MAT257 Assignment 17 (Manifolds)

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March 25, 2022

1. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function, the graph of f is defined to be $\Gamma_f = \{(x, y) \ : \ y = f(x)\} \subseteq \mathbb{R}^{n+m}$.
(a) Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$, we will show that Γ_f is a smooth n-manifold.
We plan to show that Γ_f satisfies definition (Z) for manifolds. First, for all $p \in \Gamma_f$, let us pick the open neighbourhood $U := \mathbb{R}^n \times \mathbb{R}^m$ around p, and let us pick the function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ defined by $g(x, y) := y - f(x)$. Note that g is smooth because f is smooth. Then, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, we have that $g(x, y) = 0$ if and only if $y = f(x)$, which occurs if and only if $(x, y) \in \Gamma_f$. In other words, $g^{-1}(0) = \Gamma_f$. As a result:

$$U \cap \Gamma_f = \mathbb{R}^{n+m} \cap \Gamma_f = \Gamma_f = g^{-1}(0) = \mathbb{R}^{n+m} \cap g^{-1}(0) = U \cap g^{-1}(0),$$

so $U \cap \Gamma_f = U \cap g^{-1}(0)$, as desired.
Next, for all $(x, y) \in \Gamma_f$, we have:

$$\begin{aligned} g'(x,y) &= \begin{pmatrix} \frac{\partial(y_1 - f_1(x))}{\partial x_1} & \cdots & \frac{\partial(y_1 - f_1(x))}{\partial x_n} & \frac{\partial(y_1 - f_1(x))}{\partial y_1} & \cdots & \frac{\partial(y_1 - f_1(x))}{\partial y_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(y_m - f_m(x))}{\partial x_1} & \cdots & \frac{\partial(y_m - f_m(x))}{\partial x_n} & \frac{\partial(y_m - f_m(x))}{\partial y_1} & \cdots & \frac{\partial(y_m - f_m(x))}{\partial y_m} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\partial f_1(x)}{\partial x_1} & \cdots & -\frac{\partial f_1(x)}{\partial x_n} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial f_m(x)}{\partial x_1} & \cdots & -\frac{\partial f_m(x)}{\partial x_n} & 0 & \cdots & 1 \end{pmatrix}. \end{aligned}$$

Since the last m columns of $g'(x, y)$ form the identity matrix of size m, we conclude that g' has a rank of $m = (n + m) - n$, as desired.
Overall, Γ_f satisfies definition (Z) for manifolds, so Γ_f is a smooth manifold, as required. □

(b) Consider the function $f : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $f(x) := x^{\frac{1}{3}}$. Then, we will show that f is a non-smooth function for which nevertheless Γ_f is a smooth manifold.
First, f is not differentiable at 0 because:

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = \infty.$$

Thus, f is non-smooth, as required.
Next, we plan to show that Γ_f satisfies definition (Z) for manifolds. For all $p \in \Gamma_f$, let us pick the open neighbourhood $U := \mathbb{R}^2$ around p, and let us pick the function $g : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $g(x, y) := x - y^3$. Note that g is smooth because g is a polynomial. Then, for all $(x, y) \in \mathbb{R}^2$, we have the following chain of equivalences:

$$g(x, y) = 0 \Leftrightarrow y^3 = x \Leftrightarrow y = x^{\frac{1}{3}} \Leftrightarrow y = f(x) \Leftrightarrow (x, y) \in \Gamma_f.$$

In other words, $g(x, y) = 0$ if and only if $(x, y) \in \Gamma_f$, so $g^{-1}(0) = \Gamma_f$, so:

$$U \cap \Gamma_f = \mathbb{R}^2 \cap \Gamma_f = \Gamma_f = g^{-1}(0) = \mathbb{R}^2 \cap g^{-1}(0) = U \cap g^{-1}(0),$$

so $U \cap \Gamma_f = U \cap g^{-1}(0)$, as desired.
Next, for all $(x, y) \in \Gamma_f$, we have $g'(x, y) = (\frac{\partial(x - y^3)}{\partial x}, \frac{\partial(x - y^3)}{\partial y}) = (1, -3y^2)$. Since the first entry of $g'(x, y)$ is always nonzero, we conclude that g' always has a rank of $1 = 2 - 1$, as desired.
Overall, we proved that Γ_f satisfies definition (Z) for manifolds. Therefore, f is an example of a non-smooth function for which Γ_f is a smooth manifold, as required. □

2. Given a 12-dimensional manifold M in $\mathbb{R}^{22}$, we will prove that M is of measure 0 in $\mathbb{R}^{22}$.
First, for all $p \in M$, we will construct an open neighbourhood U_p around p such that $U_p \cap M$ is of measure 0. To begin, since M satisfies definition (M) for manifolds, there exists an open neighbourhood U_p' around p, an open set V_p' in $\mathbb{R}^{22}$, and a diffeomorphism $h : U_p' \to V_p'$ such that $h(U_p' \cap M) = V_p' \cap (\mathbb{R}^{12} \times \{0\})$. Since $p \in U_p'$, we have $h(p) \in V_p'$. Then, there exists a closed rectangle C_p centred at $h(p)$ that is contained in V_p'. Then, this closed rectangle is compact. Moreover, h^{-1} is smooth, so $\det(h^{-1})'$ is continuous, and it follows that $\det(h^{-1})'$ is bounded on C_p because C_p is compact. Then, if we consider the open rectangle $V_p := \mathrm{interior}(C_p)$ contained in C_p, then $\det(h^{-1})'$ is also bounded on V_p. In other words, there exists $N \in \mathbb{R}$ such that $\left|\det(h^{-1})'(y)\right| \leq N$ for all $y \in V_p$. Moreover, since C_p is centred at $h(p)$ by construction, V_p is also centred at $h(p)$, so V_p contains $h(p)$.
Now, let us pick $U_p := h^{-1}(V_p)$. Since h^{-1} is invertible, it follows that $V_p = h(U_p)$. Since h is continuous and V_p is open, it also follows that $U_p = h^{-1}(V_p)$ is also open. Moreover, since $h(p) \in V_p$, we obtain that $U_p = h^{-1}(V_p)$ contains p, so U_p is an open neighbourhood of p. Additionally, $U_p = h^{-1}(V_p) \subseteq h^{-1}(V_p') = U_p'$. Then, we obtain $h(U_p \cap M) = V_p \cap (\mathbb{R}^{12} \times \{0\})$ because:

$$\begin{aligned}
h(U_p \cap M) &= h((U_p \cap U_p') \cap M) && \text{(Since } U_p \subseteq U_p'\text{)} \\
&= (h^{-1})^{-1}(U_p \cap U_p' \cap M) \\
&= (h^{-1})^{-1}(U_p) \cap (h^{-1})^{-1}(U_p' \cap M) && \text{(Since preimages behave well with intersections)} \\
&= h(U_p) \cap h(U_p' \cap M) \\
&= V_p \cap V_p' \cap (\mathbb{R}^{12} \times \{0\}) \\
&= V_p \cap (\mathbb{R}^{12} \times \{0\}). && \text{(Since } V_p \subseteq V_p'\text{)}
\end{aligned}$$

