MAT257 Assignment 18 (Forms and Orientations on Manifolds)

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1. We are given the set of orthogonal 3×3 matrices $O(3) := \{A \in M_{3\times 3}(\mathbb{R}) \ : \ A^T A = I\}$, the set of 3-dimensional rotations $SO(3) := \{A \in O(3) \ : \ \det A = 1\}$, and the unit sphere $S^2 := \{p \in \mathbb{R}^3 \ : \ |p| = 1\}$. We are also given specific matrices $A \in O(3)$ and $B \in SO(3)$.
(a) We will prove that $A(S^2) = S^2$.
First, we will show that $A(S^2) \subseteq S^2$.
Let us write A in the form $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then, from $A^T A = I$, we obtain:

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11}^2 + a_{21}^2 + a_{31}^2 & a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} & a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} \\ a_{12}a_{11} + a_{22}a_{21} + a_{32}a_{31} & a_{12}^2 + a_{22}^2 + a_{32}^2 & a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} \\ a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} & a_{13}a_{12} + a_{23}a_{22} + a_{33}a_{32} & a_{13}^2 + a_{23}^2 + a_{33}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Comparing the $(1,1)$-entries of both sides, we obtain:

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \tag{1}$$

Comparing the $(2,2)$-entries of both sides, we obtain:

$$a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \tag{2}$$

Comparing the $(3,3)$-entries of both sides, we obtain:

$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \tag{3}$$

Comparing the $(1,2)$-entries or the $(2,1)$-entries of both sides, we obtain:

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \tag{4}$$

Comparing the $(1,3)$-entries or the $(3,1)$-entries of both sides, we obtain:

$$a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \tag{5}$$

Comparing the $(2,3)$-entries or the $(3,2)$-entries of both sides, we obtain:

$$a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \tag{6}$$

Then, for all $(x, y, z)^T \in S^2$, we obtain:

$$\begin{aligned}
&\left|A \cdot (x, y, z)^T\right|^2 \\
&= \left|\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right|^2 \\
&= \left|\begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}\right|^2 \\
&= (a_{11}x + a_{12}y + a_{13}z)^2 + (a_{21}x + a_{22}y + a_{23}z)^2 + (a_{31}x + a_{32}y + a_{33}z)^2 \\
&= (a_{11}^2x^2 + a_{12}^2y^2 + a_{13}^2z^2 + 2a_{11}a_{12}xy + 2a_{11}a_{13}xz + 2a_{12}a_{13}yz) \\
&\quad + (a_{21}^2x^2 + a_{22}^2y^2 + a_{23}^2z^2 + 2a_{21}a_{22}xy + 2a_{21}a_{23}xz + 2a_{22}a_{23}yz) \\
&\quad + (a_{31}^2x^2 + a_{32}^2y^2 + a_{33}^2z^2 + 2a_{31}a_{32}xy + 2a_{31}a_{33}xz + 2a_{32}a_{33}yz) \\
&= (a_{11}^2 + a_{21}^2 + a_{31}^2)x^2 + (a_{12}^2 + a_{22}^2 + a_{32}^2)y^2 + (a_{13}^2 + a_{23}^2 + a_{33}^3)z^2 \\
&\quad + 2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})xy + 2(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})xz + 2(a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})yz \\
&= x^2 + y^2 + z^2 + 0xy + 0xz + 0yz \qquad \text{(Applying equations (1) through (6))} \\
&= 1. \qquad \text{(Since } (x, y, z)^T \in S^2\text{)}
\end{aligned}$$

Then, we obtain $\left|A \cdot (x, y, z)^T\right|^2 = 1$, so $\left|A \cdot (x, y, z)^T\right| = 1$ since $\left|A \cdot (x, y, z)^T\right|$ is nonnegative. As a result, $A \cdot (x, y, z)^T \in S^2$ for all $(x, y, z)^T \in S^2$, so $A(S^2) \subseteq S^2$, as desired.
Next, given that $A^TA = I$, we will explain why $AA^T = I$. Since $A^TA = I$, where I is surjective, A^T must also be surjective. Then, since A^T is a square matrix, it must also be invertible. Taking the equation $A^TA = I$, if we multiply it by $(A^T)^{-1}$ to the left and A^T to the right, we obtain $AA^T = (A^T)^{-1}A^T = I$, as desired.
Next, we will show that $A(S^2) \supseteq S^2$. Since $(A^T)^TA^T = AA^T = I$, we have $A^T \in O(3)$. Then, the same proof above shows that $A^T(S^2) \subseteq S^2$. Next, for all $(x, y, z)^T \in S^2$, we have $A \cdot (A^T \cdot (x, y, z)^T) = (A \cdot A^T) \cdot (x, y, z)^T = (x, y, z)^T$. Since $A^T(S^2) \subseteq S^2$, we also have $A^T \cdot (x, y, z)^T \in S^2$, so we obtain $(x, y, z)^T = A \cdot (A^T \cdot (x, y, z)^T) \in A(S^2)$. This shows that $A(S^2) \supseteq S^2$.
Since $A(S^2) \subseteq S^2$ and $A(S^2) \supseteq S^2$, we conclude that $A(S^2) = S^2$, as required. □

(b) Given $\omega := xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$, we will prove $A^*\omega = (\det A)\omega$ as follows:

$$
\begin{aligned}
&A^*\omega \\
&= A^*(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \\
&= (a_{11}x + a_{12}y + a_{13}z)d(a_{21}x + a_{22}y + a_{23}z) \wedge d(a_{31}x + a_{32}y + a_{33}z) \\
&\quad + (a_{21}x + a_{22}y + a_{23}z)d(a_{31}x + a_{32}y + a_{33}z) \wedge d(a_{11}x + a_{12}y + a_{13}z) \\
&\quad + (a_{31}x + a_{32}y + a_{33}z)d(a_{11}x + a_{12}y + a_{13}z) \wedge d(a_{21}x + a_{22}y + a_{23}z) \\
&= (a_{11}x + a_{12}y + a_{13}z)(a_{21}dx + a_{22}dy + a_{23}dz) \wedge (a_{31}dx + a_{32}dy + a_{33}dz) \\
&\quad + (a_{21}x + a_{22}y + a_{23}z)(a_{31}dx + a_{32}dy + a_{33}dz) \wedge (a_{11}dx + a_{12}dy + a_{13}dz) \\
&\quad + (a_{31}x + a_{32}y + a_{33}z)(a_{11}dx + a_{12}dy + a_{13}dz) \wedge (a_{21}dx + a_{22}dy + a_{23}dz) \\
&= (a_{11}x + a_{12}y + a_{13}z)((a_{21}a_{32} - a_{22}a_{31})dx \wedge dy + (a_{22}a_{33} - a_{23}a_{32})dy \wedge dz + (a_{23}a_{31} - a_{21}a_{33})dz \wedge dx) \\
&\quad + (a_{21}x + a_{22}y + a_{23}z)((a_{12}a_{31} - a_{11}a_{32})dx \wedge dy + (a_{13}a_{32} - a_{12}a_{33})dy \wedge dz + (a_{11}a_{33} - a_{13}a_{31})dz \wedge dx) \\
&\quad + (a_{31}x + a_{32}y + a_{33}z)((a_{11}a_{22} - a_{12}a_{21})dx \wedge dy + (a_{12}a_{23} - a_{13}a_{22})dy \wedge dz + (a_{13}a_{21} - a_{11}a_{23})dz \wedge dx) \\
&= (a_{11}a_{21}a_{32} - a_{11}a_{22}a_{31} + a_{12}a_{21}a_{31} - a_{11}a_{21}a_{32} + a_{11}a_{22}a_{31} - a_{12}a_{21}a_{31})xdx \wedge dy \\
&\quad + (a_{12}a_{21}a_{32} - a_{12}a_{22}a_{31} + a_{12}a_{22}a_{31} - a_{11}a_{22}a_{32} + a_{11}a_{22}a_{32} - a_{12}a_{21}a_{32})ydx \wedge dy \\
&\quad + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} - a_{11}a_{23}a_{32} + a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33})zdx \wedge dy \\
&\quad + (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31})xdy \wedge dz \\
&\quad + (a_{12}a_{22}a_{33} - a_{12}a_{23}a_{32} + a_{13}a_{22}a_{32} - a_{12}a_{22}a_{33} + a_{12}a_{23}a_{32} - a_{13}a_{22}a_{32})ydy \wedge dz \\
&\quad + (a_{13}a_{22}a_{33} - a_{13}a_{23}a_{32} + a_{13}a_{23}a_{32} - a_{12}a_{23}a_{33} + a_{12}a_{23}a_{33} - a_{13}a_{22}a_{33})zdy \wedge dz \\
&\quad + (a_{11}a_{23}a_{31} - a_{11}a_{21}a_{33} + a_{11}a_{21}a_{33} - a_{13}a_{21}a_{31} + a_{13}a_{21}a_{31} - a_{11}a_{23}a_{31})xdz \wedge dx \\
&\quad + (a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32})ydz \wedge dx \\
&\quad + (a_{13}a_{23}a_{31} - a_{13}a_{21}a_{33} + a_{11}a_{23}a_{33} - a_{13}a_{23}a_{31} + a_{13}a_{21}a_{33} - a_{11}a_{23}a_{33})zdz \wedge dx \\
&= (0x + 0y + (\det A)z)dx \wedge dy + ((\det A)x + 0y + 0z)dy \wedge dz + (0x + (\det A)y + 0z)dz \wedge dx \\
&= (\det A)(zdx \wedge dy + xdy \wedge dz + ydz \wedge dx) \\
&= (\det A)\omega.
\end{aligned}
$$

Therefore, $A^*\omega = (\det A)\omega$, as required. □

(c) We will conclude that $B^*\omega = \omega$ and hence that B is orientation preserving.
First, since $B \in SO(3) \subseteq O(3)$, the same arguments above show that $B^*\omega = (\det B)\omega$.
Next, since $B \in SO(3)$, we also have $\det B = 1$, so $B^*\omega = 1 \cdot \omega = \omega$, as required.
Next, we will explain why ω is nowhere zero. Let $p = (x, y, z)$ be an arbitrary point in S^2. Then, let $v_1 = a_1\partial_x + b_1\partial_y + c_1\partial_z$ and $v_2 = a_2\partial_x + b_2\partial_y + c_2\partial_z$ be two linearly independent vectors in the 2-dimensional vector space $T_p(S^2)$. We obtain:

$$((dy \wedge dz)(v_1, v_2), (dz \wedge dx)(v_1, v_2), (dx \wedge dy)(v_1, v_2)) = (b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1),$$

which is nonzero since v_1 and v_2 are linearly independent. Moreover, this vector is orthogonal to v_1 because:

$$
\begin{aligned}
\langle v_1, (b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)\rangle &= a_1(b_1c_2 - b_2c_1) + b_1(c_1a_2 - c_2a_1) + c_1(a_1b_2 - a_2b_1) \\
&= a_1b_1c_2 - a_1b_2c_1 + a_2b_1c_1 - a_1b_1c_2 + a_1b_2c_1 - a_2b_1c_1 \\
&= 0.
\end{aligned}
$$

Similarly, it is also orthogonal to v_2. Since we are on S^2, (x,y,z) is also orthogonal to v_1 and v_2. Thus, (x,y,z) is parallel to $((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))$, so we obtain:

$$\begin{aligned}\left|\omega(v_1,v_2)\right| &= \left|x(dy\wedge dz)(v_1,v_2)+y(dz\wedge dx)(v_1,v_2)+z(dx\wedge dy)(v_1,v_2)\right|\\ &= \left|\langle(x,y,z),((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))\rangle\right|\\ &= \left\|(x,y,z)\right\|\cdot\left\|((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))\right\|\\ &= 1\cdot\left\|((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))\right\|,\end{aligned}$$

which is nonzero since $((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))$ is nonzero. Thus, $\omega(v_1,v_2)$ is nonzero, so ω is nonzero at every point $p\in S^2$, so ω is nowhere zero.
Finally, ω is a top form on S^2 that is nonzero everywhere, so ω is an orientation on M. Then, since B^* pulls this orientation ω onto itself, B is orientation preserving, as required. $\square$

2. (a) We will show that the following relations hold on $S^2 \subseteq \mathbb{R}^3_{x,y,z}$:

$$xdz \wedge dx = ydy \wedge dz, \qquad ydx \wedge dy = zdz \wedge dx, \qquad zdy \wedge dz = xdx \wedge dy.$$

