MAT257 Assignment 13 (Wedge Products)

(Author's name here)

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1. We will find a good way of identifying $\Lambda^1(\mathbb{R}^3)$ and $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$.
   First, let $(e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1))$ be the standard basis for $\mathbb{R}^3$, and let $(\varphi_1, \varphi_2, \varphi_3)$ be the corresponding dual basis for $\Lambda^1(\mathbb{R}^3)$. Then, since $(\varphi_1, \varphi_2, \varphi_3)$ is a basis for $\Lambda^1(\mathbb{R}^3)$, we can identify $\Lambda^1(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying $\varphi_1, \varphi_2, \varphi_3$ with e_1, e_2, e_3, respectively.
   Next, the standard basis $(\omega_I)_{I \in \underline{3}^2_a}$ for $\Lambda^2(\mathbb{R}^3)$ contains the alternating tensors $\omega_{23} = \varphi_2 \wedge \varphi_3$, $\omega_{13} = \varphi_1 \wedge \varphi_3$, and $\omega_{12} = \varphi_1 \wedge \varphi_2$. Then, $(\varphi_2 \wedge \varphi_3, \varphi_1 \wedge \varphi_3, \varphi_1 \wedge \varphi_2)$ is a basis for $\Lambda^2(\mathbb{R}^3)$. Since $\varphi_3 \wedge \varphi_1 = -\varphi_1 \wedge \varphi_3$, we obtain that $(\varphi_2 \wedge \varphi_3, \varphi_3 \wedge \varphi_1, \varphi_1 \wedge \varphi_2)$ is another basis for $\Lambda^2(\mathbb{R}^3)$. As a result, we can identify $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying $\varphi_2 \wedge \varphi_3, \varphi_3 \wedge \varphi_1, \varphi_1 \wedge \varphi_2$ with e_1, e_2, e_3, respectively.
   Next, under these identifications, the wedge product $\wedge : \Lambda^1(\mathbb{R}^3) \times \Lambda^1(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^3)$ becomes a map $P : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Then, we will check that P is the cross product on $\mathbb{R}^3$.
   Let $(a_1, a_2, a_3), (b_1, b_2, b_3)$ be any two vectors in $\mathbb{R}^3$. Then, by definition, their cross product is:

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

   Additionally, we can compute $P((a_1, a_2, a_3), (b_1, b_2, b_3))$ as follows:

$$\begin{aligned}
&P((a_1, a_2, a_3), (b_1, b_2, b_3)) \\
&= P(a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3) \\
&= (a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3) \wedge (b_1\varphi_1 + b_2\varphi_2 + b_3\varphi_3) \qquad \text{(Applying identification)} \\
&= a_1b_1\varphi_1 \wedge \varphi_1 + a_1b_2\varphi_1 \wedge \varphi_2 + a_1b_3\varphi_1 \wedge \varphi_3 + a_2b_1\varphi_2 \wedge \varphi_1 + a_2b_2\varphi_2 \wedge \varphi_2 + a_2b_3\varphi_2 \wedge \varphi_3 \\
&\quad + a_3b_1\varphi_3 \wedge \varphi_1 + a_3b_2\varphi_3 \wedge \varphi_2 + a_3b_3\varphi_3 \wedge \varphi_3 \qquad \text{(Wedge product is bilinear)} \\
&= 0 + a_1b_2\varphi_1 \wedge \varphi_2 - a_1b_3\varphi_3 \wedge \varphi_1 - a_2b_1\varphi_1 \wedge \varphi_2 + 0 + a_2b_3\varphi_2 \wedge \varphi_3 \\
&\quad + a_3b_1\varphi_3 \wedge \varphi_1 - a_3b_2\varphi_2 \wedge \varphi_3 + 0 \\
&= (a_2b_3 - a_3b_2)\varphi_2 \wedge \varphi_3 + (a_3b_1 - a_1b_3)\varphi_3 \wedge \varphi_1 + (a_1b_2 - a_2b_1)\varphi_1 \wedge \varphi_2 \\
&= (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3 \qquad \text{(Applying identification)} \\
&= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \\
&= (a_1, a_2, a_3) \times (b_1, b_2, b_3).
\end{aligned}$$

   Therefore, P is the cross product on $\mathbb{R}^3$, as required. $\square$

2. For each of the linear maps given below, we will determine whether it is orientation preserving or reversing, assuming the standard orientation for each $\mathbb{R}^n$.

(a) $L_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $L_1(x, y) = (-x, y)$.
First, consider the standard basis $\beta_1 := ((1, 0), (0, 1))$ of $\mathbb{R}^2$. Then, L_1 pushes this basis to the basis $\beta_2 := (L_1(1, 0), L_1(0, 1))$ of $\mathbb{R}^2$. If we use the same orientation for both instances of $\mathbb{R}^2$, then L_1 is orientation preserving if and only if β_1 and β_2 have the same orientation. Moreover, the change of basis matrix between β_2 and β_1 is the matrix representing L_1 (under the standard basis of $\mathbb{R}^2$), so β_1 and β_2 have the same orientation if and only if $\det L_1 > 0$. Thus, L_1 is orientation preserving if $\det L_1 > 0$, and L_1 is orientation reversing if $\det L_1 < 0$.
Next, since $L_1(1, 0) = (-1, 0)$, and since $L_1(0, 1) = (0, 1)$, the matrix representing L_1 (under the standard basis of $\mathbb{R}^2$) is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This matrix has a determinant of $(-1)(1) - (0)(0) = -1 < 0$.
Thus, $\det L_1 < 0$, so L_1 is orientation reversing, as required. $\square$

(b) $L_2 : \mathbb{R}^2 \to \mathbb{R}^2$, $L_2(x, y) = (y, x)$.
Similarly to part (a), L_2 is orientation preserving if $\det L_2 > 0$, and L_2 is orientation reversing if $\det L_2 < 0$. Next, since $L_2(1, 0) = (0, 1)$ and $L_2(0, 1) = (1, 0)$, the matrix representing L_2 (under the standard basis of $\mathbb{R}^2$) is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This matrix has a determinant of $(0)(0) - (1)(1) = -1 < 0$.
Thus, $\det L_2 < 0$, so L_2 is orientation reversing, as required. $\square$