Next, let us define $f : V_p \to \mathbb{R}$ by $f(y) := \chi_{\mathbb{R}^{12}\times\{0\}}(y) \cdot \left|\det(h^{-1})'(y)\right|$. Then, since $\mathbb{R}^{12} \times \{0\}$ is of measure 0 in $\mathbb{R}^{22}$, we know that $\int_{V_p} \chi_{\mathbb{R}^{12}\times\{0\}} = 0$. Then, we can show that the upper integral of f over V_p is at most 0 as follows:

$$\begin{aligned}
U(f) &= U(\chi_{\mathbb{R}^{12}\times\{0\}} \cdot \left|\det(h^{-1})'\right|) \\
&\leq U(\chi_{\mathbb{R}^{12}\times\{0\}} \cdot N) && \text{(Since } \left|\det(h^{-1})'\right| \text{ is bounded by } N \text{ on } V_p\text{)} \\
&= N \cdot U(\chi_{\mathbb{R}^{12}\times\{0\}}) \\
&= N \cdot 0 \\
&= 0.
\end{aligned}$$

Moreover, $L(f) \geq 0$ since f is nonnegative. Therefore, since $L(f) \leq U(f)$, it follows that $L(f) = U(f) = 0$, so f is integrable over V_p, and $\int_{h(U_p)} f = \int_{V_p} f = 0$.
Now, we wish to apply the Change of Variables formula, so we will check that its hypotheses hold. First, U_p is open. Additionally, since $h : U_p' \to V_p'$ is 1-1 and continuously differentiable, $h|_{U_p} : U_p \to V_p$ is also 1-1 and continuously differentiable. Thus, we may apply the version of

Change of Variables without the requirement that $\det h' \neq 0$. It follows that:

$$\begin{aligned}
\int_{h(U_p)} f &= \int_{U_p} (f \circ h)\left|\det h'\right| \\
&= \int_{U_p} (\chi_{\mathbb{R}^{12}\times\{0\}} \circ h) \cdot \left|\det((h^{-1})' \circ h)\right| \cdot \left|\det h'\right| \\
&= \int_{U_p} (\chi_{\mathbb{R}^{12}\times\{0\}} \circ h) \cdot \left|\det(((h^{-1})' \circ h) \cdot h')\right| \\
&= \int_{U_p} (\chi_{\mathbb{R}^{12}\times\{0\}} \circ h) \cdot \left|\det((h^{-1} \circ h)')\right| \qquad \text{(Chain rule)} \\
&= \int_{U_p} (\chi_{\mathbb{R}^{12}\times\{0\}} \circ h) \cdot \left|\det \mathrm{id}'\right| \qquad \text{(Where id denotes identity map)} \\
&= \int_{U_p} (\chi_{\mathbb{R}^{12}\times\{0\}} \circ h) \cdot 1 \\
&= \int_{U_p} \chi_{h^{-1}(\mathbb{R}^{12}\times\{0\})} \qquad \text{(Since } h(x) \in \mathbb{R}^{12} \times \{0\} \text{ if and only if } x \in h^{-1}(\mathbb{R}^{12} \times \{0\})\text{)} \\
&= \int_{U_p} \chi_{U_p \cap h^{-1}(\mathbb{R}^{12}\times\{0\})} \qquad \text{(Since we are integrating over } U_p\text{)} \\
&= \int_{U_p} \chi_{h^{-1}(V_p) \cap h^{-1}(\mathbb{R}^{12}\times\{0\})} \\
&= \int_{U_p} \chi_{h^{-1}(V_p \cap (\mathbb{R}^{12}\times\{0\}))} \\
&= \int_{U_p} \chi_{U_p \cap M}.
\end{aligned}$$

On the other hand, we showed above that $\int_{h(U_p)} f = 0$, so it follows that $\int_{U_p} \chi_{U_p \cap M} = 0$. Therefore, we found an open neighbourhood U_p of p such that $U_p \cap M$ is of measure 0, as desired.

Next, we will show that countably many such U_p cover M. For all $p \in M$, let R_p be an open rectangle around p contained in U_p. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we may assume that all coondinates of all vertices of R_p are rational; otherwise, we may shrink R_p until this is true. If we do this over all $p \in M$, then we form countably many distinct rectangles because there are only countably many rational numbers available for the coordinates of their vertices. Then, these rectangles form a countable open cover of M because each $p \in R_p$ for all $p \in M$. Next, for each rectangle that we form, select one open set U_p containing the rectangle. Since there are countably many rectangles, we will select countably many sets U_p, and since the rectangles cover M, the countably many selected sets U_p also cover M. This gives us a countable cover $\bigcup_{p \in M'} U_p$ of M, where M' is a countable subset of M, as desired.

Finally, this gives us $M = M \cap \bigcup_{p \in M'} U_p = \bigcup_{p \in M'} (U_p \cap M)$, where each $U_p \cap M$ is of measure 0. Therefore, M is measure-0 as a countable union of measure-0 sets, as required. ☐

3. We are given the following set:

$$U(2) := \{A \in M_{2\times 2}(\mathbb{C}) \ : \ \overline{A}^T A = I\} \subseteq M_{2\times 2}(\mathbb{C}) = \mathbb{C}^4 = \mathbb{R}^8.$$

Then, we will prove that $U(2)$ is a 4-dimensional manifold if treated as a subset of $\mathbb{R}^8$.
First, we identify $\mathbb{R}^8$ with $M_{2\times 2}(\mathbb{C})$ as follows:

$$(a_{11}, b_{11}, a_{12}, b_{12}, a_{21}, b_{21}, a_{22}, b_{22}) = \begin{pmatrix} a_{11} + b_{11}i & a_{12} + b_{12}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i \end{pmatrix}.$$

Next, an element $A = \begin{pmatrix} a_{11} + b_{11}i & a_{12} + b_{12}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i \end{pmatrix}$ is in $U(2)$ if and only if:

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \\ &= \overline{A}^T A \\ &= \begin{pmatrix} a_{11} - b_{11}i & a_{21} - b_{21}i \\ a_{12} - b_{12}i & a_{22} - b_{22}i \end{pmatrix} \begin{pmatrix} a_{11} + b_{11}i & a_{12} + b_{12}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i \end{pmatrix} \\ &= \begin{pmatrix} (a_{11} - b_{11}i)(a_{11} + b_{11}i) + (a_{21} - b_{21}i)(a_{21} + b_{21}i) & (a_{11} - b_{11}i)(a_{12} + b_{12}i) + (a_{21} - b_{21}i)(a_{22} + b_{22}i) \\ (a_{12} - b_{12}i)(a_{11} + b_{11}i) + (a_{22} - b_{22}i)(a_{21} + b_{21}i) & (a_{12} - b_{12}i)(a_{12} + b_{12}i) + (a_{22} - b_{22}i)(a_{22} + b_{22}i) \end{pmatrix}. \end{aligned}$$

