MAT257 Assignment 12 (Alternating tensors)

(Author's name here)

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1. We are given the following functions from $\mathbb{R}^4 \times \mathbb{R}^4$ to $\mathbb{R}$:

$$f(x,y) := x_1y_2 - x_2y_1 + x_1y_1,$$

$$g(x,y) := x_1y_3 - x_3y_2,$$

$$h(x,y) := (x_1)^3(y_2)^3 - (x_2)^3(y_1)^3.$$

We will determine whether or not these functions are alternating tensors.

First, we claim that f is not an alternating tensor. Consider the vectors $u = (1,0,0,0)$ and $v = (1,0,0,0)$. Then, we can compare $f(u,v)$ with $f(v,u)$ as follows:

$$\begin{aligned} f(u,v) &= u_1v_2 - u_2v_1 + u_1v_1 \\ &= 1\cdot 0 - 0\cdot 1 + 1\cdot 1 \\ &= 1 \end{aligned} \qquad \begin{aligned} f(v,u) &= v_1u_2 - v_2u_1 + v_1u_1 \\ &= 1\cdot 0 - 0\cdot 1 + 1\cdot 1 \\ &= 1 \end{aligned}$$

Since $f(u,v) \neq -f(v,u)$, we conclude that f is not an alternating tensor, as required.

Next, we claim that g is not an alternating tensor. Consider the vectors $u = (1,0,0,0)$ and $v = (0,0,1,0)$. Then, we can compare $g(u,v)$ with $g(v,u)$ as follows:

$$\begin{aligned} g(u,v) &= u_1v_3 - u_3v_2 \\ &= 1\cdot 1 - 0\cdot 0 \\ &= 1 \end{aligned} \qquad \begin{aligned} g(v,u) &= v_1u_3 - v_3u_2 \\ &= 0\cdot 0 - 1\cdot 0 \\ &= 0 \end{aligned}$$

Since $g(u,v) \neq -g(v,u)$, we conclude that g is not an alternating tensor, as required.

Next, we claim that h is not an alternating tensor because h is not bilinear. Consider the vectors $u = (1,0,0,0)$ and $v = (0,1,0,0)$. Then, we can compare $h(u,v)$ with $h(2u,v)$ as follows:

$$\begin{aligned} h(u,v) &= (u_1)^3(v_2)^3 - (u_2)^3(v_1)^3 \\ &= 1^3\cdot 1^3 - 0^3\cdot 0^3 \\ &= 1 \end{aligned} \qquad \begin{aligned} h(2u,v) &= (2u_1)^3(v_2)^3 - (2u_2)^3(v_1)^3 \\ &= 2^3\cdot 1^3 - 0^3\cdot 0^3 \\ &= 8 \end{aligned}$$

Since $h(2u,v) \neq 2h(u,v)$, we conclude that h is not bilinear, so h is also not an alternating tensor, as required. $\square$

2. We are given that the determinant $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$, when viewed as a function of n column vectors of length n, is an alternating tensor. Then, we will write it in terms of the elementary alternating tensors ω_I, where $I \in \underline{n}_a^n$, assuming the standard basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$.
First, the only element of $\underline{n}_a^n$ is $(1, 2, \ldots, n)$ because an increasing sequence containing n distinct elements from $\{1, \ldots, n\}$ must contain all elements from $\{1, \ldots, n\}$ in increasing order. Then, let us denote $I_0 := (1, 2, \ldots, n)$. We will show that $\det = \omega_{I_0}$.
In lecture, we proved that two alternating tensors are equal if and only if they agree on arguments of the form v_I, where $I \in \underline{n}_a^n$. Thus, it suffices to prove that $\det(v_{I_0}) = \omega_{I_0}(v_{I_0})$. We also proved in lecture that $\omega_I(v_I) = 1$ for all $I \in \underline{n}_a^n$, so $\omega_{I_0}(v_{I_0}) = 1$. Then, it suffices to prove that $\det(v_{I_0}) = 1$, which we will prove below:

$$\begin{aligned}\det(v_{I_0}) &= \det(e_1, \ldots, e_n) \\ &= \det I_n \\ &= 1,\end{aligned}$$

where I_n denotes the identity matrix of size n. Since $\det(v_{I_0}) = 1 = \omega_{I_0}(v_{I_0})$, we conclude that $\det = \omega_{I_0}$, as required. $\square$