(c) $L_3 : \mathbb{R}^2 \to \mathbb{R}^2$, the counterclockwise rotation by $\frac{2\pi}{7}$.
Similarly to part (a), L_3 is orientation preserving if $\det L_3 > 0$, and L_3 is orientation reversing if $\det L_3 < 0$. Next, since $L_3(1, 0) = (\cos(\frac{2\pi}{7}), \sin(\frac{2\pi}{7}))$ and $L_3(0, 1) = (-\sin(\frac{2\pi}{7}), \cos(\frac{2\pi}{7}))$, the matrix representing L_3 (under the standard basis of $\mathbb{R}^2$) is $\begin{pmatrix} \cos(\frac{2\pi}{7}) & -\sin(\frac{2\pi}{7}) \\ \sin(\frac{2\pi}{7}) & \cos(\frac{2\pi}{7}) \end{pmatrix}$. This matrix has a determinant of:

$$\cos(\frac{2\pi}{7})\cos(\frac{2\pi}{7}) - (-\sin(\frac{2\pi}{7}))\sin(\frac{2\pi}{7}) = \cos^2(\frac{2\pi}{7}) + \sin^2(\frac{2\pi}{7}) = 1 > 0.$$

Thus, $\det L_3 > 0$, so L_3 is orientation preserving, as required. $\square$

(d) $L_4 : \mathbb{R}^2 \to \mathbb{R}^2$, the clockwise rotation by $\frac{2\pi}{7}$.
Similarly to part (a), L_4 is orientation preserving if $\det L_4 > 0$, and L_4 is orientation reversing if $\det L_4 < 0$. Next, since $L_4(1, 0) = (\cos(\frac{2\pi}{7}), -\sin(\frac{2\pi}{7}))$, and since $L_4(0, 1) = (\sin(\frac{2\pi}{7}), \cos(\frac{2\pi}{7}))$, the matrix representing L_4 (under the standard basis of $\mathbb{R}^2$) is $\begin{pmatrix} \cos(\frac{2\pi}{7}) & \sin(\frac{2\pi}{7}) \\ -\sin(\frac{2\pi}{7}) & \cos(\frac{2\pi}{7}) \end{pmatrix}$. This matrix has a determinant of:

$$\cos(\frac{2\pi}{7})\cos(\frac{2\pi}{7}) - \sin(\frac{2\pi}{7})(-\sin(\frac{2\pi}{7})) = \cos^2(\frac{2\pi}{7}) + \sin^2(\frac{2\pi}{7}) = 1 > 0.$$

Thus, $\det L_4 > 0$, so L_4 is orientation preserving, as required. $\square$

(e) $L_5 : \mathbb{R}^2 \to \mathbb{R}^2$, the complex conjugation map $z \mapsto \overline{z}$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ in the standard way.
Similarly to part (a), L_5 is orientation preserving if $\det L_5 > 0$, and L_5 is orientation reversing

if $\det L_5 < 0$. Next, identifying $\mathbb{R}^2$ with $\mathbb{C}$, since $L_5(1,0) = L_5(1+0i) = 1+0i = (1,0)$ and $L_5(0,1) = L_5(0+i) = 0-i = (0,-1)$, the matrix representing L_5 (under the standard basis of $\mathbb{R}^2$) is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This matrix has a determinant of $(1)(-1)-(0)(0) = -1 < 0$. Thus, $\det L_5 < 0$, so L_5 is orientation reversing, as required. $\square$

(f) $L_6 : \mathbb{R}^3 \to \mathbb{R}^3$, $L_6(x,y,z) = (y,z,x)$.
Similarly to part (a), L_6 is orientation preserving if $\det L_6 > 0$, and L_6 is orientation reversing if $\det L_6 < 0$. Next, since $L_6(1,0,0) = (0,0,1)$, $L_6(0,1,0) = (1,0,0)$, and $L_6(0,0,1) = (0,1,0)$, the matrix representing L_6 (under the standard basis of $\mathbb{R}^3$) is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. This matrix has a determinant of:

$$(0)(0)(0)-(0)(1)(0)-(1)(0)(0)+(1)(1)(1)+(0)(0)(0)-(0)(0)(1) = 1 > 0.$$

Thus, $\det L_6 > 0$, so L_6 is orientation preserving, as required. $\square$

(g) $L_7 : \mathbb{R}^n \to \mathbb{R}^n$, $L_7(v) = -v$.
Similarly to part (a), L_7 is orientation preserving if $\det L_7 > 0$, and L_7 is orientation reversing if $\det L_7 < 0$. Next, let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$. Then, L_7 pushes this standard basis to $(L_7e_1, \dots, L_7e_n) = (-e_1, \dots, -e_n)$, so the matrix representing L_7 (under the standard basis of $\mathbb{R}^n$) is $-I_n$, where I_n represents the identity matrix of size n. Then, since $-I_n$ is a diagonal matrix with only (-1)'s on the main diagonal, its determinant is $(-1)^n$. We obtain that $\det L_7 = (-1)^n = 1 > 0$ if n is even, and $\det L_7 = (-1)^n = -1 < 0$ if n is odd. Thus, L_7 is orientation preserving if n is even, and L_7 is orientation reversing if n is odd, as required. $\square$

(h) $L_8 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, $L_8(u,v) = (v,u)$.
Similarly to part (a), L_8 is orientation preserving if $\det L_8 > 0$, and L_8 is orientation reversing if $\det L_8 < 0$. Next, let $(e_1, \dots, e_m)$ and $(f_1, \dots, f_n)$ be the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then, the standard basis of $\mathbb{R}^{n+m}$ is $((e_1,0), \dots, (e_m,0), (0,f_1), \dots, (0,f_n))$, and L_8 pushes this basis to:

$$(L_8(e_1,0), \dots, L_8(e_m,0), L_8(0,f_1), \dots, L_8(0,f_n)) = ((0,e_1), \dots, (0,e_m), (f_1,0), \dots, (f_n,0)).$$

