V \cap \mathbb{R}^1$, as desired. Next, M is not compact because it is not closed. Also, we have that $\int_{\partial M} \omega = 0$ because M is without boundary. However, $\int_M d\omega = \int_M dx = \int_0^1 dx = 1 \neq 0$. Thus, for our example, $\int_{\partial M} \omega \neq \int_M d\omega$, so Stokes' theorem fails if the manifold is not compact, as desired.

Next, suppose that the oriented k-dimensional manifold-with-boundary M is not compact, but the $(k-1)$-form ω has compact support. Then, we will show that Stokes' theorem holds again. Our proof will closely follow the proof of the regular Stokes' theorem presented in lecture.

To begin, consider the open cover of M formed by open subsets of M that can be covered by "good k-cubes" on M (i.e., singular k-cubes which are orientation preserving, smooth, 1-1, and have differentials representing 1-1 linear maps). Then, since $\operatorname{supp} \omega$ is compact, we can find a finite subcover $\{U_1, \ldots, U_m\}$ of $\operatorname{supp} \omega$, and we can obtain a finite partition of unity $\{\varphi_1, \ldots, \varphi_\ell\}$ over $\operatorname{supp} \omega$ subordinate to the finite subcover.

Next, we proved in lecture that for all $(k-1)$-forms η on M, if the support of η can be covered by a single "good cube", then Stokes' theorem is true for η: $\int_M d\eta = \int_{\partial M} \eta$. The proof did not depend on M being compact, so the statement continues to be true when M is not compact. In particular, for all $1 \leq i \leq \ell$, since $\operatorname{supp}(\varphi_i \cdot \omega)$ is contained in $\operatorname{supp} \varphi_i$, which is contained in some U_j, which is covered by a "good cube", it follows that $\int_M d(\varphi_i \cdot \omega) = \int_{\partial M} \varphi_i \cdot \omega$. Moreover, whenever we use the summation $\sum_{i=1}^{\ell}$, this is a finite summation, so we are allowed to move the summation in and out of integrals. Finally, we have at all points $p \in M$ that $(\sum_{i=1}^{\ell} \varphi_i(p) - 1)\omega(p) = 0$ because either $p \in \operatorname{supp} \omega$, in which case we get $\sum_{i=1}^{\ell} \varphi_i(p) = 1$ from the partition of unity, or $p \notin \operatorname{supp} \omega$, in which case $\omega(p) = 0$. In other words, $\sum_{i=1}^{\ell} \varphi_i \cdot \omega = \omega$. Similarly, $\sum_{i=1}^{\ell} \varphi_i \cdot d\omega = d\omega$. Then, we obtain:

$$\begin{aligned}
\int_{\partial M} \omega &= \int_{\partial M} \sum_{i=1}^{\ell} \varphi_i \cdot \omega \\
&= \sum_{i=1}^{\ell} \int_{\partial M} \varphi_i \cdot \omega \\
&= \sum_{i=1}^{\ell} \int_{M} d(\varphi_i \cdot \omega) \\
&= \sum_{i=1}^{\ell} \int_{M} (d\varphi_i \cdot \omega + \varphi_i \cdot d\omega) \\
&= \sum_{i=1}^{\ell} \int_{M} d\varphi_i \cdot \omega + \sum_{i=1}^{\ell} \int_{M} \varphi_i \cdot d\omega \\
&= \int_{M} d(\sum_{i=1}^{\ell} \varphi_i) \cdot \omega + \int_{M} \sum_{i=1}^{\ell} \varphi_i \cdot d\omega \\
&= \int_{\operatorname{supp}\omega} d(\sum_{i=1}^{\ell} \varphi_i) \cdot \omega + \int_{M-\operatorname{supp}\omega} d(\sum_{i=1}^{\ell} \varphi_i) \cdot \omega + \int_{M} \sum_{i=1}^{\ell} \varphi_i \cdot d\omega \\
&= \int_{\operatorname{supp}\omega} d(1) \cdot \omega + \int_{M-\operatorname{supp}\omega} d(\sum_{i=1}^{\ell} \varphi_i) \cdot 0 + \int_{M} d\omega \\
&= \int_{\operatorname{supp}\omega} 0 \cdot \omega + 0 + \int_{M} d\omega \\
&= 0 + 0 + \int_{M} d\omega.
\end{aligned}$$