3. We are given a linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ represented by a matrix $A = (a_{i,j}) \in M_{n\times m}(\mathbb{R})$ relative to the standard basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and the standard basis $(f_1, \dots, f_m)$ of $\mathbb{R}^m$. Given an elementary alternating k-tensor ω_I on $\mathbb{R}^n$, where $I = (i_1, \dots, i_k) \in \underline{n}^k_a$, we will write $L^*\omega_I$ as a linear combination $\sum_{J\in \underline{m}^k_a} c_J\omega_J$ of the elementary alternating k-tensors ω_J on $\mathbb{R}^m$.
First, let us verify that $L^*\omega_I$ is an alternating k-tensor on $\mathbb{R}^m$. For all $v_1, \dots, v_k \in \mathbb{R}^m$ and all $\sigma \in S_k$, we have:

$$\begin{aligned}((L^*\omega_I) \circ \sigma^*)(v_1, \dots, v_k) &= (L^*\omega_I)(v_{\sigma(1)}, \dots, v_{\sigma(k)})\\ &= \omega_I(Lv_{\sigma(1)}, \dots, Lv_{\sigma(k)})\\ &= (\omega_I \circ \sigma^*)(Lv_1, \dots, Lv_k)\\ &= (-1)^\sigma\omega_I(Lv_1, \dots, Lv_k) \qquad \text{(Since } \omega_I \text{ is alternating)}\\ &= (-1)^\sigma(L^*\omega_I)(v_1, \dots, v_k).\end{aligned}$$

Since $((L^*\omega_I)\circ\sigma^*)(v_1, \dots, v_k) = (-1)^\sigma(L^*\omega_I)(v_1, \dots, v_k)$ for all $v_1, \dots, v_k \in \mathbb{R}^m$ and all $\sigma \in S_k$, we conclude that $L^*\omega_I$ is alternating, as desired.
Next, for all $J = (j_1, \dots, j_k) \in \underline{m}^k_a$, let us define the matrix $A_{I,J}$ as follows:

$$A_{I,J} := \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_k}\\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_k}\\ \vdots & \vdots & \ddots & \vdots\\ a_{i_k,j_1} & a_{i_k,j_2} & \cdots & a_{i_k,j_k}\end{pmatrix}.$$

Then, we will prove that $L^*\omega_I = \sum_{J\in\underline{m}^k_a} \det(A_{I,J})\omega_J$.
First, we proved in lecture that two alternating tensors on $\mathbb{R}^m$ are equal if and only if they agree on arguments of the form $v_J \in (\mathbb{R}^m)^k$, where $J \in \underline{m}^k_a$. Then, to prove that $L^*\omega_I$ equals $\sum_{J\in\underline{m}^k_a} \det(A_{I,J})\omega_J$, it suffices to show that $(L^*\omega_I)(v_J) = (\sum_{J'\in\underline{m}^k_a} \det(A_{I,J'})\omega_{J'})(v_J)$ for all $J = (j_1, \dots, j_k) \in \underline{m}^k_a$. First, we can evaluate $(L^*\omega_I)(v_J)$ as follows:

$$\begin{aligned}(L^*\omega_I)(v_J) &= (L^*\omega_I)(f_{j_1}, \dots, f_{j_k})\\ &= \omega_I(Lf_{j_1}, \dots, Lf_{j_k})\\ &= \omega_I(Af_{j_1}, \dots, Af_{j_k})\\ &= \sum_{\sigma\in S_k}(-1)^\sigma(\varphi_I \circ \sigma^*)(Af_{j_1}, \dots, Af_{j_k}) \qquad \text{(Definition of } \omega_I)\\ &= \sum_{\sigma\in S_k}(-1)^\sigma\varphi_I(Af_{j_{\sigma(1)}}, \dots, Af_{j_{\sigma(k)}})\\ &= \sum_{\sigma\in S_k}(-1)^\sigma\varphi_I(\sum_{i=1}^n a_{i,j_{\sigma(1)}}e_i, \dots, \sum_{i=1}^n a_{i,j_{\sigma(k)}}e_i)\\ &= \sum_{\sigma\in S_k}(-1)^\sigma\varphi_{i_1}(\sum_{i=1}^n a_{i,j_{\sigma(1)}}e_i)\varphi_{i_2}(\sum_{i=1}^n a_{i,j_{\sigma(2)}}e_i)\cdots\varphi_{i_k}(\sum_{i=1}^n a_{i,j_{\sigma(k)}}e_i)\\ &= \sum_{\sigma\in S_k}(-1)^\sigma a_{i_1,j_{\sigma(1)}}a_{i_2,j_{\sigma(2)}}\cdots a_{i_k,j_{\sigma(k)}}\\ &= \sum_{\sigma\in S_k}(-1)^\sigma(A_{I,J})_{1,\sigma(1)}(A_{I,J})_{2,\sigma(2)}\cdots(A_{I,J})_{k,\sigma(k)}\\ &= \det(A_{I,J}),\end{aligned}$$