As a result, the matrix representing L_8 (under the standard basis of $\mathbb{R}^{n+m}$) is the block matrix $\begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$. Next, if $\sigma \in S_k$ denotes the permutation:

$$(\sigma(1) = n+1, \sigma(2) = n+2 \dots, \sigma(m) = n+m, \sigma(m+1) = 1, \sigma(m+2) = 2, \dots, \sigma(n+m) = n),$$

then the above matrix only has nonzero entries at locations of the form $(i, \sigma(i))$. Moreover, all nonzero entries in the matrix are 1. Thus, its determinant is $(-1)^\sigma$.
Next, to compute $(-1)^\sigma$, we will show how to perform multiple transpositions in a row to construct σ. Beginning with the identity permutation:

$$(1, 2, \dots, n, n+1, \dots, n+m),$$

we first need to transport n to position $n+m$ because $\sigma(n+m) = n$. To do this, we use m transpositions, where each transposition shifts n one position to the right. This results in the following permutation:

$$(1, 2, \ldots, n-1, n+1, \ldots, n+m, n).$$

Similarly, we apply the same procedure to transport $n-1, \ldots, 1$, in descending order. This results in the permutation:

$$(n+1, \ldots, n+m, 1, 2, \ldots, n),$$

which is the permutation σ that we needed. Since we needed to transport n elements, and since each transportation procedure required m transpositions, we obtain that σ is a composition of mn transpositions. As a result, $\det L_8 = (-1)^\sigma = (-1)^{mn}$. Moreover, $(-1)^{mn} = 1 > 0$ if m or n is even, and $(-1)^{mn} = -1 < 0$ otherwise. Thus, L_8 is orientation preserving if m or n is even, and L_8 is orientation reversing otherwise, as required. $\square$

3. We are given an n-dimensional vector space V and an isomorphism $\chi : \Lambda^n(V) \to \mathbb{R}$. We are also given an integer $0 \le k \le n$. Then, we will construct an isomorphism $\psi_k : \Lambda^{n-k}(V) \to (\Lambda^k(V))^*$ that does not depend on any additional choices.
First, let us define a map $\psi_k : \Lambda^{n-k}(V) \to (\Lambda^k(V))^*$ by $(\psi_k(\lambda))(\eta) = \chi(\lambda \wedge \eta)$ for all $\lambda \in \Lambda^{n-k}(V)$ and all $\eta \in \Lambda^k(V)$. Note that ψ_k does not depend on any additional choices. (However, we will have to choose bases later on in our proof.)
Next, for this definition to be valid, we must check that $\psi_k(\lambda) \in (\Lambda^k(V))^*$ for all $\lambda \in \Lambda^{n-k}(V)$; in other words, we must check that $\psi_k(\lambda) : \Lambda^k(V) \to \mathbb{R}$ is linear. Indeed, we will show that $(\psi_k(\lambda))(c_1\eta_1 + c_2\eta_2) = c_1(\psi_k(\lambda))(\eta_1) + c_2(\psi_k(\lambda))(\eta_2)$ for all $\eta_1, \eta_2 \in \Lambda^k(V)$ and all $c_1, c_2 \in \mathbb{R}$:

$$\begin{aligned}(\psi_k(\lambda))(c_1\eta_1 + c_2\eta_2) &= \chi(\lambda \wedge (c_1\eta_1 + c_2\eta_2)) \\ &= \chi(c_1\lambda \wedge \eta_1 + c_2\lambda \wedge \eta_2) \qquad \text{(Wedge product is bilinear)} \\ &= c_1\chi(\lambda \wedge \eta_1) + c_2\chi(\lambda \wedge \eta_2) \qquad (\chi \text{ is linear}) \\ &= c_1(\psi_k(\lambda))(\eta_1) + c_2(\psi_k(\lambda))(\eta_2).\end{aligned}$$

Thus, $\psi_k(\lambda) \in (\Lambda^k(V))^*$ for all $\lambda \in \Lambda^{n-k}(V)$, as required.
Next, we will show that ψ_k is linear by showing that $\psi_k(c_1\lambda_1 + c_2\lambda_2) = c_1\psi_k(\lambda_1) + c_2\psi_k(\lambda_2)$ for all $\lambda_1, \lambda_2 \in \Lambda^{n-k}(V)$ and all $c_1, c_2 \in \mathbb{R}$. Indeed, for all $\eta \in \Lambda^k(V)$, we have:

$$\begin{aligned}(\psi_k(c_1\lambda_1 + c_2\lambda_2))(\eta) &= \chi((c_1\lambda_1 + c_2\lambda_2) \wedge \eta) \\ &= \chi(c_1\lambda_1 \wedge \eta + c_2\lambda_2 \wedge \eta) \qquad \text{(Wedge product is bilinear)} \\ &= c_1\chi(\lambda_1 \wedge \eta) + c_2\chi(\lambda_2 \wedge \eta) \qquad (\chi \text{ is linear}) \\ &= c_1(\psi_k(\lambda_1))(\eta) + c_2(\psi_k(\lambda_2))(\eta) \\ &= (c_1\psi_k(\lambda_1) + c_2\psi_k(\lambda_2))(\eta)\end{aligned}$$

Since $(\psi_k(c_1\lambda_1 + c_2\lambda_2))(\eta) = (c_1\psi_k(\lambda_1) + c_2\psi_k(\lambda_2))(\eta)$ for all $\eta \in \Lambda^k(V)$, we obtain that $\psi_k(c_1\lambda_1 + c_2\lambda_2) = c_1\psi_k(\lambda_1) + c_2\psi_k(\lambda_2)$. Therefore, ψ_k is linear, as required.
Next, we will show that ψ_k is injective. Since ψ_k is linear, it suffices to show that $\psi_k(\lambda)$ is nonzero for all nonzero $\lambda \in \Lambda^{n-k}(V)$.
Let λ be any nonzero element of $\Lambda^{n-k}(V)$. Then, there exist vectors $v_1, \dots, v_{n-k} \in V$ such that $\lambda(v_1, \dots, v_{n-k}) \neq 0$. Moreover, we claim that $v_1, \dots, v_{n-k}$ must be linearly independent. If we assume for contradiction that they are linearly dependent, then there exists $1 \le i \le n-k$ such that v_i is a linear combination of $v_1, \dots, v_{i-1}$, so we can write $v_i = a_1v_1 + \dots + a_{i-1}v_{i-1}$ for some $a_1, \dots, a_{i-1} \in \mathbb{R}$. Then, we would obtain:

$$\begin{aligned}&\lambda(v_1, \dots, v_i, \dots, v_{n-k}) \\ &= \lambda(v_1, \dots, a_1v_1 + \dots + a_{i-1}v_{i-1}, \dots, v_{n-k}) \\ &= a_1\lambda(v_1, \dots, v_1, \dots, v_{n-k}) + \dots + a_{i-1}\lambda(v_1, \dots, v_{i-1}, \dots, v_{n-k}) \qquad (\lambda \text{ is } (n-k)\text{-linear}) \\ &= a_1 \cdot 0 + \dots + a_{i-1} \cdot 0 \qquad (\lambda \text{ kills repetitions}) \\ &= 0,\end{aligned}$$