Comparing the top left corners of both sides, we obtain:

$$(a_{11} - b_{11}i)(a_{11} + b_{11}i) + (a_{21} - b_{21}i)(a_{21} + b_{21}i) = 1,$$

which gives us:

$$a_{11}^2 + b_{11}^2 + a_{21}^2 + b_{21}^2 - 1 = 0. \tag{1}$$

Comparing the bottom right corners of both sides, we obtain:

$$(a_{12} - b_{12}i)(a_{12} + b_{12}i) + (a_{22} - b_{22}i)(a_{22} + b_{22}i) = 1,$$

which gives us:

$$a_{12}^2 + b_{12}^2 + a_{22}^2 + b_{22}^2 - 1 = 0. \tag{2}$$

Comparing the top right corners of both sides, we obtain:

$$\begin{aligned} (a_{11} - b_{11}i)(a_{12} + b_{12}i) + (a_{21} - b_{21}i)(a_{22} + b_{22}i) &= 0 \\ (a_{11}a_{12} + b_{11}b_{12} + a_{21}a_{22} + b_{21}b_{22}) + (a_{11}b_{12} - a_{12}b_{11} + a_{21}b_{22} - a_{22}b_{21})i &= 0. \end{aligned}$$

The real part gives us:

$$a_{11}a_{12} + b_{11}b_{12} + a_{21}a_{22} + b_{21}b_{22} = 0, \tag{3}$$

and the imaginary part gives us:

$$a_{11}b_{12} - a_{12}b_{11} + a_{21}b_{22} - a_{22}b_{21} = 0. \tag{4}$$

Finally, comparing the bottom left corners of both sides we obtain:

$$(a_{12} - b_{12}i)(a_{11} + b_{11}i) + (a_{22} - b_{22}i)(a_{21} + b_{21}i) = 0$$
$$(a_{11}a_{12} + b_{11}b_{12} + a_{21}a_{22} + b_{21}b_{22}) - (a_{11}b_{12} - a_{12}b_{11} + a_{21}b_{22} - a_{22}b_{21})i = 0.$$

Then, we obtain (3) and (4) again.
As a result, a matrix $\begin{pmatrix} a_{11} + b_{11}i & a_{12} + b_{12}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i \end{pmatrix}$ is in $U(2)$ if and only if equations (1) through (4) are satisfied.
Next, let us define $g : \mathbb{R}^8 \to \mathbb{R}^4$ by:

$$g(a_{11}, b_{11}, a_{12}, b_{12}, a_{21}, b_{21}, a_{22}, b_{22}) = \begin{pmatrix} a_{11}^2 + b_{11}^2 + a_{21}^2 + b_{21}^2 - 1 \\ a_{12}^2 + b_{12}^2 + a_{22}^2 + b_{22}^2 - 1 \\ a_{11}a_{12} + b_{11}b_{12} + a_{21}a_{22} + b_{21}b_{22} \\ a_{11}b_{12} - a_{12}b_{11} + a_{21}b_{22} - a_{22}b_{21} \end{pmatrix}.$$

Then, as discussed above, a matrix $A \in M_{2\times 2}(\mathbb{C})$ is in $U(2)$ if and only if it satisfies (1) through (4), so $A \in U(2)$ if and only if $g(A) = 0$. As a result, $U(2) = g^{-1}(0)$.
Next, we will prove that $g'(A)$ is of rank 4 for all $A \in U(2)$. We begin by computing $g'(A)$ as follows:

$$g'(A) = \begin{pmatrix} \frac{\partial g_1(A)}{\partial a_{11}} & \frac{\partial g_1(A)}{\partial b_{11}} & \frac{\partial g_1(A)}{\partial a_{12}} & \frac{\partial g_1(A)}{\partial b_{12}} & \frac{\partial g_1(A)}{\partial a_{21}} & \frac{\partial g_1(A)}{\partial b_{21}} & \frac{\partial g_1(A)}{\partial a_{22}} & \frac{\partial g_1(A)}{\partial b_{22}} \\ \frac{\partial g_2(A)}{\partial a_{11}} & \frac{\partial g_2(A)}{\partial b_{11}} & \frac{\partial g_2(A)}{\partial a_{12}} & \frac{\partial g_2(A)}{\partial b_{12}} & \frac{\partial g_2(A)}{\partial a_{21}} & \frac{\partial g_2(A)}{\partial b_{21}} & \frac{\partial g_2(A)}{\partial a_{22}} & \frac{\partial g_2(A)}{\partial b_{22}} \\ \frac{\partial g_3(A)}{\partial a_{11}} & \frac{\partial g_3(A)}{\partial b_{11}} & \frac{\partial g_3(A)}{\partial a_{12}} & \frac{\partial g_3(A)}{\partial b_{12}} & \frac{\partial g_3(A)}{\partial a_{21}} & \frac{\partial g_3(A)}{\partial b_{21}} & \frac{\partial g_3(A)}{\partial a_{22}} & \frac{\partial g_3(A)}{\partial b_{22}} \\ \frac{\partial g_4(A)}{\partial a_{11}} & \frac{\partial g_4(A)}{\partial b_{11}} & \frac{\partial g_4(A)}{\partial a_{12}} & \frac{\partial g_4(A)}{\partial b_{12}} & \frac{\partial g_4(A)}{\partial a_{21}} & \frac{\partial g_4(A)}{\partial b_{21}} & \frac{\partial g_4(A)}{\partial a_{22}} & \frac{\partial g_4(A)}{\partial b_{22}} \end{pmatrix}$$
$$= \begin{pmatrix} 2a_{11} & 2b_{11} & 0 & 0 & 2a_{21} & 2b_{21} & 0 & 0 \\ 0 & 0 & 2a_{12} & 2b_{12} & 0 & 0 & 2a_{22} & 2b_{22} \\ a_{12} & b_{12} & a_{11} & b_{11} & a_{22} & b_{22} & a_{21} & b_{21} \\ b_{12} & -a_{12} & -b_{11} & a_{11} & b_{22} & -a_{22} & -b_{21} & a_{21} \end{pmatrix}.$$

Now, assume for contradiction that this matrix is not of rank 4 at some point $A \in U(2)$. Then, the four rows of $g'(A)$ are linearly dependent. In other words, there exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$, not all zero, such that we obtain the following 8 equations, corresponding to the 8 columns of $g'(A)$:

$$2a_{11}c_1 + a_{12}c_3 + b_{12}c_4 = 0 \tag{5}$$
$$2b_{11}c_1 + b_{12}c_3 - a_{12}c_4 = 0 \tag{6}$$
$$2a_{12}c_2 + a_{11}c_3 - b_{11}c_4 = 0 \tag{7}$$
$$2b_{12}c_2 + b_{11}c_3 + a_{11}c_4 = 0 \tag{8}$$
$$2a_{21}c_1 + a_{22}c_3 + b_{22}c_4 = 0 \tag{9}$$
$$2b_{21}c_1 + b_{22}c_3 - a_{22}c_4 = 0 \tag{10}$$
$$2a_{22}c_2 + a_{21}c_3 - b_{21}c_4 = 0 \tag{11}$$
$$2b_{22}c_2 + b_{21}c_3 + a_{21}c_4 = 0 \tag{12}$$