where $(A_{I,J})_{r,c}$ denotes the entry in the r^{th} row and c^{th} column of $A_{I,J}$.
We also showed in lecture that, for all $J_1, J_2 \in \underline{m}_a^k$, we have $\omega_{J_1}(v_{J_2}) = 1$ if $J_1 = J_2$ and $\omega_{J_1}(v_{J_2}) = 0$ otherwise. Then, we can evaluate $(\sum_{J' \in \underline{m}_a^k} \det(A_{I,J'})\omega_{J'})(v_J)$ as follows:

$$\begin{aligned}
(\sum_{J' \in \underline{m}_a^k} \det(A_{I,J'})\omega_{J'})(v_J) &= \sum_{J' \in \underline{m}_a^k} \det(A_{I,J'})\omega_{J'}(v_J) \\
&= \det(A_{I,J})\omega_J(v_J) + \sum_{\substack{J' \in \underline{m}_a^k \\ J' \neq J}} \det(A_{I,J'})\omega_{J'}(v_J) \\
&= \det(A_{I,J}) + 0 \\
&= \det(A_{I,J}) \\
&= (L^*\omega_I)(v_J).
\end{aligned}$$

Therefore, since $(\sum_{J' \in \underline{m}_a^k} \det(A_{I,J'})\omega_{J'})(v_J) = (L^*\omega_I)(v_J)$ for all $J \in \underline{m}_a^k$, we conclude that $L^*\omega_I = \sum_{J \in \underline{m}_a^k} \det(A_{I,J})\omega_J$. In other words, we have written $L^*\omega_I$ as a linear combination $\sum_{J \in \underline{m}_a^k} c_J\omega_J$ with coefficients $\boxed{c_J = \det(A_{I,J})}$, as required. $\square$

4. We will develop a theory of symmetric tensors. First, let V be an n-dimensional vector space, and let k be any positive integer. Then, we define a *symmetric* k-tensor on V to be a k-tensor on V whose outputs are unchanged if the arguments are permuted. Formally, we define a k-tensor T on V to be *symmetric* if we have:

$$T(v_1, \ldots, v_k) = T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for all arguments $v_1, \ldots, v_k \in V$ and all permutations $\sigma \in S_k$.
Finally, let us define $S^k(V)$ to be the set of all symmetric k-tensors on V.
Now, we will show that $S^k(V)$ is a vector subspace of $\mathcal{T}^k(V)$. First, $S^k(V)$ is nonempty because it contains the zero tensor. Indeed, if we pick $T_0 \in \mathcal{T}^k(V)$ defined by $T_0(v_1, \ldots, v_k) = 0$, then we obtain:

$$T_0(v_1, \ldots, v_k) = 0 = T_0(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for all $v_1, \ldots, v_k \in V$ and all $\sigma \in S_k$, which shows that $T_0 \in S^k(V)$. Next, to show that $S^k(V)$ is closed under addition and scalar multiplication, it suffices to show that $\lambda S + T \in S^k(V)$ for all $S, T \in S^k(V)$ and all $\lambda \in \mathbb{R}$. Indeed, for all $v_1, \ldots, v_k \in V$ and all $\sigma \in S_k$, we have:

$$\begin{aligned}
&(\lambda S + T)(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \\
&= \lambda S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) + T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \\
&= \lambda S(v_1, \ldots, v_k) + T(v_1, \ldots, v_k) \qquad \text{(Since } S, T \text{ are symmetric)} \\
&= (\lambda S + T)(v_1, \ldots, v_k).
\end{aligned}$$