Therefore, $\int_{\partial M} \omega = \int_M d\omega$, so Stokes' theorem holds if ω has compact support, as required. □

(b) We will show that the integral of an exact form on a compact oriented manifold with no boundary vanishes, and we will give a counterexample where the manifold is not compact.
First, suppose we are given a compact oriented k-dimensional manifold M with no boundary, and suppose we are given an exact form $d\omega \in \Omega^k(M)$. Then, since M has no boundary, $\int_{\partial M} \omega = 0$. Thus, since M is compact, we can apply Stokes' theorem to obtain $\int_M d\omega = \int_{\partial M} \omega = 0$, as desired.
Next, if M is not compact, consider the same counterexample as above: $M = (0,1) \subseteq \mathbb{R}^1_x$ and $\omega(x) = x$. Then, M has no boundary, but the integral of the exact form $d\omega$ on M does not vanish because $\int_M d\omega = \int_0^1 dx = 1 \neq 0$, as required. □

2. We are given a 2-dimensional manifold $M := \{(x,y,z) \ : \ 4x^2+y^2+4z^2 = 4 \quad \& \quad y \geq 0\} \subseteq \mathbb{R}^3$. We are also given the coordinate map $\alpha(u,v) = (u, 2(1-u^2-v^2)^{1/2}, v)$ from the open ball $B_1(0) \subseteq \mathbb{R}^2$ to $M - \partial M$. We also orient M so that α is orientation preserving, and we give ∂M the induced orientation. Finally, we are given $\omega = ydx + 3xdz \in \Omega^1(M)$.
(a) Given any point $p = (x_0, 0, z_0)$ of $\partial M = \{(x,0,z) \ : \ x^2+z^2 = 1\}$, we will write a tangent vector that defines the induced orientation of ∂M.
First, let us define the coordinate map $h_p : (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0,1) \to M$ by:

$$h_p(\theta, y) = \left(\sqrt{1-\frac{y^2}{4}}(x_0\cos\theta - z_0\sin\theta), y, \sqrt{1-\frac{y^2}{4}}(z_0\cos\theta + x_0\sin\theta)\right).$$

We can check that $h_p(\theta,y) \in M$ because $y \geq 0$ and:

$$\begin{aligned}
&4\left(\sqrt{1-\frac{y^2}{4}}(x_0\cos\theta - z_0\sin\theta)\right)^2 + y^2 + 4\left(\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta + x_0\sin\theta)\right)^2 \\
&= 4(1-\frac{y^2}{4})((x_0\cos\theta - z_0\sin\theta)^2 + (z_0\cos\theta + x_0\sin\theta)^2) + y^2 \\
&= (4-y^2)(x_0^2(\cos^2\theta + \sin^2\theta) + z_0^2(\cos^2\theta + \sin^2\theta)) + y^2 \\
&= (4-y^2)(x_0^2 + z_0^2) + y^2 \\
&= (4-y^2)\cdot 1 + y^2 \qquad\qquad \text{(Since } (x_0,0,z_0) \in \partial M\text{)} \\
&= 4.
\end{aligned}$$

Also, h_p is smooth, and h_p covers p because $h_p(0,0) = (x_0,0,z_0) = p$. Finally, h_p has the smooth inverse map:

$$h_p^{-1}(x,y,z) = (\arcsin(\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}}), y),$$

because we can compute that:

$$\begin{aligned}
(h_p^{-1} \circ h_p)(\theta, y) &= h_p^{-1}\left(\sqrt{1-\frac{y^2}{4}}(x_0\cos\theta - z_0\sin\theta), y, \sqrt{1-\frac{y^2}{4}}(z_0\cos\theta + x_0\sin\theta)\right) \\
&= \left(\arcsin\left(\frac{\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta + x_0\sin\theta)x_0 - \sqrt{1-\frac{y^2}{4}}(x_0\cos\theta - z_0\sin\theta)z_0}{\sqrt{1-\frac{y^2}{4}}}\right), y\right) \\
&= (\arcsin((z_0^2 + x_0^2)\sin\theta), y) \\
&= (\arcsin(1\cdot\sin\theta), y) \qquad \text{(Since } (x_0,0,z_0) \in \partial M\text{)} \\
&= (\theta, y), \qquad \text{(Since } \theta \text{ is restricted to } (-\frac{\pi}{2}, \frac{\pi}{2})\text{)}
\end{aligned}$$