Then, the linear combination $a_{11} \cdot (5) + b_{11} \cdot (6) + a_{21} \cdot (9) + b_{21} \cdot (10)$ gives us:

$$\begin{aligned} 0 &= 2(a_{11}^2 + b_{11}^2 + a_{21}^2 + b_{21}^2)c_1 + (a_{11}a_{12} + b_{11}b_{12} + a_{21}a_{22} + b_{21}b_{22})c_3 \\ &\quad + (a_{11}b_{12} - a_{12}b_{11} + a_{21}b_{22} - a_{22}b_{21})c_4 \\ &= 2c_1 + 0c_3 + 0c_4. \qquad \text{(Applying (1), (3), and (4))} \end{aligned}$$

As a result, $c_1 = 0$.
Next, the linear combination $a_{12} \cdot (7) + b_{12} \cdot (8) + a_{22} \cdot (11) + b_{22} \cdot (12)$ gives us:

$$\begin{aligned} 0 &= 2(a_{12}^2 + b_{12}^2 + a_{22}^2 + b_{22}^2)c_2 + (a_{11}a_{12} + b_{11}b_{12} + a_{21}a_{22} + b_{21}b_{22})c_3 \\ &\quad + (-a_{12}b_{11} + a_{11}b_{12} - a_{22}b_{21} + a_{21}b_{22})c_4 \\ &= 2c_2 + 0c_3 + 0c_4. \qquad \text{(Applying (2), (3), and (4))} \end{aligned}$$

As a result, $c_2 = 0$.
Next, the linear combination $a_{12} \cdot (5) + b_{12} \cdot (6) + a_{22} \cdot (9) + b_{22} \cdot (10)$ gives us:

$$\begin{aligned} 0 &= 2(a_{11}a_{12} + b_{11}b_{12} + a_{21}a_{22} + b_{21}b_{22})c_1 + (a_{12}^2 + b_{12}^2 + a_{22}^2 + b_{22}^2)c_3 \\ &\quad + (a_{12}b_{12} - a_{12}b_{12} + a_{22}b_{22} - a_{22}b_{22})c_4 \\ &= 0c_1 + c_3 + 0c_4. \qquad \text{(Applying (3) and (2))} \end{aligned}$$

As a result, $c_3 = 0$.
Finally, the linear combination $b_{12} \cdot (5) - a_{12} \cdot (6) + b_{22} \cdot (9) - a_{22} \cdot (10)$ gives us:

$$\begin{aligned} 0 &= 2(a_{11}b_{12} - a_{12}b_{11} + a_{21}b_{22} - a_{22}b_{21})c_1 + (a_{12}b_{12} - a_{12}b_{12} + a_{22}b_{22} - a_{22}b_{22})c_3 \\ &\quad + (b_{12}^2 + a_{12}^2 + b_{22}^2 + a_{22}^2)c_4 \\ &= 0c_1 + 0c_3 + c_4. \qquad \text{(Applying (4) and (2))} \end{aligned}$$

As a result, $c_4 = 0$.
Overall, we proved that $c_1 = c_2 = c_3 = c_4 = 0$, contradicting the condition that c_1, c_2, c_3, c_4 are not all zero. Thus, by contradiction, $g'(A)$ is of rank 4 for all $A \in U(2)$, as desired.
Finally, let $p \in U(2)$ be arbitrary. Then, let us select the open neighbourhood $V := M_{2\times 2}(\mathbb{C})$ around p. As discussed above, the rank of $g'(p)$ is 4. Moreover, we have:

$$V \cap U(2) = M_{2\times 2}(\mathbb{C}) \cap U(2) = U(2) = g^{-1}(0) = M_{2\times 2}(\mathbb{C}) \cap g^{-1}(0) = V \cap g^{-1}(0).$$

Thus, $U(2)$ satisfies definiton (Z) for manifolds. Moreover, since g maps from the 8-dimensional space $M_{2\times 2}(\mathbb{C})$ to the 4-dimensional space $\mathbb{R}^4$, the dimension of $U(2)$ is $8-4 = 4$, as required. $\square$

4. We are given open subsets U, V of $\mathbb{R}^k_+ := \{x \in \mathbb{R}^k : x_k \geq 0\}$; in other words, $U = \mathbb{R}^k_+ \cap U'$ and $V = \mathbb{R}^k_+ \cap V'$ for some open subsets U', V' of $\mathbb{R}^k$. We are also given a diffeomorphism $\phi : U \to V$. Then, we will prove that $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) = V \cap (\mathbb{R}^{k-1} \times \{0\})$.
First, we will prove that $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) \subseteq V \cap (\mathbb{R}^{k-1} \times \{0\})$. Assume for contradiction that this is false, so there exists $y_0 \in \phi(U \cap (\mathbb{R}^{k-1} \times \{0\}))$ such that $y_0 \notin V \cap (\mathbb{R}^{k-1} \times \{0\})$. In other words, y_0 is of the form $\phi(x_0)$ for some $x_0 \in U$ such that $(x_0)_k = 0$, but $(y_0)_k > 0$. Then, let us define the open neighbourhood $V'' := \{V' \cap \{y \in \mathbb{R}^k : y_k > 0\}\}$ around y_0, where V'' is open in $\mathbb{R}^k$ as the intersection of the open sets V' and $\{y \in \mathbb{R}^k : y_k > 0\}$. Note that $V'' \subseteq V$ because we have for all $y \in V''$ that $y_k > 0$ and thus $y \in \mathbb{R}^k_+$.
Next, for all $y \in V'' \subseteq V$, we have that $\phi^{-1}(y) \in U \subseteq \mathbb{R}^+_k$, so $(\phi^{-1})_k(y) \geq 0$. In particular, since $(\phi^{-1})_k(y_0) = (x_0)_k = 0$, we have that $(\phi^{-1})_k$ exhibits a local minimum at y_0. Then, at y_0, we obtain for all $1 \leq j \leq k$ that $\frac{\partial(\phi^{-1})_k(y)}{\partial y_j} = 0$.
Next, since ϕ is smooth at x_0, it has an extension $\overline{\phi} : U'' \to \mathbb{R}^m$ on some open neighbourhood U'' of x_0 such that $\phi|_{U'' \cap U} = \overline{\phi}|_{U'' \cap U}$ and such that $\overline{\phi}$ is differentiable at x_0. Then, since U'' is open in $\mathbb{R}^k$, there exists some $\varepsilon > 0$ such that the open ball $B_\varepsilon(x_0)$ is contained in U''. Next, since V'' is an open neighbourhood of y_0, and since ϕ^{-1} is continuous at y_0, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}^k$ such that $|y - y_0| < \delta$, we have $y \in V''$ and $\left|\phi^{-1}(y) - \phi^{-1}(y_0)\right| < \varepsilon$. In other words, $\left|\phi^{-1}(y) - x_0\right| < \varepsilon$, so $\phi^{-1}(y) \in B_\varepsilon(x_0) \subseteq U''$. We also have for all such y that $\phi^{-1}(y) \in U$ because the domain of ϕ is U. Overall, this gives us $\phi^{-1}(y) \in U'' \cap U$, so we obtain:

$$\overline{\phi}(\phi^{-1}(y)) = \overline{\phi}|_{U'' \cap U}(\phi^{-1}(y)) = \phi|_{U'' \cap U}(\phi^{-1}(y)) = y.$$