contradicting our condition that $\lambda(v_1, \dots, v_{n-k}) \neq 0$. Thus, by contradiction, $v_1, \dots, v_{n-k}$ are linearly independent.
Next, we can extend $(v_1, \dots, v_{n-k})$ to a basis $(v_1, \dots, v_n)$ of V. Then, let $(\varphi_1, \dots, \varphi_n)$ be the corresponding dual basis for $\Lambda^1(V)$, and let $\{\omega_I\}_{I \in \underline{n}^k_a}$ be the corresponding basis for $\Lambda^k(V)$. Let us pick $I = (n-k+1, \dots, n) \in \underline{n}^k_a$, and let us also pick $\eta := \omega_I \in \Lambda^k(V)$. Then, we can compute

$(\lambda \wedge \eta)(v_1, \ldots, v_n)$ as follows:

$$\begin{aligned}
&(\lambda \wedge \eta)(v_1, \ldots, v_n) \\
= &\sum_{\substack{\sigma \in S_n \\ \sigma(1) < \cdots < \sigma(n-k) \\ \sigma(n-k+1) < \cdots < \sigma(n)}} (-1)^\sigma \lambda(v_{\sigma(1)}, \ldots, v_{\sigma(n-k)}) \eta(v_{\sigma(n-k+1)}, \ldots, v_{\sigma(n)}) \qquad \text{(Definition of wedge product)} \\
= &\sum_{\substack{\sigma \in S_n \\ \sigma(1) < \cdots < \sigma(n-k) \\ \sigma(n-k+1) < \cdots < \sigma(n)}} (-1)^\sigma \lambda(v_{\sigma(1)}, \ldots, v_{\sigma(n-k)}) \omega_I(v_{\sigma(n-k+1)}, \ldots, v_{\sigma(n)}).
\end{aligned}$$

Since $\sigma(n-k+1) < \cdots < \sigma(n)$, we have $(\sigma(n-k+1), \ldots, \sigma(n)) \in \underline{n}_a^k$. As a result, $\omega_I(v_{\sigma(n-k+1)}, \ldots, v_{\sigma(n)}) = 1$ if $(\sigma(n-k+1), \ldots, \sigma(n)) = I = (n-k+1, \ldots, n)$, and $\omega_I(v_{\sigma(n-k+1)}, \ldots, v_{\sigma(n)}) = 0$ otherwise. Thus, a term in the summation above is nonzero only if $\sigma(n-k+1) = n-k+1, \sigma(n-k+2) = n-k+2, \ldots, \sigma(n) = n$. Since we also require $\sigma(1), \ldots, \sigma(n-k)$ to be in increasing order, the only permutation σ that satisfies this is the identity permutation, which we will denote id. Then, we obtain:

$$\begin{aligned}
(\lambda \wedge \eta)(v_1, \ldots, v_n) &= \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \cdots < \sigma(n-k) \\ \sigma(n-k+1) < \cdots < \sigma(n)}} (-1)^\sigma \lambda(v_{\sigma(1)}, \ldots, v_{\sigma(n-k)}) \omega_I(v_{\sigma(n-k+1)}, \ldots, v_{\sigma(n)}) \\
&= (-1)^{\mathrm{id}} \lambda(v_{\mathrm{id}(1)}, \ldots, v_{\mathrm{id}(n-k)}) \omega_I(v_{\mathrm{id}(n-k+1)}, \ldots, v_{\mathrm{id}(n)}) \\
&= 1 \cdot \lambda(v_1, \ldots, v_{n-k}) \omega_I(v_{n-k+1}, \ldots, v_n) \\
&= 1 \cdot \lambda(v_1, \ldots, v_{n-k}) \cdot 1 \\
&= \lambda(v_1, \ldots, v_{n-k}) \\
&\neq 0.
\end{aligned}$$

Since $(\lambda \wedge \eta)(v_1, \ldots, v_n)$ is nonzero, we obtain that $\lambda \wedge \eta$ is nonzero. Then, since $\chi : \Lambda^n(V) \to \mathbb{R}$ is an isomorphism, $\chi(\lambda \wedge \eta)$ is also nonzero. As a result, $(\psi_k(\lambda))(\eta) = \chi(\lambda \wedge \eta)$ is also nonzero. Finally, since we found $\eta \in \Lambda^k(V)$ such that $(\psi_k(\lambda))(\eta)$ is nonzero, we conclude that $\psi_k(\lambda)$ is nonzero. This is true for all nonzero $\lambda \in \Lambda^{n-k}(V)$, so ψ_k is an injective linear map, as desired.
Next, the dimension of $\Lambda^{n-k}(V)$ is $\binom{n}{n-k}$. Additionally, the dimension of $(\Lambda^k(V))^*$ equals the dimension of $\Lambda^k(V)$, which is $\binom{n}{k} = \binom{n}{n-k}$. Then, $\Lambda^{n-k}(V)$ has the same dimension as $(\Lambda^k(V))^*$. Therefore, since $\psi_k : \Lambda^{n-k}(V) \to (\Lambda^k(V))^*$ is an injective linear map, it follows that ψ_k is an isomorphism, as required. $\square$

4. We are given an n-dimensional vector space V with a basis (v_i) and a dual basis (φ_j). We are also given an integer k with $0 \le k \le n$. Finally, we are given an inner product on $\Lambda^k(V)$ defined by $\langle \omega_I, \omega_J \rangle = \delta_{IJ}$, and we define $\omega_n \in \Lambda^n(V)$ by $\omega_n := \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$.

