MAT257 Assignment 14 (Tangent vectors)

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1. We are given differentiable functions $f, g : \mathbb{R}^n \to \mathbb{R}$. Then, we will show that $d(fg) = fdg + gdf$. Indeed, we can evaluate $d(fg)$ as follows:

$$\begin{aligned}
d(fg) &= \sum_{i=1}^{n} \frac{\partial(fg)}{\partial x_i}\varphi_i \\
&= \sum_{i=1}^{n}(f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i})\varphi_i \qquad \text{(Product rule for partial derivatives)} \\
&= f\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\varphi_i + g\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\varphi_i \\
&= fdg + gdf,
\end{aligned}$$

as required. □

2. We are given a differentiable function $f : \mathbb{R} \to \mathbb{R}$, and we define the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ by $\gamma(t) = (t, f(t))$. Then, we will show that the endpoint of the tangent vector to γ at t lies on the tangent line of f at $(t, f(t))$.
First, we will find the tangent vector to γ at t. Let the coordinate representation of $\gamma(t)$ be $(\gamma^1(t), \gamma^2(t))$. Then, by definition, the tangent vector to γ at t is $((\gamma^1)'(t), (\gamma^2)'(t))_{\gamma(t)}$, which equals:

$$\begin{aligned}((\gamma^1)'(t), (\gamma^2)'(t))_{\gamma(t)} &= (\frac{d}{dt}t, \frac{d}{dt}f(t))_{\gamma(t)} \\ &= (1, f'(t))_{\gamma(t)}.\end{aligned}$$

Then, the endpoint of this tangent vector is $(\gamma^1(t)+1, \gamma^2(t)+f'(t)) = (t+1, f(t)+f'(t))$.
Next, by definition, the tangent line of f at $(t, f(t))$ has the equation $y - f(t) = f'(t)(x-t)$.
We will show that the point $(t+1, f(t)+f'(t))$ is on this line using the following LS-RS proof:

$$\begin{aligned}\text{LS} &= y - f(t) \\ &= (f(t)+f'(t)) - f(t) \\ &= f'(t)\end{aligned} \qquad \begin{aligned}\text{RS} &= f'(t)(x-t) \\ &= f'(t)((t+1)-t) \\ &= f'(t).\end{aligned}$$

Therefore, LS = RS, so the endpoint of the tangent vector to γ at t lies on the tangent line of f at $(t, f(t))$, as required. $\square$

3. We are given a differentiable curve $\gamma : [0,1] \to \mathbb{R}^2$ such that $\left|\gamma(t)\right| = 1$ for all t. Then, we will show that the tangent vector to $\gamma(t)$ at t is perpendicular to $(\gamma(t))_{\gamma(t)}$.

   First, since $\left|\gamma(t)\right| = 1$, we also know $\left|\gamma(t)\right|^2 = 1$. Next, if we split $\gamma(t)$ into its coordinates $(\gamma^1(t), \gamma^2(t))$, it follows that $(\gamma^1(t))^2 + (\gamma^2(t))^2 = 1$. Taking $\frac{d}{dt}$ of both sides, we obtain:

$$\begin{aligned} \frac{d}{dt}[(\gamma^1(t))^2 + (\gamma^2(t))^2] &= \frac{d}{dt}1 \\ 2\gamma^1(t)(\gamma^1)'(t) + 2\gamma^2(t)(\gamma^2)'(t) &= 0 \\ \gamma^1(t)(\gamma^1)'(t) + \gamma^2(t)(\gamma^2)'(t) &= 0. \qquad (*) \end{aligned}$$

   Next, by definition, the tangent vector to $\gamma(t)$ at t is $((\gamma^1)'(t), (\gamma^2)'(t))_{\gamma(t)}$. Then, the inner product of this tangent vector with $\gamma(t)_{\gamma(t)}$ (using the dot product in $T_{\gamma(t)}\mathbb{R}^2$) is:

$$\begin{aligned} \langle((\gamma^1)'(t), (\gamma^2)'(t))_{\gamma(t)}, \gamma(t)_{\gamma(t)}\rangle &= \langle((\gamma^1)'(t), (\gamma^2)'(t))_{\gamma(t)}, (\gamma^1(t), \gamma^2(t))_{\gamma(t)}\rangle \\ &= (\gamma^1)'(t)\gamma^1(t) + (\gamma^2)'(t)\gamma^2(t) \\ &= 0. \qquad \text{(Applying } (*)\text{)} \end{aligned}$$

   Therefore, since this inner product equals zero, the tangent vector to $\gamma(t)$ at t is perpendicular to $\gamma(t)_{\gamma(t)}$, as required. □

4. Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we define a vector field $\operatorname{grad} f$ by:

$$(\operatorname{grad} f)(p) := \frac{\partial}{\partial x_1} f(p)(e_1)_p + \cdots + \frac{\partial}{\partial x_n} f(p)(e_n)_p.$$

Then, if v_p is some other tangent vector at p, we will prove that $D_{v_p} f = \langle (\operatorname{grad} f)(p), v_p \rangle$. First, let the coordinate representation of v_p be $v_p = \sum_{i=1}^n v_i (e_i)_p = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$. Then, we will prove that $D_{v_p} f = \langle (\operatorname{grad} f)(p), v_p \rangle$ using the following LS-RS proof:

$$\begin{aligned} \text{LS} &= D_{v_p} f \\ &= \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} f(p) \end{aligned}$$

$$\begin{aligned} \text{RS} &= \langle (\operatorname{grad} f)(p), v_p \rangle \\ &= \langle \sum_{i=1}^n \frac{\partial}{\partial x_i} f(p)(e_i)_p, \sum_{i=1}^n v_i (e_i)_p \rangle \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} f(p) \cdot v_i \\ &= \text{LS}. \end{aligned}$$

Therefore, $D_{v_p} f = \langle (\operatorname{grad} f)(p), v_p \rangle$, as required.
For a geometric interpretation, we know that $\langle (\operatorname{grad} f)(p), v_p \rangle$ is maximized when v_p points in the same direction as $(\operatorname{grad} f)(p)$. Then, $D_{v_p} f$ is maximized when v_p points in the same direction as $(\operatorname{grad} f)(p)$, so f increases most quickly in the direction of $(\operatorname{grad} f)(p)$, as required. $\square$

Notes on intuition

I do not have much to comment on for this assignment. The problems can mostly be solved by tracing definitions.