First, we showed in class that the following relation is true on S^2:

$$xdx + ydy + zdz = 0. \tag{1}$$

Next, if we wedge (1) with dz, we obtain:

$$\begin{aligned} dz \wedge (xdx + ydy + zdz) &= dz \wedge 0 \\ xdz \wedge dx + ydz \wedge dy + zdz \wedge dz &= 0 \\ xdz \wedge dx - ydy \wedge dz + 0 &= 0 \\ xdz \wedge dx &= ydy \wedge dz, \end{aligned}$$

as required.
Next, if we wedge (1) with dx, we obtain:

$$\begin{aligned} dx \wedge (xdx + ydy + zdz) &= dx \wedge 0 \\ xdx \wedge dx + ydx \wedge dy + zdx \wedge dz &= 0 \\ 0 + ydx \wedge dy - zdz \wedge dx &= 0 \\ ydx \wedge dy &= zdz \wedge dx, \end{aligned}$$

as required.
Finally, if we wedge (1) with dy, we obtain:

$$\begin{aligned} dy \wedge (xdx + ydy + zdz) &= dy \wedge 0 \\ xdy \wedge dx + ydy \wedge dy + zdy \wedge dz &= 0 \\ -xdx \wedge dy + 0 + zdy \wedge dz &= 0 \\ zdy \wedge dz &= xdx \wedge dy, \end{aligned}$$

as required. $\square$

(b) Given $\omega := xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2)$, we will show that on S^2 away from the north and south poles (where $x = y = 0$),

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \wedge dz.$$

Indeed, we can compute that:

$$\begin{aligned}
\omega &= xdy\wedge dz + ydz\wedge dx + zdx\wedge dy \\
&= xdy\wedge dz - ydx\wedge dz + \frac{z(x^2+y^2)}{x^2+y^2}dx\wedge dy \\
&= xdy\wedge dz - ydx\wedge dz + \frac{xz}{x^2+y^2}\cdot xdx\wedge dy + \frac{yz}{x^2+y^2}\cdot ydx\wedge dy \\
&= xdy\wedge dz - ydx\wedge dz + \frac{xz}{x^2+y^2}\cdot zdy\wedge dz + \frac{yz}{x^2+y^2}\cdot zdz\wedge dx \qquad \text{(Applying part (a))} \\
&= xdy\wedge dz - ydx\wedge dz + \frac{xz^2}{x^2+y^2}dy\wedge dz - \frac{yz^2}{x^2+y^2}dx\wedge dz \\
&= xdy\wedge dz - ydx\wedge dz + x\cdot\frac{1-x^2-y^2}{x^2+y^2}dy\wedge dz - y\cdot\frac{1-x^2-y^2}{x^2+y^2}dx\wedge dz \qquad (x^2+y^2+z^2=1 \text{ on } S^2) \\
&= xdy\wedge dz - ydx\wedge dz + \frac{x}{x^2+y^2}dy\wedge dz - xdy\wedge dz - \frac{y}{x^2+y^2}dx\wedge dz + ydx\wedge dz \\
&= \frac{x}{x^2+y^2}dy\wedge dz - \frac{y}{x^2+y^2}dx\wedge dz \\
&= \frac{xdy-ydx}{x^2+y^2}\wedge dz,
\end{aligned}$$

as required. □

(c) We will provide a physical interpretation of this result.
First, we will relate ω with the standard area in $\mathbb{R}^3$. At any point $p=(x,y,z)\in S^2$, if we have two tangent vectors $v_1 = a_1\partial_x + b_1\partial_y + c_1\partial_z, v_2 = a_2\partial_x + b_2\partial_y + c_2\partial_z \in T_p(S^2)$, then we obtain:

$$((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2)) = (b_1c_2-b_2c_1, c_1a_2-c_2a_1, a_1b_2-a_2b_1),$$

which is the cross product of v_1 and v_2. Moreover, this vector is orthogonal to v_1 because:

$$\begin{aligned}
\langle v_1,(b_1c_2-b_2c_1,c_1a_2-c_2a_1,a_1b_2-a_2b_1)\rangle &= a_1(b_1c_2-b_2c_1)+b_1(c_1a_2-c_2a_1)+c_1(a_1b_2-a_2b_1) \\
&= a_1b_1c_2 - a_1b_2c_1 + a_2b_1c_1 - a_1b_1c_2 + a_1b_2c_1 - a_2b_1c_1 \\
&= 0.
\end{aligned}$$

Similarly, it is also orthogonal to v_2. Since we are on S^2, (x,y,z) is also orthogonal to v_1 and v_2. Thus, (x,y,z) is parallel to $((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))$, so we obtain:

$$\begin{aligned}
\left|\omega(v_1,v_2)\right| &= \left|x(dy\wedge dz)(v_1,v_2)+y(dz\wedge dx)(v_1,v_2)+z(dx\wedge dy)(v_1,v_2)\right| \\
&= \left|\langle (x,y,z),((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))\rangle\right| \\
&= \left\|(x,y,z)\right\|\cdot\left\|((dy\wedge dz)(v_1,v_2),(dz\wedge dx)(v_1,v_2),(dx\wedge dy)(v_1,v_2))\right\| \\
&= 1\cdot\|v_1\times v_2\| \\
&= \|v_1\times v_2\|,
\end{aligned}$$

which is the standard area formed by v_1 and v_2. Thus, ω can be treated as a volume form, and we can find the area of a part of S^2 by integrating ω on it, as desired.
Now, suppose we place a spherical loaf of bread into a bread cutting machine, and suppose that it cuts a slice between $z=a$ and $z=b$, with $a<b$. Then, let us define the 2-cube $c: I^2\to\mathbb{R}^3$ by:

$$c(t_1,t_2) := (\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1), \sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1), a+(b-a)t_2).$$