Since $(\lambda S + T)(v_1, \ldots, v_k) = (\lambda S + T)(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$, we find that $\lambda S + T$ is also symmetric, so $S^k(V)$ is closed under addition and scalar multiplication. Thus, since $S^k(V)$ is nonempty and closed under addition and scalar multiplication, $S^k(V)$ is a vector subspace of $\mathcal{T}^k(V)$, as required.
Next, we will construct a basis for $S^k(V)$. To begin, let $(e_1, \ldots, e_n)$ be a basis for V, and let $\{\varphi_I : I \in \underline{n}^k\}$ be the corresponding basis for $\mathcal{T}^k(V)$. Also, let us define $\underline{n}^k_{nd}$ to be the set of all non-decreasing sequences of length k with elements in $\{1, \ldots, n\}$, as was done in lecture. Then, for all $I \in \underline{n}^k_{nd}$, let us define the tensor $\sigma_I \in \mathcal{T}^k(V)$ by:

$$\sigma_I := \sum_{\sigma \in S_k} \varphi_I \circ \sigma^*.$$

Then, for all $I \in \underline{n}^k_{nd}$, we will prove that σ_I is symmetric.
First, permuting arguments of σ_I is equivalent to composing σ_I with τ^*, where $\tau \in S_k$ is a permutation. Then, to prove that the outputs of σ_I are unchanged if its arguments are permuted, we need to prove that $\sigma_I = \sigma_I \circ \tau^*$ for all permutations τ:

$$\begin{aligned}
\sigma_I \circ \tau^* &= \left(\sum_{\sigma \in S_k} \varphi_I \circ \sigma^* \right) \circ \tau^* \qquad \text{(Definition of } \sigma_I\text{)} \\
&= \sum_{\sigma \in S_k} (\varphi_I \circ \sigma^* \circ \tau^*) \\
&= \sum_{\sigma \in S_k} (\varphi_I \circ (\tau \circ \sigma)^*)
\end{aligned}$$

In this summation, we claim that $\tau \circ \sigma$ takes on the value of every permutation in S_k exactly once as σ ranges over all permutations in S_k. Indeed, for all $\sigma' \in S_k$, we claim that $\tau \circ \sigma = \sigma'$ if and only if $\sigma = \tau^{-1} \circ \sigma'$. For the "$\Rightarrow$" direction, if $\tau \circ \sigma = \sigma'$, then we can apply τ^{-1} to both sides to obtain $\tau^{-1} \circ \tau \circ \sigma = \tau^{-1} \circ \sigma'$, so $\sigma = \tau^{-1} \circ \sigma'$. For the "$\Leftarrow$" direction, if $\sigma = \tau^{-1} \circ \sigma'$, then we can compute $\tau \circ \sigma = \tau \circ \tau^{-1} \circ \sigma' = \sigma'$. Thus, $\tau \circ \sigma = \sigma'$ if and only if σ equals the permutation $\tau^{-1} \circ \sigma'$, so $\tau \circ \sigma$ equals every permutation in S_k exactly once as σ ranges over all permutations in S_k, as desired. Then, the above summation can be evaluated as:

$$\begin{aligned}\sigma_I \circ \tau^* &= \sum_{\sigma \in S_k} (\varphi_I \circ (\tau \circ \sigma)^*) \\ &= \sum_{\sigma' \in S_k} (\varphi_I \circ \sigma') \\ &= \sigma_I.\end{aligned}$$

Therefore, $\sigma_I = \sigma_I \circ \tau^*$ for all permutations τ, so $\sigma_I \in S^k(V)$, as required.
Next, we will prove that:

$$\{\sigma_I : I \in \underline{n}^k_{nd}\}$$

is a basis for $S^k(V)$. We will do this using the four following steps:
Step 1: For all $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_k) \in \underline{n}^k_{nd}$, we will prove that $\sigma_I(v_J) > 0$ if $I = J$ and $\sigma_I(v_J) = 0$ otherwise.
First, suppose that $I = J$. Then, we obtain:

$$\begin{aligned}\sigma_I(v_J) &= \sum_{\sigma \in S_k} \varphi_I(\sigma^*(v_J)) \\ &= \varphi_I(\mathrm{id}^*(v_J)) + \sum_{\substack{\sigma \in S_k \\ \sigma \neq \mathrm{id}}} \varphi_I(\sigma^*(v_J)) \qquad \text{(Where id denotes the identity permutation)} \\ &= \varphi_I(e_{j_{\mathrm{id}(1)}}, \ldots, e_{j_{\mathrm{id}(1)}}) + \sum_{\substack{\sigma \in S_k \\ \sigma \neq \mathrm{id}}} \varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) \\ &= \varphi_I(e_{j_1}, \ldots, e_{j_k}) + \sum_{\substack{\sigma \in S_k \\ \sigma \neq \mathrm{id}}} \varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) \\ &= \varphi_I(e_J) + \sum_{\substack{\sigma \in S_k \\ \sigma \neq \mathrm{id}}} \varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}).\end{aligned}$$

Since $I = J$, we have $\varphi_I(e_J) = 1$. Moreover, for all $\sigma \in S_k$ such that $\sigma \neq \mathrm{id}$, we have $\varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) = 1$ if $I = (j_{\sigma(1)}, \ldots, j_{\sigma(k)})$, and $\varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) = 0$ otherwise. Either way, $\varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}})$ is nonnegative. Thus, the entire summation is at least 1, so $\sigma_I(v_J) > 0$ if $I = J$, as desired.
Next, suppose that $I \neq J$. Then, we obtain:

$$\begin{aligned}\sigma_I(v_J) &= \sum_{\sigma \in S_k} \varphi_I(\sigma^*(v_J)) \\ &= \sum_{\sigma \in S_k} \varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}).\end{aligned}$$

For each $\sigma \in S_k$, we have the following two cases: $(j_{\sigma(1)}, \ldots, j_{\sigma(k)})$ is non-decreasing, or it is not non-decreasing. If $(j_{\sigma(1)}, \ldots, j_{\sigma(k)})$ is non-decreasing, then it equals J, so it does not equal I

since $I \neq J$. Otherwise, if $(j_{\sigma(1)}, \ldots, j_{\sigma(k)})$ is not non-decreasing, then it does not equal I since I is non-decreasing. Either way, $(j_{\sigma(1)}, \ldots, j_{\sigma(k)}) \neq I$, so $\varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) = 0$. Since this is true for all $\sigma \in S_k$, we obtain:

$$\sigma_I(v_J) = \sum_{\sigma \in S_k} \varphi_I(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) = \sum_{\sigma \in S_k} 0 = 0,$$

as desired.

Step 2: For all $S, T \in S^k(V)$, we will prove that $S = T$ if and only if, for all $I \in \underline{n}^k_{nd}$, we have $S(v_I) = T(v_I)$.

The "$\Rightarrow$" direction is clear. For the "$\Leftarrow$" direction, suppose that $S(v_I) = T(v_I)$ for all $I \in \underline{n}^k_{nd}$. First, we will show that $S(v_J) = T(v_J)$ for all $J \in \underline{n}^k$. Given $J = (j_1, \ldots, j_k) \in \underline{n}^k$, it is possible to sort the elements of J into non-decreasing order using some permutation $\sigma \in S_k$ to form a sequence $(j_{\sigma(1)}, \ldots, j_{\sigma(k)}) \in \underline{n}^k_{nd}$. Then, we obtain:

$$\begin{aligned} S(v_J) &= S(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) && \text{(Since } S \text{ is symmetric)} \\ &= T(e_{j_{\sigma(1)}}, \ldots, e_{j_{\sigma(k)}}) && \text{(Since } (j_{\sigma(1)}, \ldots, j_{\sigma(k)}) \in \underline{n}^k_{nd}\text{)} \\ &= T(v_J). && \text{(Since } S \text{ is symmetric)} \end{aligned}$$

Thus, $S(v_J) = T(v_J)$ for all $J \in \underline{n}^k$.