and we can also compute that:

$$
\begin{aligned}
(h_p \circ h_p^{-1})(x,y,z) &= h_p(\arcsin(\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}}), y) \\
&= \Bigg(\sqrt{1-\frac{y^2}{4}}(x_0\cos(\arcsin(\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}})) - z_0\sin(\arcsin(\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}}))), y, \\
&\quad \sqrt{1-\frac{y^2}{4}}(z_0\cos(\arcsin(\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}})) + x_0\sin(\arcsin(\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}})))\Bigg) \\
&= \Bigg(\sqrt{1-\frac{y^2}{4}}(x_0\frac{\sqrt{1-\frac{y^2}{4}-(zx_0-xz_0)^2}}{\sqrt{1-\frac{y^2}{4}}} - z_0\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}}), y, \\
&\quad \sqrt{1-\frac{y^2}{4}}(z_0\frac{\sqrt{1-\frac{y^2}{4}-(zx_0-xz_0)^2}}{\sqrt{1-\frac{y^2}{4}}} + x_0\frac{zx_0 - xz_0}{\sqrt{1-\frac{y^2}{4}}})\Bigg) \\
&= \Bigg(x_0\sqrt{1-\frac{y^2}{4}-(zx_0-xz_0)^2} - z_0(zx_0-xz_0), y, \\
&\quad z_0\sqrt{1-\frac{y^2}{4}-(zx_0-xz_0)^2} + x_0(zx_0-xz_0)\Bigg) \\
&= \Bigg(x_0\sqrt{1-\frac{4-4x^2-4z^2}{4}-(zx_0-xz_0)^2} - z_0(zx_0-xz_0), y, \\
&\quad z_0\sqrt{1-\frac{4-4x^2-4z^2}{4}-(zx_0-xz_0)^2} + x_0(zx_0-xz_0)\Bigg) \qquad \text{(Since } (x,y,z)\in M\text{)} \\
&= (x_0\sqrt{(x^2+z^2)\cdot 1-(zx_0-xz_0)^2} - z_0(zx_0-xz_0), y, \\
&\quad z_0\sqrt{(x^2+z^2)\cdot 1-(zx_0-xz_0)^2} + x_0(zx_0-xz_0)) \\
&= (x_0\sqrt{(x^2+z^2)(x_0^2+z_0^2)-(zx_0-xz_0)^2} - z_0(zx_0-xz_0), y, \\
&\quad z_0\sqrt{(x^2+z^2)(x_0^2+z_0^2)-(zx_0-xz_0)^2} + x_0(zx_0-xz_0)) \qquad \text{(Since } (x_0,0,z_0)\in \partial M\text{)} \\
&= (x_0\sqrt{x^2x_0^2+2xx_0zz_0+z^2z_0^2} - z_0(zx_0-xz_0), y, \\
&\quad z_0\sqrt{x^2x_0^2+2xx_0zz_0+z^2z_0^2} + x_0(zx_0-xz_0)) \\
&= (x_0(xx_0+zz_0) - z_0(zx_0-xz_0), y, z_0(xx_0+zz_0)+x_0(zx_0-xz_0)) \\
&= (x(x_0^2+z_0^2), y, z(x_0^2+z_0^2)) \\
&= (x,y,z).
\end{aligned}
$$

Next, we will check that h_p is orientation preserving. Let us pick any $(u,v)\in B_1(0)$ and any $(\theta,y)\in(-\frac{\pi}{2},\frac{\pi}{2})\times[0,1)$ such that $\alpha(u,v)=h_p(\theta,y)$. (Note: We know that the images of α and h_p intersect because $h_p(\theta,y)$ has a positive y-coordinate whenever y is positive.) Then, since α is orientation preserving, we have that the following basis for $T_{\alpha(u,v)}(M)$ has positive orientation on M:

$$
\begin{aligned}
&(\alpha_*((u,v),(1,0)),\alpha_*((u,v),(0,1)))\\
&=((\alpha(u,v),\alpha'(u,v)\cdot\begin{pmatrix}1\\0\end{pmatrix}),(\alpha(u,v),\alpha'(u,v)\cdot\begin{pmatrix}0\\1\end{pmatrix}))\\
&=((\alpha(u,v),\frac{\partial\alpha(u,v)}{\partial u}),(\alpha(u,v),\frac{\partial\alpha(u,v)}{\partial v}))\\
&=((\alpha(u,v),\begin{pmatrix}\frac{\partial}{\partial u}u\\ \frac{\partial}{\partial u}2\sqrt{1-u^2-v^2}\\ \frac{\partial}{\partial u}v\end{pmatrix}),(\alpha(u,v),\begin{pmatrix}\frac{\partial}{\partial v}u\\ \frac{\partial}{\partial v}2\sqrt{1-u^2-v^2}\\ \frac{\partial}{\partial v}v\end{pmatrix}))\\
&=((\alpha(u,v),\begin{pmatrix}1\\ \frac{-2u}{\sqrt{1-u^2-v^2}}\\ 0\end{pmatrix}),(\alpha(u,v),\begin{pmatrix}0\\ \frac{-2v}{\sqrt{1-u^2-v^2}}\\ 1\end{pmatrix})).
\end{aligned}
$$

Next, consider the following basis for $T_{h_p(\theta,y)}(M)$:

$$
\begin{aligned}
&((h_p)_*((\theta,y),(1,0)),(h_p)_*((\theta,y),(0,1)))\\
&=((h_p(\theta,y),h_p'(\theta,y)\cdot\begin{pmatrix}1\\0\end{pmatrix}),(h_p(\theta,y),h_p'(\theta,y)\cdot\begin{pmatrix}0\\1\end{pmatrix}))\\
&=((h_p(\theta,y),\frac{\partial h_p(\theta,y)}{\partial\theta}),(h_p(\theta,y),\frac{\partial h_p(\theta,y)}{\partial y}))\\
&=((h_p(\theta,y),\begin{pmatrix}\frac{\partial}{\partial\theta}\sqrt{1-\frac{y^2}{4}}(x_0\cos\theta-z_0\sin\theta)\\ \frac{\partial}{\partial\theta}y\\ \frac{\partial}{\partial\theta}\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta+x_0\sin\theta)\end{pmatrix}),(h_p(\theta,y),\begin{pmatrix}\frac{\partial}{\partial y}\sqrt{1-\frac{y^2}{4}}(x_0\cos\theta-z_0\sin\theta)\\ \frac{\partial}{\partial y}y\\ \frac{\partial}{\partial y}\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta+x_0\sin\theta)\end{pmatrix}))\\
&=((h_p(\theta,y),\begin{pmatrix}-\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta+x_0\sin\theta)\\ 0\\ \sqrt{1-\frac{y^2}{4}}(x_0\cos\theta-z_0\sin\theta)\end{pmatrix}),(h_p(\theta,y),\begin{pmatrix}\frac{-y}{4\sqrt{1-\frac{y^2}{4}}}(x_0\cos\theta-z_0\sin\theta)\\ 1\\ \frac{-y}{4\sqrt{1-\frac{y^2}{4}}}(z_0\cos\theta+x_0\sin\theta)\end{pmatrix})).
\end{aligned}
$$

Comparing the first and third "rows" of each basis, since the first and third "rows" of the first basis form the identity matrix, we find that the change of basis matrix must be:

$$
\begin{pmatrix}-\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta+x_0\sin\theta) & \frac{-y}{4\sqrt{1-\frac{y^2}{4}}}(x_0\cos\theta-z_0\sin\theta)\\ \sqrt{1-\frac{y^2}{4}}(x_0\cos\theta-z_0\sin\theta) & \frac{-y}{4\sqrt{1-\frac{y^2}{4}}}(z_0\cos\theta+x_0\sin\theta)\end{pmatrix},
$$

which has a determinant of:

$$\begin{aligned}
&-\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta + x_0\sin\theta)\cdot\frac{-y}{4\sqrt{1-\frac{y^2}{4}}}(z_0\cos\theta + x_0\sin\theta)\\
&-\sqrt{1-\frac{y^2}{4}}(x_0\cos\theta - z_0\sin\theta)\cdot\frac{-y}{4\sqrt{1-\frac{y^2}{4}}}(x_0\cos\theta - z_0\sin\theta)\\
=&\frac{y}{4}((z_0\cos\theta + x_0\sin\theta)^2 + (x_0\sin\theta - z_0\cos\theta)^2)\\
\geq& 0.
\end{aligned}$$