Here, $a+(b-a)t_2$ represents the z-coordinate sliding from a to b, and $\sqrt{1-(a+(b-a)t_2)^2}$ is the radius of the xy-cross section at the z-coordinate. Then, this cube maps onto the crust of the slice of the bread that gets cut, so the amount of crust on this slice is:

$$\begin{aligned}
\int_c \omega &= \int_c \frac{xdy - ydx}{x^2+y^2} \wedge dz \qquad\qquad \text{(Applying part (b))} \\
&= \int_{I^2} c^*\left(\frac{xdy-ydx}{x^2+y^2} \wedge dz\right) \\
&= \int_{I^2} \Bigg(\frac{\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1) d(\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \wedge d(a+(b-a)t_2) \\
&\qquad - \frac{\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1) d(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \wedge d(a+(b-a)t_2) \Bigg) \\
&= \int_{I^2} \Bigg(\frac{\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1) d(\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \wedge (b-a)dt_2 \\
&\qquad - \frac{\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1) d(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \wedge (b-a)dt_2 \Bigg) \\
&= \int_{I^2} \Bigg(\frac{\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1) \frac{\partial}{\partial t_1}(\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \cdot (b-a)dt_1 \wedge dt_2 \\
&\qquad + \frac{\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1) \frac{\partial}{\partial t_2}(\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \cdot (b-a)dt_2 \wedge dt_2 \\
&\qquad - \frac{\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1) \frac{\partial}{\partial t_1}(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \cdot (b-a)dt_1 \wedge dt_2 \\
&\qquad - \frac{\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1) \frac{\partial}{\partial t_2}(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \cdot (b-a)dt_2 \wedge dt_2 \Bigg) \\
&= \int_{I^2} \Bigg(\frac{\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1) \cdot 2\pi\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1)}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \cdot (b-a)dt_1 \wedge dt_2 + 0 \\
&\qquad - \frac{\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1) \cdot 2\pi\sqrt{1-(a+(b-a)t_2)^2}(-\sin(2\pi t_1))}{(\sqrt{1-(a+(b-a)t_2)^2}\cos(2\pi t_1))^2 + (\sqrt{1-(a+(b-a)t_2)^2}\sin(2\pi t_1))^2} \cdot (b-a)dt_1 \wedge dt_2 - 0 \Bigg) \\
&= \int_{I^2} \frac{2\pi(1-(a+(b-a)t_2)^2)(\cos^2(2\pi t_1)+\sin^2(2\pi t_1))}{(1-(a+(b-a)t_2)^2)(\cos^2(2\pi t_1)+\sin^2(2\pi t_1))} \cdot (b-a)dt_1 \wedge dt_2 \\
&= \int_{I^2} 2\pi(b-a)dt_1 \wedge dt_2 \\
&= 2\pi(b-a)\int_{I^2} dt_1 \wedge dt_2 \\
&= 2\pi(b-a).
\end{aligned}$$

Thus, the amount of crust that a slice has only depends on the width of the slice. If we assume that the machine cuts slices of equal width, we conclude that every slice has the same amount of crust, as required. □

3. We are given a k-dimensional manifold-with-boundary M with an orientation $\mathcal{O}^M$. Then, we defined the induced orientation $\mathcal{O}^{\partial M}$ on ∂M as follows: For all $p \in \partial M$, we picked a normal vector $\nu(p) \in T_p(M)$ perpendicular to $T_p(\partial M)$, then we defined $\mathcal{O}_p^{\partial M}$ such that, for all bases $(v_1, \ldots, v_{k-1})$ of $T_p(\partial M)$, $(v_1, \ldots, v_{k-1})$ has orientation $\mathcal{O}_p^{\partial M}$ if and only if $(\nu(p), v_1, \ldots, v_{k-1})$ has orientation $\mathcal{O}_p^M$. Then, we will prove that $\mathcal{O}_p^{\partial M}$ is well-defined.
For all bases $(v_1, \ldots, v_{k-1}), (w_1, \ldots, w_{k-1})$ of $T_p(\partial M)$, we must check that they have the same orientation in $T_p(\partial M)$ if and only if our definition says that they do. In other words, we must show that they have the same orientation in $T_p(\partial M)$ if and only if $(\nu(p), v_1, \ldots, v_{k-1})$ and $(\nu(p), w_1, \ldots, w_{k-1})$ have the same orientation in $T_p(M)$. First, for all $1 \leq i \leq k-1$, let us write w_i in the form $\sum_{j=1}^{k-1} a_{i,j} v_j$. Then, we obtain the change of basis matrix $A := (a_{i,j})$ between $(v_1, \ldots, v_{k-1}), (w_1, \ldots, w_{k-1})$. Next, the change of basis matrix between $(\nu(p), v_1, \ldots, v_{k-1})$ and $(\nu(p), w_1, \ldots, w_{k-1})$ is the block matrix $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$, which has the same determinant as A. As a result, one change of basis matrix has positive determinant if and only if the other change of basis matrix also has positive determinant. Therefore, $(v_1, \ldots, v_{k-1})$ and $(w_1, \ldots, w_{k-1})$ have the same orientation if and only if $(\nu(p), v_1, \ldots, v_{k-1})$ and $(\nu(p), w_1, \ldots, w_{k-1})$ have the same orientation, as required. ☐