Next, for all $(v_1, \ldots, v_k) \in V^k$, we can write each v_i in the form $v_i = \sum_{j_i=1}^n a_{i,j_i} e_{j_i}$, where a_{i,j_i} are real coefficients. Then, we obtain:

$$\begin{aligned} S(v_1, \ldots, v_k) &= S(\sum_{j_1=1}^n a_{1,j_1} e_{j_1}, \sum_{j_2=1}^n a_{2,j_2} e_{j_2}, \ldots, \sum_{j_k=1}^n a_{k,j_k} e_{j_k}) \\ &= \sum_{j_1=1}^n a_{1,j_1} S(e_{j_1}, \sum_{j_2=1}^n a_{2,j_2} e_{j_2}, \ldots, \sum_{j_k=1}^n a_{k,j_k} e_{j_k}) && \text{(Since } S \text{ is } k\text{-linear)} \\ &= \cdots \\ &= \sum_{j_1=1}^n a_{1,j_1} \sum_{j_2=1}^n a_{2,j_2} \cdots \sum_{j_k=1}^n a_{k,j_k} S(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) \\ &= \sum_{j_1=1}^n a_{1,j_1} \cdots \sum_{j_k=1}^n a_{k,j_k} S(v_{(j_1,\ldots,j_k)}) \\ &= \sum_{j_1=1}^n a_{1,j_1} \cdots \sum_{j_k=1}^n a_{k,j_k} T(v_{(j_1,\ldots,j_k)}) && \text{(Since } S(v_J) = T(v_J) \text{ for all } J \in \underline{n}^k\text{)} \\ &= \sum_{j_1=1}^n a_{1,j_1} \cdots \sum_{j_k=1}^n a_{k,j_k} T(e_{j_1}, \ldots, e_{j_k}) \\ &= \cdots \\ &= T(\sum_{j_1=1}^n a_{1,j_1} e_{j_1}, \ldots, \sum_{j_k=1}^n a_{k,j_k} e_{j_k}) \\ &= T(v_1, \ldots, v_k). \end{aligned}$$

Therefore, $S(v_1, \ldots, v_k) = T(v_1, \ldots, v_k)$ for all $v_1, \ldots, v_k \in V$, so $S = T$. This concludes our proof that $S = T$ if and only if $S(v_I) = T(v_I)$ for all $I \in \underline{n}^k_{nd}$, as desired.

Step 3: We will prove that $\{\sigma_I : I \in \underline{n}^k_{nd}\}$ spans $S^k(V)$.
Let T be any element of $S^k(V)$. For all $I \in \underline{n}^k_{nd}$, let us define the real coefficient $c_I := \frac{T(v_I)}{\sigma_I(v_I)}$; we can divide by $\sigma_I(v_I)$ because we proved in Step 1 that $\sigma_I(v_I) > 0$. Then, we claim that $T = \sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I$. Indeed, for all $J \in \underline{n}^k_{nd}$, we have:

$$\begin{aligned}
(\sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I)(v_J) &= \sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I(v_J) \\
&= c_J\sigma_J(v_J) \qquad \text{(Since } \sigma_I(v_J) = 0 \text{ if } I \neq J\text{)} \\
&= \frac{T(v_J)}{\sigma_J(v_J)} \cdot \sigma_J(v_J) \qquad \text{(Definition of } c_J\text{)} \\
&= T(v_J).
\end{aligned}$$

Since $(\sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I)(v_J) = T(v_J)$ for all $J \in \underline{n}^k_{nd}$, it follows from Step 2 that $\sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I = T$. Thus, we have expressed T as a linear combination of elements from $\{\sigma_I : I \in \underline{n}^k_{nd}\}$. Since this is possible for all $T \in S^k(V)$, we conclude that $\{\sigma_I : I \in \underline{n}^k_{nd}\}$ spans $S^k(V)$, as desired.
Step 4: We will prove that $\{\sigma_I : I \in \underline{n}^k_{nd}\}$ is linearly independent.
Suppose that we have real coefficients c_I for all $I \in \underline{n}^k_{nd}$ such that $\sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I = 0$. Then, for all $J \in \underline{n}^k_{nd}$, we obtain:

$$(\sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I)(v_J) = 0$$
$$\sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I(v_J) = 0$$