Therefore, $(\alpha_*((u,v),(1,0)),\alpha_*((u,v),(0,1)))$ and $((h_p)_*((\theta,y),(1,0)),(h_p)_*((\theta,y),(0,1)))$ have the same orientation, so $((h_p)_*((\theta,y),(1,0)),(h_p)_*((\theta,y),(0,1)))$ has the positive orientation on M, which means that h_p is orientation preserving, as desired.
Next, consider the following vector in $T_p(M)$:

$$\begin{aligned}
(h_p)_*((0,0),(0,-1)) &= (p, h_p'(0,0)\cdot\begin{pmatrix}0\\-1\end{pmatrix})\\
&= (p, -\frac{\partial h_p(\theta,y)}{\partial y})\Big|_{(\theta,y)=(0,0)}\\
&= (p, \begin{pmatrix}\frac{y}{4\sqrt{1-\frac{y^2}{4}}}(x_0\cos\theta - z_0\sin\theta)\\ -1\\ \frac{y}{4\sqrt{1-\frac{y^2}{4}}}(z_0\cos\theta + x_0\sin\theta)\end{pmatrix})\Big|_{(\theta,y)=(0,0)}\\
&= (p, \begin{pmatrix}0\\-1\\0\end{pmatrix}).
\end{aligned}$$

This vector is perpendicular to $T_p(\partial M)$ because $T_p(\partial M)$ lies in the xz-plane. Moreover, it points away from M because $((0,0),(0,-1))$ points away from $\mathbb{R}^2_{\geq 0}$. Thus, $(h_p)_*((0,0),(0,-1))$ is the unit outward normal vector at $T_p(\partial M)$. Next, since $(((0,0),(0,-1)),((0,0),(1,0)))$ has the standard orientation on $\mathbb{R}^2$, $(h_p)_*((0,0),(0,-1)),(h_p)_*((0,0),(1,0))$ has the positive orientation on M. Then, we have that $(h_p)_*((0,0),(1,0)) \in T_p(\partial M)$ because $((0,0),(1,0)) \in T_{(0,0)}(\partial\mathbb{R}^2_{\geq 0})$, so we obtain that the following vector has the positive orientation on ∂M at p:

$$\begin{aligned}
(h_p)_*((0,0),(1,0)) &= (p, h_p'(0,0)\cdot\begin{pmatrix}1\\0\end{pmatrix})\\
&= (p, \frac{\partial h_p(\theta,y)}{\partial\theta})\Big|_{(\theta,y)=(0,0)}\\
&= (p, \begin{pmatrix}-\sqrt{1-\frac{y^2}{4}}(z_0\cos\theta + x_0\sin\theta)\\ 0\\ \sqrt{1-\frac{y^2}{4}}(x_0\cos\theta - z_0\sin\theta)\end{pmatrix})\Big|_{(\theta,y)=(0,0)}\\
&= (p, \begin{pmatrix}-z_0\\0\\x_0\end{pmatrix}).
\end{aligned}$$

Therefore, the vector $\boxed{(-z_0, 0, x_0)}$ represents the induced orientation of ∂M at p, as required. $\square$

(b) We will compute $\int_{\partial M} \omega$.
First, let us define the 1-cube $c : [0,1] \to \partial M$ by $c(t) := (\cos 2\pi t, 0, \sin 2\pi t)$. Then, we can check that c is orientation-preserving by computing $c_*((0,1))$, where $(0,1)$ is the positive unit vector in $\mathbb{R}^1$ starting at 0:

$$\begin{aligned} c_*((0,1)) &= (c(0), c'(0) \cdot 1) \\ &= ((\cos 2\pi t, 0, \sin 2\pi t), (-2\pi \sin 2\pi t, 0, 2\pi \cos 2\pi t))\Big|_{t=0} \\ &= ((1,0,0),(0,0,2\pi)). \end{aligned}$$