In other words, near y_0, the map $\overline{\phi} \circ \phi^{-1}$ is well-defined and is equal to the identity map. This gives us that $(\overline{\phi} \circ \phi^{-1})'(y_0)$ is the identity matrix Id. Moreover, by the chain rule, we have:

$$(\overline{\phi} \circ \phi^{-1})'(y_0) = \overline{\phi}'(\phi^{-1}(y_0)) \cdot (\phi^{-1})'(y_0) = \overline{\phi}'(x_0) \cdot (\phi^{-1})'(y_0).$$

On the other hand, we argued above that $\frac{\partial(\phi^{-1})_k(y)}{\partial y_j} = 0$ at y_0 for all $1 \leq j \leq k$, so the k^{th} row of $(\phi^{-1})'(y_0)$ consists of zeroes, which means that $(\phi^{-1})'(y_0)$ is not invertible. This implies that $(\overline{\phi} \circ \phi^{-1})'(y_0) = \overline{\phi}'(x_0) \cdot (\phi^{-1})'(y_0)$ is also not invertible, so it cannot equal Id, a contradiction. Thus, by contradiction, $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) \subseteq V \cap (\mathbb{R}^{k-1} \times \{0\})$, as desired.
Next, we will show that $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) \supseteq V \cap (\mathbb{R}^{k-1} \times \{0\})$. Indeed, since $\phi^{-1} : V \to U$ is also a diffeomorphism, the same proof above also shows that $\phi^{-1}(V \cap (\mathbb{R}^{k-1} \times \{0\})) \subseteq U \cap (\mathbb{R}^{k-1} \times \{0\})$. Then, for all $y \in V \cap (\mathbb{R}^{k-1} \times \{0\})$, we have $\phi^{-1}(y) \in \phi^{-1}(V \cap \mathbb{R}^{k-1} \times \{0\}) \subseteq U \cap (\mathbb{R}^{k-1} \times \{0\})$. Since $\phi^{-1}(y) \in U \cap (\mathbb{R}^{k-1} \times \{0\})$, it follows that $y = \phi(\phi^{-1}(y)) \in \phi(U \cap (\mathbb{R}^{k-1} \times \{0\}))$. Since we showed that $y \in \phi(U \cap (\mathbb{R}^{k-1} \times \{0\}))$ for all $y \in V \cap (\mathbb{R}^{k-1} \times \{0\})$, it follows that $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) \supseteq V \cap (\mathbb{R}^{k-1} \times \{0\})$, as desired.
Finally, from $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) \subseteq V \cap (\mathbb{R}^{k-1} \times \{0\})$ and $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) \supseteq V \cap (\mathbb{R}^{k-1} \times \{0\})$, we conclude that $\phi(U \cap (\mathbb{R}^{k-1} \times \{0\})) = V \cap (\mathbb{R}^{k-1} \times \{0\})$, as required. □

5. (a) Given a k-manifold $M \subseteq \mathbb{R}^m$ and an l-manifold $N \subseteq \mathbb{R}^n$, both without boundary, we will prove that $M \times N$ is a $(k+l)$-dimensional manifold without boundary in $\mathbb{R}^m \times \mathbb{R}^n$.
We plan to show that $M \times N$ satisfies definition (C) for manifolds. Let $p = (p_1, p_2) \in M \times N$ be arbitrary. Then, since M satisfies definition (C) for manifolds, we obtain an open neighbourhood U_1 around p_1, an open set $W_1 \subseteq \mathbb{R}^k$, and a smooth 1-1 function $f_1 : W_1 \to \mathbb{R}^m$ such that:

i) $f_1(W_1) = M \cap U_1$.
ii) $f_1^{-1} : M \cap U_1 \to W_1$ is continuous.
iii) For all $a \in W_1$, $\operatorname{rank} f_1'(a) = k$.

Since N satisfies definition (C) for manifolds, we also obtain an open neighbuorhood U_2 around p_2, an open set $W_2 \subseteq \mathbb{R}^l$, and a smooth 1-1 function $f_2 : W_2 \to \mathbb{R}^n$ such that:

i) $f_2(W_2) = N \cap U_2$.
ii) $f_2^{-1} : N \cap U_2 \to W_2$ is continuous.
iii) For all $b \in W_2$, $\operatorname{rank} f_2'(b) = l$.

Next, let us define the open neighbourhood $U := U_1 \times U_2$ around p, and let us define the open set $W := W_1 \times W_2 \subseteq \mathbb{R}^k \times \mathbb{R}^l$. Let us also define $f : W \to \mathbb{R}^m \times \mathbb{R}^n$ by $f(x, y) := (f_1(x), f_2(y))$ for all $x \in W_1$ and all $y \in W_2$. Then, f is smooth because f_1 and f_2 are smooth. Moreover, f is 1-1 because f_1 and f_2 are 1-1. (In more detail: If we have $(x_1, y_1), (x_2, y_2) \in W$ such that $f(x_1, y_1) = f(x_2, y_2)$, then $(f_1(x_1), f_2(y_1)) = (f_1(x_2), f_2(y_2))$, so we get $f_1(x_1) = f_1(x_2)$ and $f_2(y_1) = f_2(y_2)$, which gives us $x_1 = x_2$ and $y_1 = y_2$ because f_1 and f_2 are 1-1.)
Now, we must show that f satisfies the following three properties:

i) We will show that $f(W) = (M \times N) \cap U$.
To begin, we will show that $f(W) \subseteq (M \times N) \cap U$. For all $f(x, y) \in f(W)$, where $(x, y) \in W$, we have that:

$$f(x,y) = (f_1(x), f_2(y)) \in f_1(W_1) \times f_2(W_2) = (M \cap U_1) \times (N \cap U_2) = (M \times N) \cap (U_1 \times U_2) = (M \times N) \cap U.$$

Thus, $f(x, y) \in (M \times N) \cap U$ for all $f(x, y) \in f(W)$, so $f(W) \subseteq (M \times N) \cap U$.
Next, we will show that $f(W) \supseteq (M \times N) \cap U$. For all $(a, b) \in (M \times N) \cap U$, we have:

$$(a, b) \in (M \times N) \cap U = (M \times N) \cap (U_1 \times U_2) = (M \cap U_1) \times (N \cap U_2).$$

Then, $a \in M \cap U_1 = f_1(W_1)$, so there exists $x \in W_1$ such that $f_1(x) = a$. Moreover, $b \in N \cap U_2 = f_2(W_2)$, so there exists $y \in W_2$ such that $f_2(y) = b$. This gives us $f(x, y) = (f_1(x), f_2(y)) = (a, b)$, where $(x, y) \in W$, so $(a, b) \in f(W)$. Thus, $(a, b) \in f(W)$ for all $(a, b) \in (M \times N) \cap U$, so $f(W) \supseteq (M \times N) \cap U$.
Since we also showed that $f(W) \subseteq (M \times N) \cap U$, we conclude that $f(W) = (M \times N) \cap U$, as required.

ii) We will show that $f^{-1} : (M \times N) \cap U \to W$ is continuous. Let any $(a, b) \in (M \times N) \cap U$ be given, and let any $\varepsilon > 0$ be given. Then, we have:

$$(a, b) \in (M \times N) \cap U = (M \times N) \cap (U_1 \times U_2) = (M \cap U_1) \times (N \cap U_2).$$

In other words, $a \in M \cap U_1$ and $b \in N \cap U_2$. Then, since f_1^{-1} is continuous, there exists $\delta_1 > 0$ such that for all $a' \in M \cap U_1$ such that $\left|a' - a\right| < \delta_1$, we obtain $\left|f_1^{-1}(a') - f_1^{-1}(a)\right| < \frac{\varepsilon}{2}$.