(a) We will show that there is a unique isomorphism $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ that satisfies $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$ for all $\lambda, \eta \in \Lambda^k(V)$.
Step 0: We will introduce some useful notation.
First, for all $I \in \underline{n}_a^k$, let I^c be the ascending sequence in $\underline{n}_a^{n-k}$ containing all integers in $\{1, \ldots, n\}$ which do not appear in I. Also, let $\sigma_I \in S_n$ be the permutation on $\{1, \ldots, n\}$ that maps $1, \ldots, k$ to the elements of I in ascending order and maps $k+1, \ldots, n$ to the elements of I^c in ascending order.
Step 1: We will show that if $*$ exists, then it must be unique. To do this, we will compute the value of $*\omega_I$ for every basic element ω_I of $\Lambda^k(V)$.
First, $*\omega_I$ can be written in terms of basic elements of $\Lambda^{n-k}(V)$ as $\sum_{J \in \underline{n}_a^{n-k}} a_J \omega_J$, where each a_J is a real coefficient. Then, for all $I' \in \underline{n}_a^k$, we obtain:

$$\begin{aligned}
&(\omega_{I'} \wedge (*\omega_I))(v_1, \ldots, v_n) \\
&= (\omega_{I'} \wedge \sum_{J \in \underline{n}_a^{n-k}} a_J \omega_J)(v_1, \ldots, v_n) \\
&= \sum_{J \in \underline{n}_a^{n-k}} a_J (\omega_{I'} \wedge \omega_J)(v_1, \ldots, v_n) \qquad \text{(Wedge product is bilinear)} \\
&= \sum_{J \in \underline{n}_a^{n-k}} \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \cdots < \sigma(k) \\ \sigma(k+1) < \cdots < \sigma(n)}} a_J (-1)^\sigma \omega_{I'}(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \omega_J (v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}) \qquad \text{(Definition of wedge product)}
\end{aligned}$$

Since I' and $(\sigma(1), \ldots, \sigma(k))$ are both elements of $\underline{n}_a^k$, we obtain that $\omega_{I'}(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 1$ if $I' = (\sigma(1), \ldots, \sigma(k))$, and $\omega_{I'}(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 0$ otherwise. Then, a term in the above summation is nonzero only if σ maps $1, \ldots, k$ to the elements of I' in ascending order. Next, since $(\sigma(k+1), \ldots, \sigma(n))$ must also be in increasing order, we obtain that σ maps $k+1, \ldots, n$ to elements of $(I')^c$ in increasing order, so $\sigma = \sigma_{I'}$. Finally, since $(\sigma(k+1), \ldots, \sigma(n)) = (I')^c$ and J are both in $\underline{n}_a^{n-k}$, we obtain $\omega_J(v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}) = 1$ if $J = (I')^c$, and we obtain $\omega_J(v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}) = 0$ otherwise. Overall, a term in the above summation is nonzero only if $J = (I')^c$ and $\sigma = \sigma_{I'}$, so we obtain:

$$\begin{aligned}
(\omega_{I'} \wedge (*\omega_I))(v_1, \ldots, v_n) &= a_{(I')^c} (-1)^{\sigma_{I'}} \omega_{I'}(v_{I'}) \omega_{(I')^c}(v_{(I')^c}) \\
&= a_{(I')^c} (-1)^{\sigma_{I'}} \cdot 1 \cdot 1 \\
&= a_{(I')^c} (-1)^{\sigma_{I'}}. \qquad (1)
\end{aligned}$$

Next, we need $\omega_{I'} \wedge (*\omega_I) = \langle \omega_{I'}, \omega_I \rangle \omega_n$, so we obtain:

$$\begin{aligned}
(\omega_{I'} \wedge (*\omega_I))(v_1, \ldots, v_n) &= \langle \omega_{I'}, \omega_I \rangle \omega_n(v_1, \ldots, v_n) \\
&= \langle \omega_{I'}, \omega_I \rangle \cdot 1 \\
&= \langle \omega_{I'}, \omega_I \rangle.
\end{aligned}$$

It follows from (1) that $a_{(I')^c}(-1)^{\sigma_{I'}} = \langle \omega_{I'}, \omega_I \rangle$, so $a_{(I')^c} = (-1)^{\sigma_{I'}} \langle \omega_{I'}, \omega_I \rangle$. Also, we are given that $\langle \omega_{I'}, \omega_I \rangle = 1$ if $I' = I$ and $\langle \omega_{I'}, \omega_I \rangle = 0$ otherwise. Then, $a_{I^c} = (-1)^{\sigma_I} \cdot 1 = (-1)^{\sigma_I}$, and

$a_{(I')^c} = (-1)^{\sigma_{I'}} \cdot 0 = 0$ for all $I' \neq I$. Additionally, $(I')^c$ takes the value of every sequence in $\underline{n}_a^{n-k}$ exactly once as I' ranges across $\underline{n}_a^k$; indeed, for all $J \in \underline{n}_a^{n-k}$, we have $(I')^c = J$ if and only if I' is the unique ascending sequence containing all integers in $\{1, \dots, n\}$ excluded from J. As a result, $a_J = 0$ for all $J \neq I^c$. Therefore, $*\omega_I$ must equal $\sum_{J \in \underline{n}_a^{n-k}} a_J \omega_J = a_{I^c}\omega_{I^c} = (-1)^{\sigma_I}\omega_{I^c}$. Since $*\omega_I$ is uniquely determined for all $I \in \underline{n}_a^k$, and since $\{\omega_I\}_{I \in \underline{n}_a^k}$ is a basis for $\Lambda^k(V)$, it follows that $*$ must be unique if it exists, as required.

Step 2: We will show that $*$ exists by constructing $*$.

Let $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ be the linear map defined by $*\omega_I := (-1)^{\sigma_I}\omega_{I^c}$ for all basic elements ω_I of $\Lambda^k(V)$. First, as shown above, I^c takes the value of every sequence in $\underline{n}_a^{n-k}$ exactly once as I ranges across $\underline{n}_a^k$. As a result, $\{\omega_{I^c}\}_{I \in \underline{n}_a^k} = \{\omega_J\}_{J \in \underline{n}_a^{n-k}}$ is the standard basis for $\Lambda^{n-k}(V)$, so $\{*\omega_I\}_{I \in \underline{n}_a^k}$ is also a basis for $\Lambda^{n-k}(V)$. This shows that $*$ is an isomorphism, as required. (Formally, $*$ is surjective because $\{*\omega_I\}_{I \in \underline{n}_a^k}$ spans $\Lambda^{n-k}(V)$, then $*$ is an isomorphism because $\Lambda^k(V)$ and $\Lambda^{n-k}(V)$ have the same dimension of $\binom{n}{k} = \binom{n}{n-k}$.)