According to part (a), $((1,0,0),(0,0,1))$ has positive orientation on ∂M. Then, $c_*((0,1))$ is a positive scalar multiple of $((1,0,0),(0,0,1))$, so $c_*((0,1))$ also has positive orientation, so c is orientation preserving, as desired. This means that c is approximately a "good cube" that covers ∂M. (Its image contains the point $(1,0,0)$ twice, but this point has content 0 in ∂M, so we ignore this anomaly.) Then, we obtain:

$$\begin{aligned} \int_{\partial M} \omega &= \int_c \omega \\ &= \int_{I^1} c_*(\omega) \\ &= \int_{I^1} c_*(ydx + 3xdz) \\ &= \int_{I^1} (0d(\cos 2\pi t) + 3\cos 2\pi t d(\sin 2\pi t)) \\ &= \int_{I^1} 6\pi \cos^2 2\pi t dt \\ &= \int_0^1 6\pi \cos^2 2\pi t dt \\ &= \int_0^1 (3\pi(2\cos^2 2\pi t - 1) + 3\pi) dt \\ &= \int_0^1 (3\pi \cos 4\pi t + 3\pi) dt \qquad \text{(Applying double angle formula)} \\ &= (\frac{1}{4\pi} \cdot 3\pi \sin 4\pi t + 3\pi t)\Big|_0^1 \\ &= (0 + 3\pi) - (0 + 0) \\ &= \boxed{3\pi}. \end{aligned}$$

$\square$

(c) We will compute $\int_M d\omega$.
First, we will compute $d\omega$ as follows:

$$d\omega = d(ydx + 3xdz) = dy \wedge dx + 3dx \wedge dz.$$

Next, consider the 2-cube $b : (0,1)^2 \to M$ defined by $b(r,t) = \alpha(r\cos 2\pi t, r \sin 2\pi t)$, and consider the transition map $\phi : (0,1)^2 \to B_1(0)$ defined by $\phi(r,t) = \alpha^{-1}(b(r,t)) = (r\cos 2\pi t, r\sin 2\pi t)$.

Then, since α is orientation preserving, we can check that b is also orientation preserving by checking that the transition function has a differential with a positive determinant:

$$\begin{aligned}
&\det\begin{pmatrix} \frac{\partial}{\partial r}(r\cos 2\pi t) & \frac{\partial}{\partial t}(r\cos 2\pi t) \\ \frac{\partial}{\partial r}(r\sin 2\pi t) & \frac{\partial}{\partial t}(r\sin 2\pi t) \end{pmatrix} \\
&= \det\begin{pmatrix} \cos 2\pi t & -2\pi r\sin 2\pi t \\ \sin 2\pi t & 2\pi r\cos 2\pi t \end{pmatrix} \\
&= (\cos 2\pi t)(2\pi r\cos 2\pi t) - (-2\pi r\sin 2\pi t)(\sin 2\pi t) \\
&= 2\pi r(\cos^2 2\pi t + \sin^2 2\pi t) \\
&> 0,
\end{aligned}$$

as desired. Then, b is approximately a "good cube" that covers M. (It does not cover some content-0 sets, including ∂M, the point $(0, 2, 0)$, and the intersection of M with the positive xy-plane.) Then, we can compute $\int_M d\omega$ on the next page:

$$
\begin{aligned}
\int_M d\omega &= \int_b d\omega \\
&= \int_{I^2} b_*(d\omega) \\
&= \int_{I^2} (a \circ \phi)_*(dy \wedge dx + 3dx \wedge dz) \\
&= \int_{I^2} \phi_*(a_*(dy \wedge dx + 3dx \wedge dz)) \\
&= \int_{I^2} \phi_*(d(2\sqrt{1-u^2-v^2}) \wedge du + 3du \wedge dv) \\
&= \int_{I^2} \phi_*((\frac{-2u}{\sqrt{1-u^2-v^2}}du + \frac{-2v}{\sqrt{1-u^2-v^2}}dv) \wedge du + 3du \wedge dv) \\
&= \int_{I^2} \phi_*(\frac{2v}{\sqrt{1-u^2-v^2}}du \wedge dv + 3du \wedge dv) \\
&= \int_{I^2} \phi_*((3 + \frac{2v}{\sqrt{1-u^2-v^2}})du \wedge dv) \\
&= \int_{I^2} (3 + \frac{2r\sin 2\pi t}{\sqrt{1-(r\cos 2\pi t)^2-(r\sin 2\pi t)^2}})d(r\cos 2\pi t) \wedge d(r \sin 2\pi t) \\
&= \int_{I^2} (3 + \frac{2r\sin 2\pi t}{\sqrt{1-r^2(\cos^2 2\pi t+\sin^2 2\pi t)}})(\cos 2\pi t dr - 2\pi r \sin 2\pi t dt) \wedge (\sin 2\pi t dr + 2\pi r \cos 2\pi t dt) \\
&= \int_{I^2} (3 + \frac{2r\sin 2\pi t}{\sqrt{1-r^2}})(2\pi r\cos^2 2\pi t dr \wedge dt + 2\pi r \sin^2 2\pi t dr \wedge dt) \\
&= \int_{I^2} (3 + \frac{2r\sin 2\pi t}{\sqrt{1-r^2}}) \cdot 2\pi r dr \wedge dt \\
&= \int_0^1 \left(\int_0^1 (3 + \frac{2r\sin 2\pi t}{\sqrt{1-r^2}}) \cdot 2\pi r dt \right) dr \qquad \text{(Applying Fubini's theorem)} \\
&= \int_0^1 \left((6\pi r t - \frac{2r^2\cos 2\pi t}{\sqrt{1-r^2}}) \Big|_{t=0}^{t=1} \right) dr \\
&= \int_0^1 ((6\pi r - \frac{2r^2}{\sqrt{1-r^2}}) - (0 - \frac{2r^2}{\sqrt{1-r^2}}))dr \\
&= \int_0^1 6\pi r dr \\
&= 3\pi r^2 \Big|_{r=0}^{r=1} \\
&= \boxed{3\pi}.
\end{aligned}
$$

This matches our answer from part (b), as expected from Stokes' theorem. □

3. For $r > 0$, we define $D^3_r = \{x \in \mathbb{R}^3 : |x| \leq r\}$ to be the 3-dimensional disk of radius r with the orientation induced from the standard orientation of $\mathbb{R}^3$, and we define $S^2_r = \partial D^3_r$ with the induced orientation. We are also given $\omega \in \Omega^2(\mathbb{R}^3 - \{0\})$ which satisfies:

$$\int_{S^2_r} \omega = a + \frac{b}{r}$$

for all $r > 0$, where a, b are real constants.

(a) Given $0 < c < d$, we will compute $\int_{D^3_d - (\operatorname{int} D^3_c)} d\omega$.

First, the boundary of D^3_d is S^2_d. Next, once we remove $\operatorname{int} D^3_c$, it adds $(-S^2_c)$ to the boundary of $D^3_d - (\operatorname{int} D^3_c)$. (Here, the negative sign means that S^2_c is oriented in the opposite direction, because on S^2_c, the outward normal vector to $D^3_d - (\operatorname{int} D^3_c)$ is opposite to the outward normal vector to D^3_c.) Then, we have that $\partial(D^3_d - (\operatorname{int} D^3_c)) = S^2_d + (-S^2_c)$. As a result:

$$\begin{aligned} \int_{D^3_d - (\operatorname{int} D^3_c)} d\omega &= \int_{\partial(D^3_d - (\operatorname{int} D^3_c))} \omega && \text{(Applying Stokes' theorem)} \\ &= \int_{S^2_d} \omega - \int_{S^2_c} \omega \\ &= (a + \frac{b}{d}) - (a + \frac{b}{c}) \\ &= \boxed{b(\frac{1}{d} - \frac{1}{c})}. \end{aligned}$$

□

(b) If ω is closed, then we claim that $b = 0$.

First, let us pick any $0 < c < d$. Then, in part (a), we found that $\int_{D^3_d - (\operatorname{int} D^3_c)} d\omega = b(\frac{1}{d} - \frac{1}{c})$. This time, we are given that $d\omega = 0$, so we also have $\int_{D^3_d - (\operatorname{int} D^3_c)} d\omega = 0$. As a result, $0 = b(\frac{1}{d} - \frac{1}{c})$. Since $c \neq d$, we divide both sides by $\frac{1}{d} - \frac{1}{c}$ to obtain $b = 0$, as required. □

(c) If ω is exact, then we claim that $a = b = 0$.