Since f_2^{-1} is also continuous, there exists $\delta_2 > 0$ such that for all $b' \in N \cap U_2$ such that $\left|b' - b\right| < \delta_2$, we obtain $\left|f_2^{-1}(b') - f_2^{-1}(b)\right| < \frac{\varepsilon}{2}$. Next, let us pick $\delta := \min(\delta_1, \delta_2) > 0$. Then, for all $(a', b') \in (M \times N) \cap U$ such that $\left|(a', b') - (a, b)\right| < \delta$, we obtain:

$$
\begin{aligned}
&\left|f^{-1}(a', b') - f^{-1}(a, b)\right| \\
=&\left|(f_1^{-1}(a') - f_1^{-1}(a), f_2^{-1}(b') - f_2^{-1}(b))\right| \\
=&\sqrt{\left|f_1^{-1}(a') - f_1^{-1}(a)\right|^2 + \left|f_2^{-1}(b') - f_2^{-1}(b)\right|^2} \\
<&\sqrt{(\frac{\varepsilon}{2})^2 + \left|f_2^{-1}(b') - f_2^{-1}(b)\right|^2} \qquad \text{(Since } \left|a' - a\right| \leq \left|(a', b') - (a, b)\right| < \delta \leq \delta_1\text{)} \\
<&\sqrt{(\frac{\varepsilon}{2})^2 + (\frac{\varepsilon}{2})^2} \qquad \text{(Since } \left|b' - b\right| \leq \left|(a', b') - (a, b)\right| < \delta \leq \delta_2\text{)} \\
<&\,\varepsilon.
\end{aligned}
$$

Thus, for all $(a, b) \in (M \times N) \cap U$ and all $\varepsilon > 0$, we found $\delta > 0$ such that for all $(a', b') \in (M \times N) \cap U$ such that $\left|(a', b') - (a, b)\right| < \delta$, we obtain $\left|f^{-1}(a', b') - f^{-1}(a, b)\right| < \varepsilon$. We conclude that f^{-1} is continuous, as required.

iii) We will show that $\operatorname{rank} f'(a, b) = k + l$ for all $(a, b) \in W$. Indeed, $f'(a, b)$ is the following block matrix:

$$
f'(a, b) = \begin{pmatrix} \frac{\partial f_1(a)}{\partial a} & \frac{\partial f_1(a)}{\partial b} \\ \frac{\partial f_2(b)}{\partial a} & \frac{\partial f_2(b)}{\partial b} \end{pmatrix} = \begin{pmatrix} f_1'(a) & 0 \\ 0 & f_2'(b) \end{pmatrix}.
$$

Since $f_1'(a)$ is of rank k and $f_2'(b)$ is of rank l, we conclude that $f'(a, b)$ is of rank $k + l$, as required.

Therefore, $M \times N$ satisfies definition (C) for a $(k + 1)$-manifold, as required. ☐

(b) We are given a k-manifold $M \subseteq \mathbb{R}^m$ and an l-manifold $N \subseteq \mathbb{R}^n$, both with boundary. Then, we will prove that $M \times N$ is the disjoint union of a $(k + l)$-manifold with boundary and a $(k + l - 2)$-manifold without boundary.

First, ∂M is a $(k - 1)$-manifold without boundary and ∂N is an $(l - 1)$-manifold without boundary, as explained in lecture. Then, applying part (a), $\partial M \times \partial N$ is a $(k + l - 2)$-manifold without boundary. To finish the proof, it suffices to show that $(M \times N) - (\partial M \times \partial N)$ is a $(k + l)$-manifold with boundary.

Let (p_1, p_2) be an arbitrary point in $(M \times N) - (\partial M \times \partial N)$. Then, p_1 is inside M and p_2 is inside N. Moreover, p_1 is outside ∂M or p_2 is outside ∂N. We have the two following cases:

Case 1: p_1 is outside ∂M. Then, we proceed similarly to part (a). First, since M is a manifold with boundary, there exists an open neighbourhood U_1 around p_1, an open subset W_1 of $\mathbb{R}^k_+$, and a smooth 1-1 function $f_1 : W_1 \to \mathbb{R}^m$ such that:

i) $f_1(W_1) = M \cap U_1$.
ii) $f_1^{-1} : M \cap U_1 \to W_1$ is continuous.
iii) $\operatorname{rank} f_1'(a) = k$ for all $a \in W_1$.

Next, since p_1 is outside ∂M, we have $(f_1^{-1})_k(p_1) > 0$. Additionally, since W_1 is open in $\mathbb{R}^k_+$, there exists an open subset W_1' of $\mathbb{R}^k$ such that $W_1 = W_1' \cap \mathbb{R}^k_+$. Then, let us define

$W_1'' := W_1' \cap \{x \in \mathbb{R}^k \ : \ x_k > 0\} \ni f_1^{-1}(p_1)$. We have that W_1'' is open in $\mathbb{R}^k$ as the intersection of the two open sets W_1' and $\{x \in \mathbb{R}^k \ : \ x_k > 0\}$. Since $W_1'' \subseteq \{x \in \mathbb{R}^k \ : \ x_k > 0\} \subseteq \mathbb{R}^k_+$, it follows that W_1'' is also open in $\mathbb{R}^k_+$. Then, since f_1^{-1} is continuous, we obtain that $f_1(W_1'')$ is open in M, so there exists an open subset U_1'' of $\mathbb{R}^n$ such that $f(W_1'') = U_1'' \cap M$. Additionally, since $f_1^{-1}(p_1) \in W_1''$, we have that $p_1 \in f(W_1'') \subseteq U_1''$, so U_1'' is an open neighbourhood of p_1.
Next, let us check that the three conditions above hold when we replace U_1 and W_1 with U_1'' and W_1'':

i) $f_1(W_1'') = M \cap U_1''$ by definition.
ii) f_1^{-1} was continuous on the domain $M \cap U_1$, so it is still continuous on the smaller domain $M \cap U_1''$.
iii) For all $a \in U_1''$, we also have $a \in U_1$, so $\operatorname{rank} f_1'(a) = k$ is still true.