Here is a proof that the outward unit normal vector $\nu(p)$ is well-defined at all $p \in \partial M$, which was not in my original graded submission but is required to fully solve this problem:
First, we review the definition for $\nu(p)$. Since $\dim T_p(M) = k$ and $\dim T_p(\partial M) = k - 1$, Axler 6.50 gives us that we have a 1-dimensional vector space in $T_p(M)$ orthogonal to $T_p(\partial M)$. Within this 1-dimensional vector space, we have exactly two vectors of unit length; let us call them (p, n) and $(p, -n)$. The next part of the definition requires us to choose a coordinate map $f : U \to M$ that covers p, where U is an open subset of $\mathbb{R}_+^k$. Given such a coordinate map, there exist two vectors $(f^{-1}(p), v), (f^{-1}(p), -v) \in T_{f^{-1}(p)}(\mathbb{R}^k)$ such that $f_*((f^{-1}(p), v)) = (p, n)$ and $f_*((f^{-1}(p), -v)) = (p, -n)$. Note that f maps $U \cap (\mathbb{R}^{k-1} \times \{0\})$ to ∂M, so f_* pushes vectors in $T_{f^{-1}(p)}(\mathbb{R}^{k-1} \times \{0\})$ to $T_p(\partial M)$. In other words, if the k^{th} coordinate of v satisfies $v_k = 0$, we would get $n \in T_p(\partial M)$, a contradiction since n is perpendicular to $T_p(\partial M)$. Then, we must have $v_k \neq 0$ instead. If $v_k < 0$, then we pick $\nu(p) = n$; otherwise, $v_k > 0$, so $(-v)_k < 0$, and we pick $\nu(p) = -n$. In other words, after we pick the coordinate map f, we define $\nu(p)$ to be the unique unit vector in $T_p(M)$ perpendicular to $T_p(\partial M)$ such that $(f_*)^{-1}(\nu(p))$ has a negative k-coordinate in $T_{f^{-1}(p)}(\mathbb{R}^k)$.
Next, we need to prove that $\nu(p)$ is well-defined by proving that this definition does not depend on the choice of f. Suppose we have two coordinate maps $f : U \to M$ and $g : V \to M$ which cover p, where U and V are open subsets of $\mathbb{R}_+^k$. Then, let (p, n_f) be the value of $\nu(p)$ given by applying the definition using f, so if we let $(f^{-1}(p), v_f)$ be the unique vector in $T_{f^{-1}(p)}(\mathbb{R}^k)$ such that $f_*(f^{-1}(p), v_f) = (p, n_f)$, then $(v_f)_k < 0$. Also, let $(g^{-1}(p), v_g)$ be the unique vector in $T_{g^{-1}(p)}(\mathbb{R}^k)$ such that $g_*(g^{-1}(p), v_g) = (p, n_f)$. Then, to prove that g gives the same definition for $\nu(p)$, it suffices to prove that $(v_g)_k < 0$.
Next, we will examine the differential $(g^{-1} \circ f)'$ at $f^{-1}(p)$. First, since $p \in \partial M$, we have that the k^{th} coordinate of $f^{-1}(p)$ is 0 and that the k^{th} coordinate of $(g^{-1} \circ f)(f^{-1}(p)) = g^{-1}(p)$ is also 0. Next, for all $1 \leq i \leq k$, if $(e_1, \ldots, e_k)$ denotes the standard basis for $\mathbb{R}^k$, then we have for small, positive h that the point $f^{-1}(p) + h \cdot e_i \in \mathbb{R}_+^k$ satisfies $f(f^{-1}(p) + h \cdot e_i) \in M \cap g(V)$. This means that $g^{-1}(f(f^{-1}(p) + h \cdot e_i)) \in \mathbb{R}_+^k$, so $(g^{-1} \circ f)(f^{-1}(p) + h \cdot e_i)$ has a nonnegative k-coordinate. As a result, $\frac{\partial (g^{-1} \circ f)_k}{\partial x_i} \geq 0$ at $f^{-1}(p)$ for all $1 \leq i \leq k$. Similarly, for all $1 \leq i \leq k-1$, we have

for small, positive h that the point $f^{-1}(p) - h \cdot e_i \in \mathbb{R}^k_+$ satisfies $(g^{-1} \circ f)(f^{-1}(p) - h \cdot e_i) \in \mathbb{R}^k_+$, so $\frac{\partial (g^{-1} \circ f)_k}{\partial x_i} \leq 0$ at $f^{-1}(p)$. Combined with $\frac{\partial (g^{-1} \circ f)_k}{\partial x_i} \geq 0$, this gives us $\frac{\partial (g^{-1} \circ f)_k}{\partial x_i} = 0$ at $f^{-1}(p)$. (Note: This does not hold for $i = k$ because $f^{-1}(p) - h \cdot e_k$ is no longer in $\mathbb{R}^k_+$). Overall, the differential $(g^{-1} \circ f)'(f^{-1}(p))$ is the following block matrix:

$$\begin{pmatrix} *_{k-1,k-1} & *_{k-1,1} \\ 0 & + \end{pmatrix},$$

where $*_{i,j}$ denotes an unknown $i \times j$ block, and $+$ denotes a single unknown nonnegative entry. Since f and g are diffeomorphisms, this differential must be invertible, so its final row cannot contain only zeroes. Thus, the entry labelled "$+$" is actually positive.
Next, since $f_*(f^{-1}(p), v_f) = (p, n_f)$, we get $n_f = f'(f^{-1}(p)) \cdot v_f$. Since $g_*(g^{-1}(p), v_g) = (p, n_f)$, we also get $(g^{-1})_*(p, n_f) = (g_*)^{-1}(p, n_f) = (g^{-1}(p), v_g)$, so:

$$\begin{aligned} v_g &= (g^{-1})'(p) \cdot n_f \\ &= (g^{-1})'(p) \cdot f'(f^{-1}(p)) \cdot v_f \\ &= (g^{-1})'(f(f^{-1}(p))) \cdot f'(f^{-1}(p)) \cdot v_f \\ &= (g^{-1} \circ f)'(f^{-1}(p)) \cdot v_f \qquad \text{(Chain rule)} \\ &= \begin{pmatrix} *_{k-1,k-1} & *_{k-1,1} \\ 0 & + \end{pmatrix} \cdot v_f. \end{aligned}$$

Then, by matrix multiplication, $(v_g)_k$ is a positive multiple of $(v_f)_k$. Since $(v_f)_k$ is negative, $(v_g)_k$ is also negative. Thus, g and f agree on the definition for $\nu(p)$, so $\nu(p)$ is well-defined, as required. □

4. We will prove that a k-manifold $M \subseteq \mathbb{R}^n$ is orientable if and only if it has an atlas for which all transition functions have differentials with positive determinants.
First, for the "⇒" direction, suppose that M is orientable. Then, let us pick an orientation for M by picking $\eta \in \Omega^k(M)$ that is nowhere zero. Let any atlas of M be given (such an atlas exists by definition (C) for manifolds), and let us modify the atlas as follows. Consider an arbitrary coordinate map $f : W \to \mathbb{R}^n$. For all $x \in W$, let $((x, e_1), \ldots, (x, e_k))$ be the standard basis for $T_x(\mathbb{R}^k)$. Then, for all $1 \leq i \leq k$, $f_*((x, e_i)) = (f(x), f'(x) \cdot e_i) = (f(x), \frac{\partial f(x)}{\partial x_i})$. Since we assumed $\operatorname{rank} f'(x) = k$ in definition (C) for manifolds, it follows that $(f_*((x, e_1)), \ldots, f_*((x, e_k)))$ is linearly independent in $T_{f(x)}(\mathbb{R}^n)$, so it is a basis for $T_{f(x)}(M)$. Then, since η is nowhere zero, we obtain $\eta(f_*((x, e_1)), \ldots, f_*((x, e_k))) \neq 0$ for all $x \in W$. Moreover, η is continuous, so it follows that $\eta(f_*((x, e_1)), \ldots, f_*((x, e_k)))$ is either always positive or always negative on W.
If it is positive, then we leave $f : W \to \mathbb{R}^n$ as it is. Otherwise, let us define the invertible linear map $T : \mathbb{R}^k \to \mathbb{R}^k$ by $T(x_1, x_2, \ldots, x_k) := (-x_1, x_2, \ldots, x_k)$ (note that T is its own inverse), and let us replace f with the coordinate map $g := f \circ T : T(W) \to \mathbb{R}^n$. Then, for all $x \in T(W)$, we obtain:

$$\begin{aligned}
&\eta(g_*((x, e_1)), g_*((x, e_2)), \ldots, g_*((x, e_k))) \\
&= \eta(f_*(T_*((x, e_1))), f_*(T_*((x, e_2))), \ldots, f_*(T_*((x, e_k)))) \\
&= \eta(f_*((T(x), T'(x) \cdot e_1)), f_*((T(x), T'(x) \cdot e_2)), \ldots, f_*((T(x), T'(x) \cdot e_k))) \\
&= \eta(f_*((T(x), T \cdot e_1)), f_*((T(x), T \cdot e_2)), \ldots, f_*((T(x), T \cdot e_k))) \qquad \text{(Since } T \text{ is a linear map)} \\
&= \eta(f_*((T(x), -e_1)), f_*((T(x), e_2)), \ldots, f_*((T(x), e_k))) \\
&= \eta(-f_*((T(x), e_1)), f_*((T(x), e_2)), \ldots, f_*((T(x), e_k))) \qquad \text{(Since } f_* \text{ is linear)} \\
&= -\eta(f_*((T(x), e_1)), f_*((T(x), e_2)), \ldots, f_*((T(x), e_k))), \qquad \text{(Since } \eta \text{ is } k\text{-linear)}
\end{aligned}$$

which is positive because $\eta(f_*((T(x), e_1)), \ldots, f_*((T(x), e_k)))$ is negative. This forms a new atlas of M such that for all coordinate maps $f : W \to \mathbb{R}^n$, and for all $x \in W$, $\eta(f_*((x, e_1)), \ldots, f_*((x, e_k)))$ is positive. In other words, if $\omega_k := dx_1 \wedge \cdots \wedge dx_k$ denotes the standard k-form on $\mathbb{R}^k$, then we have that $(f^*(\eta))((x, e_1), \ldots, (x, e_k)) > 0$, so $f^*(\eta)$ is a positive multiple of ω_k.
Next, for all coordinate maps $f_1 : W_1 \to \mathbb{R}^n$ and $f_2 : W_2 \to \mathbb{R}^n$ such that $f_1(W_1) \cap f_2(W_2) \neq \emptyset$, consider the transition map $\phi := f_2^{-1} \circ f_1 : f_1^{-1}(f_1(W_1) \cap f_2(W_2)) \to f_2^{-1}(f_1(W_1) \cap f_2(W_2))$. Then, at all points $x \in f_1^{-1}(f_1(W_1) \cap f_2(W_2))$, we obtain:

$$f_1^*(\eta) = (f_2 \circ \phi)^*(\eta) = \phi^*(f_2^*(\eta)).$$

Since $f_1^*(\eta)$ and $f_2^*(\eta)$ are both positive multiples of ω_k, it follows that $\phi^*(f_2^*(\eta))$ and $f_2^*(\eta)$ are both positive multiples of ω_k, so ϕ is orientation preserving. This means that $\det \phi' > 0$. Therefore, if M is orientable, then we showed that M has an atlas for which all transition functions have differentials with positive determinants, as required for the "⇒" direction.
Next, for the "⇐" direction, suppose that M has an atlas for which all transition functions have differentials with positive determinants. Then, let us construct an orientation for M as follows: For all $p \in M$, find a coordinate map $f : W \to \mathbb{R}^n$ in the given atlas such that $p \in f(W)$, then pick the orientation for $T_p(M)$ given by the ordered basis $(f_*((f^{-1}(p), e_1)), \ldots, f_*((f^{-1}(p), e_k)))$. Then, $f_*((f^{-1}(p), e_1)), \ldots, f_*((f^{-1}(p), e_k))$ are k smooth vector fields on W which locally represent our constructed orientation, so such a representation exists at all points on M. Then, to finish proving that we have constructed a valid orientation for M, it suffices to show that the definitions induced by overlapping coordinate patches agree with each other.
Suppose $f_1 : W_1 \to \mathbb{R}^n$ and $f_2 : W_2 \to \mathbb{R}^n$ are coordinate maps such that $f_1(W_1) \cap f_2(W_2) \neq \emptyset$.

Then, the transition map $\phi := f_2^{-1} \circ f_1 : f_1^{-1}(f_1(W_1) \cap f_2(W_2)) \to f_2^{-1}(f_1(W_1) \cap f_2(W_2))$ has a differential with a positive determinant by assumption. Next, we have:

$$\begin{aligned}((f_1)_*((f_1^{-1}(p), e_1)), \ldots, (f_1)_*((f_1^{-1}(p), e_k))) &= ((f_2 \circ \phi)_*((f_1^{-1}(p), e_1)), \ldots, (f_2 \circ \phi)_*((f_1^{-1}(p), e_k))) \\ &= ((f_2)_*(\phi_*((f_1^{-1}(p), e_1))), \ldots, (f_2)_*(\phi_*((f_1^{-1}(p), e_k)))).\end{aligned}$$

Since $\det \phi' > 0$, we obtain that ϕ is orientation preserving, so $(\phi_*((f_1^{-1}(p), e_1)), \ldots, \phi_*((f_1^{-1}(p), e_k)))$ has the same orientation as $((f_2^{-1}(p), e_1), \ldots, (f_2^{-1}(p), e_k))$ in $T_{f_2^{-1}(p)}(\mathbb{R}^k)$. As a result, $(f_2)_*$ pushes them to the same orientation in $T_p(M)$. Therefore, the orientations induced by f_1 and f_2 agree on $f_1(W_1) \cap f_2(W_2)$, so our orientation on M is well defined, as required for the "$\Leftarrow$" direction.
Since we proved both directions, we conclude that M is orientable if and only if it has an atlas whose transition functions all have differentials with positive determinants, as required. $\square$

5. Given an $(n-1)$-dimensional manifold M in $\mathbb{R}^n$, we will show that M is orientable if and only if there exists a consistent nonzero normal field ν to M in $\mathbb{R}^n$.