First, since ω is exact, ω is also closed, so part (b) gives us $b = 0$. Next, if ω is exact, then $\omega = d\eta$ for some $\eta \in \Omega^1(\mathbb{R}^3 - \{0\})$. Then, we obtain:

$$\begin{aligned} \int_{S^2_r} \omega &= a && \text{(Since } b = 0\text{)} \\ \int_{S^2_r} d\eta &= a \\ \int_{\partial S^2_r} \eta &= a && \text{(Applying Stokes' theorem)} \\ 0 &= a, && \text{(Since } S^2_r \text{ is a manifold without boundary)} \end{aligned}$$

as required. □

4. We are given a compact oriented $(k+l+1)$-dimensional manifold M without boundary, and we are also given $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$. Then, we will prove that the following formula holds:

$$\int_M \omega \wedge d\eta = s \int_M d\omega \wedge \eta,$$

where s is some sign, and we will also determine s.
First, since M has no boundary, $\int_{\partial M} \omega \wedge \eta = 0$. Next, since M is compact, Stokes' theorem gives us $\int_M d(\omega \wedge \eta) = \int_{\partial M} \omega \wedge \eta = 0$. Next, by Spivak's Theorem 4-10, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. As a result, $\int_M (d\omega \wedge \eta + (-1)^k \omega \wedge d\eta) = 0$, so $\int_M d\omega \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta$. Therefore, the desired formula is true, with the sign $s = (-1)^{k+1}$, as required. □

Notes on intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. To find counterexamples to Stokes' theorem for parts (a) and (b), the key idea is that if we remove the boundary of a compact manifold, then it eliminates $\int_{\partial M} \omega$ while making a negligible difference to $\int_M d\omega$, and then $\int_{\partial M} \omega$ and $\int_M d\omega$ are no longer equal to each other. Next, to prove that Stokes' theorem holds when the form has compact support, my solution was heavily motivated by the proof of the regular Stokes' theorem seen in lecture.
   Although trying to use Stokes' theorem on the compact set $\operatorname{supp}(\omega)$ is a promising approach, it has technical issues where $\operatorname{supp}(\omega)$ may not be a manifold. To illustrate, suppose $M = \mathbb{R}^2$, and suppose $\omega \in \Omega^1(\mathbb{R}^2)$ is defined by:
$$\omega(x, y) := \begin{cases} e^{-\frac{1}{x}} e^{-\frac{1}{1-x}} e^{-\frac{1}{y}} e^{-\frac{1}{1-y}} dx, & \text{if } 0 < x < 1 \text{ and } 0 < y < 1; \\ 0, & \text{otherwise;} \end{cases}$$
   using the bump functions discussed earlier in the course (when creating partitions of unity). ω is constructed so that ω is nonzero on the open rectangle $(0, 1)^2$. Then, $\operatorname{supp}(\omega)$ is the closed rectangle $[0, 1]^2$. As seen in Assignment 17 Question 5(b), $[0, 1]^2$ is not a manifold because of the four problematic vertices of the rectangle.

2. For part (a), most of my solution was technical computations to obtain a coordinate map covering points on ∂M. This was done so that afterwards, we can directly apply the definition for the induced orientation on ∂M to finish the problem.
   For parts (b) and (c), the main idea is that we mainly know how to integrate on cubes, so we find a 2-cube covering M and a 1-cube covering ∂M using a polar coordinates approach, then we integrate on the cubes. We have also encountered this approach earlier in the course, such as in Assignment 18 Question 2(c).

3. For part (a), we know almost nothing about $d\omega$, so this motivates us to apply Stokes' theorem so that we can integrate ω instead. Next, part (b) is a simple application of part (a). Finally, for part (c), if ω is exact, then $\omega = d\eta$ for some $\eta \in \Omega^1(\mathbb{R}^3 - \{0\})$, and we want to use η somehow, which motivates us to use Stokes' theorem again.

4. We are given that M has several properties which hint that applying Stokes' theorem would work well: M is compact and oriented (so we can actually apply Stokes' theorem), and M has no boundary (so $\int_{\partial M}(anything) = 0$). Then, we want to find a form such that if we apply d to it, we obtain the terms $\omega \wedge d\eta$ and $\pm d\omega \wedge \eta$, and Spivak's Theorem 4-10 helps us with that.