Next, for all $a \in M \cap U_1''$, we have that U_1'' is an open neighbourhood of a, W_1'' is an open subset of $\mathbb{R}^k_+$, and $f_1|_{W_1''} : W_1'' \to \mathbb{R}^m$ is a smooth, 1-1 function satisfying the three properties above. Moreover, $f_1^{-1}(a) \in W_1'' \subseteq \{x \in \mathbb{R}^k \ : \ x_k > 0\}$, so $(f_1^{-1})_k(p_1) > 0$. Thus, $a \in M - \partial M$ for all $a \in M \cap U_1''$, so $M \cap U_1'' \subseteq M - \partial M$, giving us:

$$M \cap U_1'' = (M - \partial M) \cap M \cap U_1'' = (M - \partial M) \cap U_1''.$$

Next, since N is a manifold with boundary, there exists an open neighbourhood U_2 around p_2, an open subset W_2 of $\mathbb{R}^l_+$, and a smooth 1-1 function $f_2 : W_2 \to \mathbb{R}^n$ such that:

i) $f_2(W_2) = N \cap U_2$.
ii) $f_2^{-1} : N \cap U_2 \to W_2$ is continuous.
iii) $\operatorname{rank} f_2'(b) = l$ for all $b \in W_2$.

Now, let us define the open neighbourhood $U := U_1'' \times U_2$ around p. Since W_1'' is open in $\mathbb{R}^k$ and W_2 is open in $\mathbb{R}^l_+$, we can also define the open set $W := W_1'' \times W_2$ in $\mathbb{R}^k \times \mathbb{R}^l_+ = \mathbb{R}^{k+l}_+$. Finally, let us define $f : W \to \mathbb{R}^m \times \mathbb{R}^n$ by $f(x, y) := (f_1(x), f_2(y))$ for all $x \in W_1''$ and all $y \in W_2$. Then, f is smooth and 1-1 because f_1 and f_2 are smooth and 1-1.
Now, we must show that f satisfies the following three properties:

i) We will show that $f(W) = ((M \times N) - (\partial M \times \partial N)) \cap U$.
To begin, we will show that $f(W) \subseteq ((M \times N) - (\partial M \times \partial N)) \cap U$. For all $f(x, y) \in f(W)$, where $(x, y) \in W$, we have that:

$$f(x, y) = (f_1(x), f_2(y)) \in f_1(W_1'') \times f_2(W_2) = (M \cap U_1'') \times (N \cap U_2).$$

Since $M \cap U_1'' = (M - \partial M) \cap U_1''$, we proceed with:

$$f(x, y) \in ((M - \partial M) \cap U_1'') \times (N \cap U_2) = ((M - \partial M) \times N) \cap (U_1'' \times U_2) \subseteq ((M \times N) - (\partial M \times \partial N)) \cap U.$$

Thus, $f(x, y) \in ((M \times N) - (\partial M \times \partial N)) \cap U$ for all $f(x, y) \in f(W)$, so we obtain $f(W) \subseteq ((M \times N) - (\partial M \times \partial N)) \cap U$.
Additionally, we will show that $f(W) \supseteq ((M \times N) - (\partial M \times \partial N)) \cap U$. Indeed, for all $(a, b) \in ((M \times N) - (\partial M \times \partial N)) \cap U$, we have:

$$(a, b) \in ((M \times N) - (\partial M \times \partial N)) \cap U \subseteq (M \times N) \cap U = (M \times N) \cap (U_1'' \times U_2) = (M \cap U_1'') \times (N \cap U_2).$$

Then, $a \in M \cap U_1'' = f_1(W_1'')$, so there exists $x \in W_1''$ such that $f_1(x) = a$. Moreover, $b \in N \cap U_2 = f_2(W_2)$, so there exists $y \in W_2$ such that $f_2(y) = b$. This gives us $f(x,y) = (f_1(x), f_2(y)) = (a,b)$, where $(x,y) \in W$, so $(a,b) \in f(W)$. Thus, $(a,b) \in f(W)$ for all $(a,b) \in ((M \times N) - (\partial M \times \partial N)) \cap U$, so $f(W) \supseteq ((M \times N) - (\partial M \times \partial N)) \cap U$. Since we also showed that $f(W) \subseteq ((M \times N) - (\partial M \times \partial N)) \cap U$, we conclude that $f(W) = ((M \times N) - (\partial M \times \partial N)) \cap U$, as required.

ii) We will show that $f^{-1} : ((M \times N) - (\partial M \times \partial N)) \cap U \to W$ is continuous. Let any $(a,b) \in ((M \times N) - (\partial M \times \partial N)) \cap U$ be given, and let any $\varepsilon > 0$ be given. Then, we have:

$$(a,b) \in ((M\times N)-(\partial M\times\partial N))\cap U \subseteq (M\times N)\cap U = (M\times N)\cap(U_1''\times U_2) = (M\cap U_1'')\times(N\cap U_2).$$

In other words, $a \in M \cap U_1''$ and $b \in N \cap U_2$. Then, since $f_1^{-1} : M \cap U_1'' \to W_1''$ is continuous, there exists $\delta_1 > 0$ such that for all $a' \in M \cap U_1''$ such that $\left|a' - a\right| < \delta_1$, we obtain $\left|f_1^{-1}(a') - f_1^{-1}(a)\right| < \frac{\varepsilon}{2}$. Since f_2^{-1} is also continuous, there exists $\delta_2 > 0$ such that for all $b' \in N \cap U_2$ such that $\left|b' - b\right| < \delta_2$, we obtain $\left|f_2^{-1}(b') - f_2^{-1}(b)\right| < \frac{\varepsilon}{2}$. Next, let us pick $\delta := \min(\delta_1, \delta_2) > 0$. Then, for all $(a', b') \in ((M \times N) - (\partial M \times \partial N)) \cap U$ such that $\left|(a', b') - (a,b)\right| < \delta$, we obtain:

$$\begin{aligned}
&\left|f^{-1}(a',b') - f^{-1}(a,b)\right| \\
=&\left|(f_1^{-1}(a') - f_1^{-1}(a), f_2^{-1}(b') - f_2^{-1}(b))\right| \\
=&\sqrt{\left|f_1^{-1}(a') - f_1^{-1}(a)\right|^2 + \left|f_2^{-1}(b') - f_2^{-1}(b)\right|^2} \\
<&\sqrt{(\frac{\varepsilon}{2})^2 + \left|f_2^{-1}(b') - f_2^{-1}(b)\right|^2} \qquad \text{(Since } \left|a' - a\right| \leq \left|(a',b') - (a,b)\right| < \delta \leq \delta_1\text{)} \\
<&\sqrt{(\frac{\varepsilon}{2})^2 + (\frac{\varepsilon}{2})^2} \qquad \text{(Since } \left|b' - b\right| \leq \left|(a',b') - (a,b)\right| < \delta \leq \delta_2\text{)} \\
<&\ \varepsilon.
\end{aligned}$$

Thus, for all $(a,b) \in ((M \times N) - (\partial M \times \partial N)) \cap U$ and all $\varepsilon > 0$, we found $\delta > 0$ such that for all $(a', b') \in ((M \times N) - (\partial M \times \partial N)) \cap U$ such that $\left|(a',b') - (a,b)\right| < \delta$, we obtain $\left|f^{-1}(a',b') - f^{-1}(a,b)\right| < \varepsilon$. We conclude that f^{-1} is continuous, as required.

iii) We will show that $\operatorname{rank} f'(a,b) = k + l$ for all $(a,b) \in W$. Indeed, $f'(a,b)$ is the following block matrix:

$$f'(a,b) = \begin{pmatrix} \frac{\partial f_1(a)}{\partial a} & \frac{\partial f_1(a)}{\partial b} \\ \frac{\partial f_2(b)}{\partial a} & \frac{\partial f_2(b)}{\partial b} \end{pmatrix} = \begin{pmatrix} f_1'(a) & 0 \\ 0 & f_2'(b) \end{pmatrix}.$$

Since $f_1'(a)$ is of rank k and $f_2'(b)$ is of rank l, we conclude that $f'(a,b)$ is of rank $k+l$, as required.