First, for the "$\Rightarrow$" direction, suppose that M is orientable, and suppose that an orientation $\mathcal{O}_p$ is chosen at all $p \in M$ in a consistent way. Then, let us define the normal field ν to M as follows. Given any $p \in M$, since $\dim T_p(\mathbb{R}^n) = n$ and $\dim T_p(M) = n-1$, it follows from Axler 6.50 that we have a 1-dimensional vector space of vectors perpendicular to $T_p(M)$ at p. Within this 1-dimensional vector space, we have exactly two vectors of unit length; let us call them n_1 and $-n_1$. Next, let $(v_1, \ldots, v_{n-1})$ be an ordered basis for $T_p(M)$ with orientation $\mathcal{O}_p$. Then, $(v_1, \ldots, v_{n-1}, n_1)$ is linearly independent because n_1 is orthogonal to $v_1, \ldots, v_{n-1}$, so $(dx_1 \wedge \cdots \wedge dx_n)(v_1, \ldots, v_{n-1}, n_1)$ is nonzero. As a result, either $(dx_1 \wedge \cdots \wedge dx_n)(v_1, \ldots, v_{n-1}, n_1)$ or $(dx_1 \wedge \cdots \wedge dx_n)(v_1, \ldots, v_{n-1}, -n_1)$ is positive, and we will define $\nu(p) := n_1$ or $\nu(p) := -n_1$, respectively. In other words, $\nu(p)$ is defined to be the unique unit vector perpendicular to $T_p(M)$ such that $(v_1, \ldots, v_{n-1}, \nu(p))$ has the standard orientation in $\mathbb{R}^n$.

Next, we must check that $\nu(p)$ is well-defined by showing that the definition does not depend on the choice for $(v_1, \ldots, v_{n-1})$. Suppose that we have two ordered bases $(v_1, \ldots, v_{n-1}), (w_1, \ldots, w_{n-1})$ for $T_p(M)$, both with orientation $\mathcal{O}_p$. Then, for all $1 \le i \le n-1$, we can write w_i in the form $\sum_{j=1}^{n-1} a_{i,j} v_j$. Afterward, we obtain the $(n-1) \times (n-1)$ change of basis matrix $A := (a_{i,j})$ between $(v_1, \ldots, v_{n-1})$ and $(w_1, \ldots, w_{n-1})$. These bases have the same orientation, we obtain $\det A > 0$. Additionally, if we consider n_1 and $-n_1$ from the previous paragraph (which are independent of our choice of basis for $T_p(M)$), then the change of basis matrix between $(v_1, \ldots, v_{n-1}, n_1)$ and $(w_1, \ldots, w_{n-1}, n_1)$ is the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, whose determinant is also $\det A > 0$. Thus, $(v_1, \ldots, v_{n-1}, n_1)$ has the standard orientation of $\mathbb{R}^n$ if and only if $(w_1, \ldots, w_{n-1}, n_1)$ has the standard orientation of $\mathbb{R}^n$, so the bases $(v_1, \ldots, v_{n-1})$ and $(w_1, \ldots, w_{n-1})$ induce the same definition for $\nu(p)$, as required.

Next, we must check that ν is smooth on M. Let $p \in M$ be arbitrary, and let $f : W \to \mathbb{R}^n$ be any coordinate map such that $p \in f(W)$. Then, for all $x \in W$, let $((x, e_1), \ldots, (x, e_{n-1}))$ be the standard basis for $T_x(\mathbb{R}^{n-1})$. Next, we can define the smooth vector fields $X_1, \ldots, X_{n-1}$ on $f(W)$ by $X_i(f(x)) := f_*((x, e_i))$ for all $1 \le i \le n-1$. Then, let us define $g : W \times \mathbb{R}^n \to \mathbb{R}^n$ by:

$$g(x, y) = (\langle X_1(f(x)), (f(x), y)\rangle, \ldots, \langle X_{n-1}(f(x)), (f(x), y)\rangle, |y|^2 - 1).$$

Then, g is smooth. Moreover, we have that:

$$g(f^{-1}(p), \nu(p)) = (\langle X_1(p), \nu(p)\rangle, \ldots, \langle X_{n-1}(p), \nu(p)\rangle, \left|\nu(p)\right|^2 - 1) = (0, \ldots, 0, 0),$$

where each $\langle X_i(p), \nu(p)\rangle$ equals zero since $X_i(p) \in T_p(M)$ and $\nu(p)$ is orthogonal to $T_p(M)$, and $\left|\nu(p)\right|^2 - 1 = 0$ since $\nu(p)$ is a unit vector. Finally, if we write each $X_i(p)$ in the form $(p, a_{i,1}e_1 + \cdots + a_{i,n}e_n)$, and if we write $\nu(p)$ in the form $(p, y_1 e_1 + \cdots + y_n e_n)$, we can compute the matrix $\frac{\partial g(x,y)}{\partial y}$ at $(f^{-1}(p),(y_1,\dots,y_n))$ as follows:

$$\begin{aligned}
\frac{\partial g(x,y)}{\partial y} &= \begin{pmatrix} \frac{\partial g_1(x,y)}{\partial y_1} & \cdots & \frac{\partial g_1(x,y)}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{n-1}(x,y)}{\partial y_1} & \cdots & \frac{\partial g_{n-1}(x,y)}{\partial y_n} \\ \frac{\partial g_n(x,y)}{\partial y_1} & \cdots & \frac{\partial g_n(x,y)}{\partial y_n} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial}{\partial y_1}(a_{1,1}y_1+\cdots+a_{1,n}y_n) & \cdots & \frac{\partial}{\partial y_n}(a_{1,1}y_1+\cdots+a_{1,n}y_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial y_1}(a_{n-1,1}y_1+\cdots+a_{n-1,n}y_n) & \cdots & \frac{\partial}{\partial y_n}(a_{n-1,1}y_1+\cdots+a_{n-1,n}y_n) \\ \frac{\partial}{\partial y_1}(y_1^2+\cdots+y_n^2-1) & \cdots & \frac{\partial}{\partial y_n}(y_1^2+\cdots+y_n^2-1) \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ 2y_1 & \cdots & 2y_n \end{pmatrix} \\
&= \begin{pmatrix} X_1(p) \\ \vdots \\ X_{n-1}(p) \\ 2\nu(p) \end{pmatrix}.
\end{aligned}$$