Next, we must check that $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$ for all $\lambda, \eta \in \Lambda^k(V)$. Since both sides are bilinear in terms of (λ, η), it suffices to check this equality for all λ, η in the standard basis of $\Lambda^k(V)$. Then, let $\lambda = \omega_{I'}$ and $\eta = \omega_I$, where $I', I \in \underline{n}_a^k$. We have two cases: $I' = I$ or $I' \neq I$.

Case 1: $I' = I$. Then, we have $*\omega_I = (-1)^{\sigma_I}\omega_{I^c} = (-1)^{\sigma_{I'}}\omega_{(I')^c}$, so $*\omega_I$ has an $\omega_{(I')^c}$-coefficient of $(-1)^{\sigma_{I'}}$. According to (1), it follows that:

$$(\omega_{I'} \wedge (*\omega_I))(v_1, \dots, v_n) = (-1)^{\sigma_{I'}}(-1)^{\sigma_{I'}} = 1.$$

Next, since $\omega_{I'} \wedge (*\omega_I) \in \Lambda^n(V)$, and since $\{\omega_n\}$ is a basis of the 1-dimensional space $\Lambda^n(V)$, we can write $\omega_{I'} \wedge (*\omega_I) = c\omega_n$ for some real constant c. Then, we obtain:

$$\begin{aligned}(\omega_{I'} \wedge (*\omega_I))(v_1, \dots, v_n) &= c\omega_n(v_1, \dots, v_n) \\ 1 &= c \cdot 1 \\ 1 &= c.\end{aligned}$$

Since $c = 1$, it follows that $\omega_{I'} \wedge (*\omega_I) = \omega_n$. Moreover, since $I' = I$, we have $\langle \omega_{I'}, \omega_I \rangle = 1$, so $\langle \omega_{I'}, \omega_I \rangle \omega_n = \omega_n$. Thus, $\omega_{I'} \wedge (*\omega_I) = \langle \omega_{I'}, \omega_I \rangle \omega_n$ if $I' = I$, as required.

Case 2: $I' \neq I$. Then, $*\omega_I = (-1)^{\sigma_I}\omega_{I^c}$, where $I^c \neq (I')^c$, so $*\omega_I$ has an $\omega_{(I')^c}$-coefficient of 0. According to (1), it follows that:

$$(\omega_{I'} \wedge (*\omega_I))(v_1, \dots, v_n) = 0 \cdot (-1)^{\sigma_{I'}} = 0.$$

Next, since $\omega_{I'} \wedge (*\omega_I) \in \Lambda^n(V)$, and since $\{\omega_n\}$ is a basis of the 1-dimensional space $\Lambda^n(V)$, we can write $\omega_{I'} \wedge (*\omega_I) = c\omega_n$ for some real constant c. Then, we obtain:

$$\begin{aligned}(\omega_{I'} \wedge (*\omega_I))(v_1, \dots, v_n) &= c\omega_n(v_1, \dots, v_n) \\ 0 &= c \cdot 1 \\ 0 &= c.\end{aligned}$$

Since $c = 0$, it follows that $\omega_{I'} \wedge (*\omega_I) = 0\omega_n$. Moreover, since $I' \neq I$, we have $\langle \omega_{I'}, \omega_I \rangle = 0$, so $\langle \omega_{I'}, \omega_I \rangle \omega_n = 0\omega_n$. Thus, $\omega_{I'} \wedge (*\omega_I) = \langle \omega_{I'}, \omega_I \rangle \omega_n$ if $I' \neq I$, as required.

Overall, we proved by cases that $\omega_{I'} \wedge (*\omega_I) = \langle \omega_{I'}, \omega_I \rangle \omega_n$ for all $I', I \in \underline{n}_a^k$. As a result, our construction for the linear isomorphism $*$ satisfies $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$.

Therefore, there is a unique isomorphism $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ that satisfies $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$ for all $\lambda, \eta \in \Lambda^k(V)$, as required. $\square$

(b) First, given $n = 3$ and $k = 1$, we will compute $*\omega_1$, $*\omega_2$, and ω_3.
In part (a), when we proved that $*$ is unique, we also proved the formula $*\omega_I = (-1)^{\sigma_I}\omega_{I^c}$ for all $I \in \underline{n}_a^k$. First, if $I = (1) \in \underline{3}_a^1$, then we have $I^c = (2,3)$. We get $\sigma_I(1) = 1$ since $I = (1)$, and we also get $\sigma_I(2) = 2$ and $\sigma_I(3) = 3$ since $I^c = (2,3)$. Then, σ_I is the identity permutation, so we obtain $(-1)^{\sigma_I} = 1$. Overall, if $I = (1)$, then $*\omega_1 = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = \omega_{23}$.
Next, if $I = (2) \in \underline{3}_a^1$, then we have $I^c = (1,3)$. We get $\sigma_I(1) = 2$ since $I = (2)$, and we also get $\sigma_I(2) = 1$ and $\sigma_I(3) = 3$ since $I^c = (1,3)$. Then, σ_I is the permutation that transposes 1 and 2, so $(-1)^{\sigma_I} = -1$. Overall, if $I = (2)$, then $*\omega_2 = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = -\omega_{13}$.
Next, if $I = (3) \in \underline{3}_a^1$, then we have $I^c = (1,2)$. We get $\sigma_I(1) = 3$ since $I = (3)$, and we also get $\sigma_I(2) = 1$ and $\sigma_I(3) = 2$ since $I^c = (1,2)$. Then, σ_I is the permutation that transposes 1 and 2 to get the permutation $(2,1,3)$, followed by transposing 2 and 3 to get the permutation $(3,1,2)$. As a result, $(-1)^{\sigma_I} = (-1)^2 = 1$. Overall, if $I = (3)$, then $*\omega_3 = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = \omega_{12}$.
Our results for $n = 3$ and $k = 1$ are summarized in the table below:

ω_I	$*\omega_I$
ω_1	ω_{23}
ω_2	$-\omega_{13}$
ω_3	ω_{12}

Next, given $n = 4$ and $k = 2$, we will compute $*\omega_{12}$, $*\omega_{13}$, $*\omega_{14}$, $*\omega_{23}$, $*\omega_{24}$, and $*\omega_{34}$.
First, if $I = (1,2) \in \underline{4}_a^2$, then we have $I^c = (3,4)$. We get $\sigma_I(1) = 1$ and $\sigma_I(2) = 2$ since $I = (1,2)$, and we also get $\sigma_I(3) = 3$ and $\sigma_I(4) = 4$ since $I^c = (3,4)$. Then, σ_I is the identity permutation, so $(-1)^{\sigma_I} = 1$. Overall, if $I = (1,2)$, then $*\omega_{12} = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = \omega_{34}$.
Next, if $I = (1,3) \in \underline{4}_a^2$, then we have $I^c = (2,4)$. We get $\sigma_I(1) = 1$ and $\sigma_I(2) = 3$ since $I = (1,3)$, and we also get $\sigma_I(3) = 2$ and $\sigma_I(4) = 4$ since $I^c = (2,4)$. Then, σ_I is the permutation that transposes 2 and 3, so $(-1)^{\sigma_I} = -1$. Overall, if $I = (1,3)$, then we obtain $*\omega_{13} = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = -\omega_{24}$.
Next, if $I = (1,4) \in \underline{4}_a^2$, then we have $I^c = (2,3)$. We get $\sigma_I(1) = 1$ and $\sigma_I(2) = 4$ since $I = (1,4)$, and we also get $\sigma_I(3) = 2$ and $\sigma_I(4) = 3$ since $I^c = (2,3)$. Then, σ_I is the permutation that transposes 2 and 3 to get the permutation $(1,3,2,4)$, followed by transposing 3 and 4 to get the permutation $(1,4,2,3)$. As a result, $(-1)^{\sigma_I} = (-1)^2 = 1$. Overall, if $I = (1,4)$, then $*\omega_{14} = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = \omega_{23}$.
Next, if $I = (2,3) \in \underline{4}_a^2$, then we have $I^c = (1,4)$. We get $\sigma_I(1) = 2$ and $\sigma_I(2) = 3$ since $I = (2,3)$, and we also get $\sigma_I(3) = 1$ and $\sigma_I(4) = 4$ since $I^c = (1,4)$. Then, σ_I is the permutation that transposes 1 and 3 to get the permutation $(3,2,1,4)$, followed by transposing 2 and 3 to get the permutation $(2,3,1,4)$. As a result, $(-1)^{\sigma_I} = (-1)^2 = 1$. Overall, if $I = (2,3)$, then $*\omega_{23} = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = \omega_{14}$.
Next, if $I = (2,4) \in \underline{4}_a^2$, then we have $I^c = (1,3)$. We get $\sigma_I(1) = 2$ and $\sigma_I(2) = 4$ since $I = (2,4)$, and we also get $\sigma_I(3) = 1$ and $\sigma_I(4) = 3$ since $I^c = (1,3)$. Then, σ_I is the permutation that transposes 1 and 3 to get the permutation $(3,2,1,4)$, followed by transposing 2 and 3 to get the permutation $(2,3,1,4)$, followed by transposing 3 and 4 to get the permutation $(2,4,1,3)$. As a result, $(-1)^{\sigma_I} = (-1)^3 = -1$. Overall, if $I = (2,4)$, then $*\omega_{24} = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = -\omega_{13}$.
Next, if $I = (3,4) \in \underline{4}_a^2$, then we have $I^c = (1,2)$. We get $\sigma_I(1) = 3$ and $\sigma_I(2) = 4$ since $I = (3,4)$, and we also get $\sigma_I(3) = 1$ and $\sigma_I(4) = 2$ since $I^c = (1,2)$. Then, σ_I is the permutation that transposes 1 and 3, followed by transposing 2 and 4, so $(-1)^{\sigma_I} = (-1)^2 = 1$. Overall, if $I = (3,4)$, then $*\omega_{34} = *\omega_I = (-1)^{\sigma_I}\omega_{I^c} = \omega_{12}$.

Our results for $n = 4$ and $k = 2$ are summarized in the table below:

ω_I	$*\omega_I$
ω_{12}	ω_{34}
ω_{13}	$-\omega_{24}$
ω_{14}	ω_{23}
ω_{23}	ω_{14}
ω_{24}	$-\omega_{13}$
ω_{34}	ω_{12}

$\square$

(c) We will show that $* \circ *$, which is a composition $\Lambda^k(V) \to \Lambda^{n-k}(V) \to \Lambda^k(V)$, is equal to $(-1)^{k(n-k)}\,\mathrm{id}$, where id denotes the identity map of $\Lambda^k(V)$.
First, since $* \circ *$ and $(-1)^{k(n-k)}\,\mathrm{id}$ are both linear, it suffices to show that they agree on the standard basis of $\Lambda^k(V)$. Then, it suffices to show that $(* \circ *)\omega_I = (-1)^{k(n-k)}\omega_I$ for all $I \in \underline{n}^k_a$.
First, we can evaluate $(* \circ *)\omega_I$ as follows:

$$\begin{aligned}
(* \circ *)\omega_I &= *(*\omega_I) \\
&= *((-1)^{\sigma_I}\omega_{I^c}) \\
&= (-1)^{\sigma_I}(*\omega_{I^c}) \\
&= (-1)^{\sigma_I}((-1)^{\sigma_{I^c}}\omega_{(I^c)^c}) \\
&= (-1)^{\sigma_I}(-1)^{\sigma_{I^c}}\omega_I \\
&= (-1)^{\sigma_I^{-1}}(-1)^{\sigma_{I^c}}\omega_I \\
&= (-1)^{\sigma_I^{-1}\circ\sigma_{I^c}}\omega_I
\end{aligned}$$

Next, we will examine the behaviour of the permutation $\sigma_I^{-1} \circ \sigma_{I^c}$. For all $1 \le i \le n-k$, we have that σ_{I^c} maps i to $(I^c)_i$, the i^{th} element of I^c. Meanwhile, σ_I maps $i+k$ to $(I^c)_i$ because $k+1 \le i+k \le n$, so σ_I^{-1} maps $(I^c)_i$ to $i+k$. Overall, $(\sigma_I^{-1} \circ \sigma_{I^c})(i) = \sigma_I^{-1}((I^c)_i) = i+k$ if $1 \le i \le n-k$. Next, for all $n-k+1 \le i \le n$, we have that σ_{I^c} maps i to $I_{i-(n-k)}$, the $(i-(n-k))^{\text{th}}$ element of I. Meanwhile, σ_I maps $i-(n-k)$ to $I_{i-(n-k)}$ because $1 \le i-(n-k) \le k$, so σ_I^{-1} maps $I_{i-(n-k)}$ to $i-(n-k)$. Overall, $(\sigma_I^{-1} \circ \sigma_{I^c})(i) = \sigma_I^{-1}(I_{i-(n-k)}) = i-(n-k)$ if $n-k+1 \le i \le n$.
Now, to compute $(-1)^{\sigma_I^{-1}\circ\sigma_{I^c}}$, we will show how to perform multiple transpositions in a row to obtain $\sigma_I^{-1} \circ \sigma_{I^c}$. Beginning with the identity permutation:

$$(1, 2, \ldots, k, k+1, \ldots, n),$$

we first need to transport k to position n because $(\sigma_I^{-1} \circ \sigma_{I^c})(n) = k$. To do this, we use $n-k$ transpositions, where each transposition shifts k one position to the right. This results in the following permutation:

$$(1, 2, \ldots, k-1, k+1, \ldots, n, k).$$

Similarly, we apply the same procedure to transport $k-1, \ldots, 1$, in descending order. This results in the permutation:

$$(k+1, \ldots, n, 1, 2, \ldots, k),$$

which is the permutation $\sigma_I^{-1} \circ \sigma_{I^c}$ that we needed. Since we needed to transport k elements, where each transportation procedure required $n-k$ transpositions, we obtain that $\sigma_I^{-1} \circ \sigma_{I^c}$ is a composition of $k(n-k)$ transpositions. As a result, $(-1)^{\sigma_I^{-1} \circ \sigma_{I^c}} = (-1)^{k(n-k)}$, so we obtain $(* \circ *)\omega_I = (-1)^{\sigma_I^{-1} \circ \sigma_{I^c}} \omega_I = (-1)^{k(n-k)} \omega_I$. Therefore, since $(* \circ *)\omega_I = (-1)^{k(n-k)} \omega_I$ for all basic elements ω_I of $\Lambda^k(V)$, we conclude that $(* \circ *)\omega_I = (-1)^{k(n-k)}\,\mathrm{id}$, as required. $\square$

Notes on intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. First, it was quite intuitive to identify $\Lambda^1(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying φ_1, φ_2, and φ_3 with e_1, e_2, and e_3, respectively, since this was the simplest choice possible. Next, we can "work backwards" to choose how to identify $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$. In other words, for all $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{R}^3$, we first compare $(a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3) \wedge (b_1\varphi_1 + b_2\varphi_2 + b_3\varphi_3)$ with $(a_1, a_2, a_3) \times (b_1, b_2, b_3)$, and this comparison tells us how to identify $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$.

2. The key idea was to treat the matrix representing each L_i as a change of basis matrix as L_i pushes a basis of $\mathbb{R}^n$ to another basis of $\mathbb{R}^n$. This tells us that L_i is orientation preserving if $\det L_i > 0$, and L_i is orientation reversing if $\det L_i < 0$. Afterward, we can finish by simply computing the determinant of each linear map.

3. First, to define ψ_k, we must define a linear map $\psi_k(\lambda) \in (\Lambda^k(V))^*$ for all $\lambda \in \Lambda^{n-k}(V)$, which means that we must define $(\psi_k(\lambda))(\eta) \in \mathbb{R}$ for all $\lambda \in \Lambda^{n-k}(V)$ and all $\eta \in \Lambda^k(V)$. We also wish to use the given isomorphism χ, which requires us to input an alternating n-tensor. Intuitively, this n-tensor should be $\lambda \wedge \eta$ (or $\eta \wedge \lambda$), which leads to the definition $(\psi_k(\lambda))(\eta) := \chi(\lambda \wedge \eta)$.
   Next, we must prove that ψ_k satisfies the required properties. It is easy to check that ψ_k is linear and that $\psi_k(\lambda)$ is linear for all $\lambda \in \Lambda^{n-k}(V)$, so the main challenge is to prove that ψ_k is invertible. To do so, we use the standard linear algebra trick of proving that $\psi_k(\lambda) \neq 0$ for all nonzero λ. We need to find η such that $(\psi_k(\lambda))(\eta) = \chi(\lambda \wedge \eta) \neq 0$, which is equivalent to $\lambda \wedge \eta$ being nonzero since χ is an isomorphism. To construct η, it would be convenient to have a basis for V, so we pick one by picking $v_1, \ldots, v_{n-k}$ such that $\lambda(v_1, \ldots, v_{n-k}) \neq 0$, proving that $v_1, \ldots, v_{n-k}$ are linearly independent, then extending to a basis $v_1, \ldots, v_n$. Intuitively, this process helps us ensure that λ has a nonzero $\omega_{(1,\ldots,n-k)}$-coefficient. Then, we find a matching η by picking $\eta = \omega_{(n-k+1,\ldots,n)}$. After computing that $(\lambda \wedge \eta)(v_1, \ldots, v_n) \neq 0$, we are done.

4. For part (a), our solution is motivated by the proof done in lecture for the unique existence of the wedge product. First, to prove uniqueness of $*$, we use the given conditions to compute the value of $*\omega_I$ for all $I \in \underline{n}^k_a$, thus showing that $*$ maps every element of $\Lambda^k(V)$ to exactly one possible value. After computing the values of each $*\omega_I$, we can also use them to obtain an explicit construction for $*$. Finally, we perform calculations using this construction to prove that there exists $*$ satisfying the required conditions.
   Next, we solve part (b) using the formulas we obtained in part (a).
   Next, for part (c), we begin computing $* \circ *$ by computing that $(* \circ *)\omega_I = (-1)^{\sigma_I^{-1} \circ \sigma_{I^c}} \omega_I$. Then, we must compute $(-1)^{\sigma_I^{-1} \circ \sigma_{I^c}}$. We see that σ_{I^c} maps $(1, \ldots, n-k)$ to I^c, which σ_I^{-1} maps to $(k+1, \ldots, n)$. Moreover, σ_{I^c} maps $(n-k+1, \ldots, n)$ to I, which σ_I^{-1} maps to $(1, \ldots, k)$. Intuitively, $\sigma_I^{-1} \circ \sigma_{I^c}$ breaks $(1, \ldots, n)$ into two chunks of size k and $n-k$, and then it swaps them. Next, the key idea is that this swap can also be performed via transpositions by shifting elements one-by-one. This requires $k(n-k)$ transpositions, so we get $(-1)^{\sigma_I^{-1} \circ \sigma_{I^c}} \omega_I = (-1)^{k(n-k)} \omega_I$, and we are done.