Thus, we have shown that all requirements hold if p_1 is outside ∂M.
Case 2: p_2 is outside ∂N. Then, the proof is analogous to Case 1, with the roles of M and N, as well as all of their related objects, being swapped.
Therefore, $(M \times N) - (\partial M \times \partial N)$ satisfies the definition for a $(k+l)$-manifold with boundary. Overall, we obtain that $M \times N$ is the disjoint union of $(M \times N) - (\partial M \times \partial N)$, a $(k+l)$-manifold with boundary, and $\partial M \times \partial N$, a $(k+l-2)$-manifold without boundary, as required. □

Notes on intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. For part (a), since the definition of Γ_f is in the form "the set of all points satisfying an equation", this motivates us to write Γ_f as a zero set for definition (Z). The natural way to do so is to set $g(x, y) = y - f(x)$ so that $\Gamma_f := \{(x, y) \, : \, y - f(x) = 0\}$. Then, it only remains to check that the details for definition (Z) work out.
   For part (b), my idea was that, if $g : \mathbb{R}^m \to \mathbb{R}^n$ is smooth, then the proof above would imply that $\{(x, y) \, : \, x = g(y)\}$ is also a smooth manifold. If the inverse g^{-1} exists but is not smooth, then we would be able to write $\{(x, y) \, : \, x = g(y)\} = \{(x, y) \, : \, y = g^{-1}(x)\} = \Gamma_{g^{-1}}$ and finish the problem. We can make g^{-1} non-smooth by making its derivative diverge to infinity at some point; we make this happen by defining $g(y) = y^3$ so that g is "horizontal" at 0, and then g^{-1} is "vertical" at 0. This gives us the function $f(x) = g^{-1}(x) = x^{1/3}$ that I used in the solution.

2. First, since our definitions for manifolds are entirely local, we cannot expect to prove a global statement directly. Instead, we break up the global statement into local statements, then show that the local statements combine to make the global statement true. In this case, this motivates us to prove that M is "locally of measure 0" (i.e., there is an open neighbourhood U_p around each $p \in M$ such that $U_p \cap M$ is of measure 0), then prove that M is a countable union of these measure-0 pieces so that M is of measure 0.
   To prove that M is "locally of measure 0", we know that M locally looks like $\mathbb{R}^{12}$, which is of measure 0 in $\mathbb{R}^{22}$. Then, since we are given a local diffeomorphism h mapping a piece of M onto $\mathbb{R}^{12}$, we formally relate the "volume" along M with the "volume" along $\mathbb{R}^{12}$ using Change of Variables on h.
   Finally, we face the problem of proving that finitely many of these open neighbourhoods U_p cover M. The solution is to recall that this problem also appeared (in a more general form) in last year's Test 2 rejects (as Question 15). On the sheet of Test 2 rejects, there was also a helpful hint to consider rectangles whose vertices all have rational coordinates, and we can use the hint to finish the problem.

3. Similarly to Question 1, $U(2)$ is written in the form "the set of all points satisfying an equation", so we can try to write $U(2)$ as a zero set for definition (Z). This time, it is still possible, but it is quite messy. Eventually, we reach the step where we show that $g'(A)$ is of rank 4 for all $A \in U(2)$, which was most likely the hardest step in the question. We want to combine equations (5) through (12) to obtain $c_1 = c_2 = c_3 = c_4 = 0$. To figure out which combinations of the equations will help, we first see that c_1 only appears in (5), (6), (9), and (10), with coefficients of $2a_{11}$, $2b_{11}$, $2a_{21}$, and $2b_{21}$. These coefficients remind us of equation (1): $a_{11}^2 + b_{11}^2 + a_{21}^2 + b_{21}^2 = 1$. Then, we can get a nonzero c_1-coefficient using the combination $a_{11} \cdot (5) + b_{11} \cdot (6) + a_{21} \cdot (9) + b_{21} \cdot (10)$, and we obtain $c_1 = 0$. A similar process also helps us prove that $c_2 = 0$, $c_3 = 0$, and $c_4 = 0$.

4. The main idea here is that if $\phi(x_0) = y_0$ such that x_0 is on the boundary of $\mathbb{R}^k_+$ but y_0 is not, then ϕ^{-1} is trying to compress the k-dimensional space around y_0 into the half-space around x_0, which does not have enough room. This motivates us to prove that $(\phi^{-1})'(y_0)$ is not invertible, meaning that ϕ^{-1} can no longer be a diffeomorphism. Other than that, it was also important to be careful with the difference between being "open in $\mathbb{R}^k$" and being "open in $\mathbb{R}^k_+$".

5. If we consider part (b) for a moment, we know that we used definition (C) as a framework for defining manifolds with boundary in lecture. Then, we are forced to that definition for part (b). Intuitively, parts (a) and (b) will have very similar solutions, so that motivates us to use definition (C) for part (a).
   Next, after we decide to use definition (C) for part (a), we need to give local coordinate systems on $M \times N$. There is a natural way to do so: Give some coordinates on M, then give some coordinates on N. After we accordingly define the coordinate map $f : W \to \mathbb{R}^m \times \mathbb{R}^n$ by $f(x, y) := (f_1(x), f_2(y))$, where f_1 is a coordinate map on M and f_2 is a coordinate map on N, it only remains to check in detail that definition (C) is satisfied.
   Next, for part (b), based on the hint on Crowdmark, let us first understand the case when $M = N = [0, 1]$. In this case, we have that $M \times N$ is the square $[0, 1]^2$, which looks like a manifold-with-boundary where the boundary consists of the four edges of the square. However, with a closer look, we see that there are problems at the four vertices of the square. This is because the coordinate maps require us to map the half-space $\{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 \geq 0\}$ onto $[0, 1]^2$, but at the vertices, there is only enough room to map the "quarter-space" $\{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1, x_2 \geq 0\}$. Then, our 2-dimensional manifold-with-boundary would have to exclude the four vertices of the square, corresponding to $\partial M \times \partial N$.
   From this example, we can guess that, in the general case, our $(k + l)$-dimensional manifold-with-boundary should exclude $\partial M \times \partial N$. Indeed, after solving part (a), it is easy to show that $\partial M \times \partial N$ is a $(k + l - 2)$-dimensional manifold without boundary. Then, it remains to show, using a similar proof to part (a) with slightly messier details, that $(M \times N) - (\partial M \times \partial N)$ is a $(k + l)$-dimensional manifold-with-boundary.
   (Remark: For step (ii) of each proof, where I showed that f^{-1} is continuous, I could have simplified the proof greatly by writing $f^{-1}(a, b) = (f_1^{-1}(a), f_2^{-1}(b))$, then concluding that f^{-1} is continuous because f_1^{-1} and f_2^{-1} are continuous.)