Applying Step 1, we have $\sigma_I(v_J) > 1$ if $I = J$ and $\sigma_I(v_J) = 0$ otherwise, so the above summation equals $c_J \cdot \sigma_J(v_J)$. As a result, $c_J \cdot \sigma_J(v_J) = 0$, so $c_J = \frac{0}{\sigma_J(v_J)} = 0$ for all $J \in \underline{n}^k_{nd}$. Then, since the coefficients c_I must all equal zero whenever $\sum_{I\in\underline{n}^k_{nd}} c_I\sigma_I = 0$, we conclude that $\{\sigma_I : I \in \underline{n}^k_{nd}\}$ is linearly independent, as desired.
Overall, since $\{\sigma_I : I \in \underline{n}^k_{nd}\}$ spans $S^k(V)$ and is linearly independent, we conclude that it is a basis for $S^k(V)$, as required.
Finally, the number of elements in this basis equals the number of sequences I in $\underline{n}^k_{nd}$. In lecture, we proved that $\underline{n}^k_{nd}$ has $\binom{n+k-1}{k}$ elements. Therefore, the dimension of $S^k(V)$ is $\binom{n+k-1}{k}$, as required. $\square$

Notes on intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. Our first task is to show that f is not alternating by constructing an example of $(u, v) \in \mathbb{R}^4 \times \mathbb{R}^4$ such that $f(u, v) \neq -f(v, u)$. To construct (u, v), the main idea is to find a term in $f(x, y)$ of the form $x_i y_j$ without a corresponding term $-x_j y_i$, then to set $u = e_i$ and $y = e_j$. In this case, $f(x, y)$ contains the term $x_1 y_1$ without a corresponding $-x_1 y_1$, so we set $u = v = e_1$. This same strategy also works to show that g is not alternating: Since $g(x, y)$ contains $x_1 y_3$ without a corresponding $-x_3 y_1$, we pick $u = e_1$ and $v = e_3$. Finally, the function h was a "trick question" because although h is alternating (i.e., $h(x, y) = -h(y, x)$), h is still not an alternating tensor because it is not bilinear.

2. The main idea is that $\Lambda^n(\mathbb{R}^n)$ has a dimension of 1 and only has one basic element ω_{I_0}, where $I_0 = (1, \ldots, n) \in \underline{n}^n_a$. Then, $\det$ is a multiple of ω_{I_0}. After a simple computation shows that $\det(v_{I_0}) = 1 = \omega_{I_0}(v_{I_0})$, this motivates our hypothesis that $\det = \omega_{I_0}$. Then, we can use the same computation to prove our hypothesis.

3. The main idea is to find the ω_J-coefficient of $L^*\omega_I$ by computing $(L^*\omega_I)(v_J)$. Once we reach the step:
$$(L^*\omega_I)(v_J) = \sum_{\sigma \in S_k} (-1)^\sigma a_{i_1, j_{\sigma(1)}} a_{i_2, j_{\sigma(2)}} \cdots a_{i_k, j_{\sigma(k)}},$$
we observe that the terms $a_{i_1, j_{\sigma(1)}}, \ldots, a_{i_k, j_{\sigma(k)}}$ always comes from the same submatrix of A containing Rows $i_1, \ldots, i_k$ and Columns $j_1, \ldots, j_k$. This motivates us to define $A_{I,J}$ to be this submatrix. Once we further compute that $(L^*\omega_I)(v_J) = \det(A_{I,J})$, this motivates our hypothesis that $L^*\omega_I = \sum_{J \in \underline{m}^k_a} \det(A_{I,J})\omega_J$. Then, we can prove our hypothesis using the same computation.

4. The main idea is to follow the same procedure that was used in lecture to develop a theory of alternating tensors – in fact, this strategy helps us to write almost all of the solution. The new tasks for our solution are to define σ_I and $\underline{n}^k_{nd}$. To motivate our definition for σ_I, we begin with the definition $\omega_I = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ \sigma^*$ for basic alternating tensors, and then we remove the $(-1)^\sigma$ so that the tensor is symmetric instead of alternating. Next, to motivate the definition of $\underline{n}^k_{nd}$, we first review the motivation for the definition of $\underline{n}^k_a$. When constructing basic elements ω_I for $\Lambda^k(V)$, we first restricted to our attention to non-decreasing sequences I because if I were not non-decreasing, then it could be sorted into a non-decreasing sequence I', and we could write ω_I in terms of $\omega_{I'}$. Then, we restricted our attention further to strictly increasing sequences I because if I had repeating entries, we would get $\omega_I = 0$ because ω_I kills repetitions. Transitioning to symmetric tensors σ_I, we still need to restrict our attention to non-decreasing sequences I for the same reason as above. However, σ_I no longer kills repetitions, so we no longer need I to be strictly increasing. This leads us to our definition of $\underline{n}^k_{nd}$, as desired.