This matrix has linearly independent rows, so it is invertible. Therefore, all conditions of the implicit function theorem are satisfied, so there exists a unique smooth function h defined near $f^{-1}(p)$ such that $g(x,h(x)) \equiv 0$ and such that $(p,h(f^{-1}(p))) = \nu(p)$. Then, for all $1 \le i \le n-1$, since $g_i(x,h(x)) \equiv 0$, the vector $(f(x),h(x)) \in T_{f(x)}(\mathbb{R}^n)$ is orthogonal to $X_i(f(x))$ for x near $f^{-1}(p)$. As a result, $(f(x),h(x))$ is orthogonal to $T_{f(x)}(M)$. Also, since $g_n(x,h(x)) \equiv 0$, we have that $\left|h(x)\right| \equiv 1$, so $(f(x),h(x))$ is a unit vector. Finally, if ω_n denotes the standard n-form on $\mathbb{R}^n$, we have that $\omega_n(X_1(f(x)),\dots,X_{n-1}(f(x)),(f(x),h(x)))$ is nowhere zero since $X_1(f(x)),\dots,X_{n-1}(f(x)),(f(x),h(x))$ are linearly independent (since $(f(x),h(x))$ is orthogonal to $X_1(f(x)),\dots,X_{n-1}(f(x))$). Then, $\omega_n(X_1(f(x)),\dots,X_{n-1}(f(x)),(f(x),h(x)))$ has a fixed sign since it is continuous. Also, at $x = f^{-1}(p)$, $\omega_n(X_1(p),\dots,X_{n-1}(p),\nu(p)) > 0$ because we know that $(X_1(p),\dots,X_{n-1}(p),\nu(p))$ has the standard orientation by definition of p. As a result, $\omega_n(X_1(f(x)),\dots,X_{n-1}(f(x)),(f(x),h(x)))$ is positive everywhere, so we obtain that $(X_1(f(x)),\dots,X_{n-1}(f(x)),(f(x),h(x)))$ has the standard orientation. Thus, $(f(x),h(x))$ is precisely $\nu(f(x))$. Since h is smooth, it follows that ν is smooth near p. Since this is true for all $p \in M$, ν is a smooth nonzero vector field normal to M, as required for the "$\Rightarrow$" direction.

Next, for the "$\Leftarrow$" direction, suppose that there exists a consistent nonzero normal field ν to M in $\mathbb{R}^n$. Then, let us define $\eta \in \Omega^{n-1}(M)$ as follows: For all $p \in M$ and all $v_1,\dots,v_{n-1} \in T_p(M)$, let us define $\eta(p)(v_1,\dots,v_{n-1}) := (dx_1 \wedge \cdots \wedge dx_n)(v_1,\dots,v_{n-1},\nu(p))$. Then, η is smooth because $dx_1 \wedge \cdots \wedge dx_n$ and ν are smooth. Moreover, if we pick any basis $(e_1,\dots,e_{n-1})$ for $T_p(M)$, then $(e_1,\dots,e_{n-1},\nu(p))$ is linearly independent in $T_p(\mathbb{R}^n)$ because $\nu(p)$ is orthogonal to $e_1,\dots,e_{n-1}$, which means that $\eta(p)(e_1,\dots,e_{n-1}) = (dx_1 \wedge \cdots \wedge dx_n)(e_1,\dots,e_{n-1},\nu(p)) \neq 0$. As a result, η is nonzero at every point $p \in M$, so η is a valid orientation on M. Therefore, M is orientable if there exists a consistent nonzero normal field ν to M in $\mathbb{R}^n$, as required for the "$\Leftarrow$" direction.

Since we proved both directions, we conclude that M is orientable if and only if there exists a consistent nonzero normal field ν to M in $\mathbb{R}^n$, as required. □

Notes on intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. As noted in the grader's comments, it is possible to find much cleaner solutions if one is comfortable enough with clever linear algebra tricks. Otherwise, it is also possible to brute-force this problem as in my solution, but it is much messier and more time-consuming.

2. For part (a), the given hint (wedge $xdx + ydy + zdz = 0$ with dx, dy, and dz) solves the problem immediately. It may also be possible to approach this problem without the hint by working backwards. For instance, suppose we want to prove that $xdz \wedge dx = ydy \wedge dz$ on S^2. This is equivalent to $0 = xdz \wedge dx - ydy \wedge dz = xdz \wedge dx + ydz \wedge dy = dz \wedge (xdx + ydy)$. Then, the $(xdx + ydy)$ terms could remind you of $xdx + ydy + zdz = 0$, then we proceed as in the hint.
   For part (b), we can approach this problem by working backwards. We want to prove that $\omega = \frac{x}{x^2+y^2} dy \wedge dz + \frac{y}{x^2+y^2} dz \wedge dx$, so we can compare $\frac{x}{x^2+y^2} dy \wedge dz$ and $\frac{y}{x^2+y^2} dz \wedge dx$ with the $xdy \wedge dz + ydz \wedge dx$ terms contained in ω, and we can see how much $dy \wedge dz$ and $dz \wedge dx$ is remaining, and we can see how to convert $zdx \wedge dy$ into the remaining terms.
   For part (c), if we interpret ω as the volume form, then we want to integrate ω over the crust to find the amount of crust on a slice. To do so, the main idea is that we mainly know how to integrate on cubes, so we find a 2-cube covering the crust using a polar coordinates approach, and then we integrate ω over the 2-cube to find the amount of crust.

3. For this question, the main idea was to use change-of-basis matrices to compare orientations. Relating orientations using change-of-basis matrices is a standard idea that we have also seen earlier in the course, including in Assignment 13 Question 2.

4. One possible approach is to recall that, similarly to Assignment 13 Question 2, a transition function $g^{-1} \circ f$ (where f and g are coordinate maps) has a differential with positive determinant if and only if it is orientation preserving. Then, one way for $g^{-1} \circ f$ to be orientation preserving is if both components, f and g, preserve orientations. This motivates us to pick the atlas such that all coordinate maps are orientation preserving for the "$\Rightarrow$" direction, and also to construct an orientation on M such that all coordinate maps are orientation preserving for the "$\Leftarrow$" direction.

5. The main intuition for this question is that a normal vector field on M can be used to induce an orientation of M using an orientation of $\mathbb{R}^n$, similarly to how in Question 3, a normal vector field on ∂M induces an orientation of ∂M using an orientation of M. Additionally, to ensure that the normal vector field $\nu(p)$ is defined consistently, it was helpful to define $\nu(p)$ as a unit vector so that it does not change by a scalar factor based on certain choices. This also helps to ensure that $\nu(p)$ is smooth because $\nu(p)$ does not suddenly extend or contract non-continuously on any point on the manifold.