MAT257 Assignment 11 (k-tensors)

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1. We are given a vector space V of dimension n, as well as the map $\iota : V \to (V^*)^*$ defined by:

$$\iota(v)(\phi) = \phi(v)$$

for all $v \in V$ and $\phi \in V^*$.

(a) If $(v_1, \ldots, v_n)$ is a basis of V and $(\phi_1, \ldots, \phi_n)$ is its dual basis, then we will show that $(\iota(v_1), \ldots, \iota(v_n))$ is the dual basis of $(\phi_1, \ldots, \phi_n)$ and conclude that $(\iota(\phi_1), \ldots, \iota(\phi_n))$ is a basis of $(V^*)^*$.
First, let any two indices $1 \leq i, j \leq n$ be given. Then, we are given that $\iota(v_i)(\phi_j) = \phi_j(v_i)$. Moreover, since $(\phi_1, \ldots, \phi_n)$ is the dual basis of $(v_1, \ldots, v_n)$, we know that $\phi_j(v_i) = 1$ if $i = j$ and $\phi_j(v_i) = 0$ otherwise. Then, $\iota(v_i)(\phi_j) = 1$ if $i = j$, and $\iota(v_i)(\phi_j) = 0$ otherwise. Therefore, $(\iota(v_1), \ldots, \iota(v_n))$ is the dual basis of $(\phi_1, \ldots, \phi_n)$, as required.
Finally, by Axler 3.98, it follows that $(\iota(v_1), \ldots, \iota(v_n))$ is a basis of $(V^*)^*$, as required. □

(b) We will show that ι is an isomorphism from the vector space V to the vector space $(V^*)^*$.
Step 1: We will check that ι is a linear map.
To do this, we need to check that $\iota(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \iota(v_1) + \lambda_2 \iota(v_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and all $v_1, v_2 \in V$. To check that they are the same map, we can compute for all $\phi \in V^*$ that:

$$\begin{aligned}
\iota(\lambda_1 v_1 + \lambda_2 v_2)(\phi) &= \phi(\lambda_1 v_1 + \lambda_2 v_2) \\
&= \lambda_1 \phi(v_1) + \lambda_2 \phi(v_2) && (\phi \text{ is linear}) \\
&= \lambda_1 \iota(v_1)(\phi) + \lambda_2 \iota(v_2)(\phi) \\
&= (\lambda_1 \iota(v_1) + \lambda_2 \iota(v_2))(\phi).
\end{aligned}$$

Since we computed that $\iota(\lambda_1 v_1 + \lambda_2 v_2)(\phi) = (\lambda_1 \iota(v_1) + \lambda_2 \iota(v_2))(\phi)$ for all $\phi \in V^*$, we obtain $\iota(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \iota(v_1) + \lambda_2 \iota(v_2)$. Since this is true for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and all $v_1, v_2 \in V$, we conclude that ι is a linear map, as required.
Step 2: We will check that ι is injective, and we will conclude that ι is an isomorphism between vector spaces.
Assume for contradiction that ι is not injective. Then, since ι is a linear map, there exists some nonzero element $v \in V$ such that $\iota(v) = 0$. Using the basis $(v_1, \ldots, v_n)$ of V, we can write v as $v = c_1 v_1 + \cdots + c_n v_n$, where $c_1, \ldots, c_n \in \mathbb{R}$ are not all zero. Then, we obtain:

$$\begin{aligned}
\iota(c_1 v_1 + \cdots + c_n v_n) &= 0 \\
c_1 \iota(v_1) + \cdots + c_n \iota(v_n) &= 0,
\end{aligned}$$

where $c_1, \ldots, c_n$ are not all zero. However, we proved in part (a) that $(\iota(v_1), \ldots, \iota(v_n))$ is a basis of $(V^*)^*$, so this is not possible. Thus, by contradiction, ι is an injective linear map from V to $(V^*)^*$. Since V^* has the same dimension as V, and since $(V^*)^*$ has the same dimension as V^*, we also know that $(V^*)^*$ has the same dimension as V. Therefore, we conclude that ι is a vector space isomorphism, as required. □

2. We are given that V is the vector space of polynomials p of degree at most 2 with coefficients in $\mathbb{R}$. We are also given the elements $\phi_{-1}, \phi_0, \phi_1 \in V^*$, defined as follows:

$$\phi_x(p) = p(x), \qquad\qquad x \in \{-1, 0, 1\}.$$

We will show that $\gamma := (\phi_{-1}, \phi_0, \phi_1)$ is a basis of V^*, and we will find a basis β of V whose dual is γ.

Step 1: We will construct a candidate for β.

First, let us define $f_{-1} \in V$ by $f_{-1}(x) = \frac{1}{2}x(x-1)$. Then, we obtain:

$$\begin{aligned} \phi_{-1}(f_{-1}) &= f_{-1}(-1) \\ &= \frac{1}{2}(-1)(-1-1) \\ &= 1 \end{aligned} \qquad \begin{aligned} \phi_0(f_{-1}) &= f_{-1}(0) \\ &= \frac{1}{2}(0)(0-1) \\ &= 0 \end{aligned} \qquad \begin{aligned} \phi_1(f_{-1}) &= f_{-1}(1) \\ &= \frac{1}{1}(1)(1-1) \\ &= 0 \end{aligned}$$

Next, let us define $f_0 \in V$ by $f_0(x) = -(x-1)(x+1)$. Then, we obtain:

$$\begin{aligned} \phi_{-1}(f_0) &= f_0(-1) \\ &= -(-1-1)(-1+1) \\ &= 0 \end{aligned} \qquad \begin{aligned} \phi_0(f_0) &= f_0(0) \\ &= -(0-1)(0+1) \\ &= 1 \end{aligned} \qquad \begin{aligned} \phi_1(f_0) &= f_0(1) \\ &= -(1-1)(1+1) \\ &= 0 \end{aligned}$$

Next, let us define $f_1 \in V$ by $f_1(x) = \frac{1}{2}x(x+1)$. Then, we obtain:

$$\begin{aligned} \phi_{-1}(f_1) &= f_1(-1) \\ &= \frac{1}{2}(-1)(-1+1) \\ &= 0 \end{aligned} \qquad \begin{aligned} \phi_0(f_1) &= f_1(0) \\ &= \frac{1}{2}(0)(0+1) \\ &= 0 \end{aligned} \qquad \begin{aligned} \phi_1(f_1) &= f_1(1) \\ &= \frac{1}{2}(1)(1+1) \\ &= 1 \end{aligned}$$

To summarize our results, for all $i, j \in \{-1, 0, 1\}$, we have $\phi_i(f_j) = 1$ if $i = j$, and $\phi_i(f_j) = 0$ otherwise. Then, let us define $\beta := (f_{-1}, f_0, f_1)$. If we prove that β is a basis for V, then the computations above show that γ is the dual basis for β.

Step 2: We will verify that β is a basis for V.

First, we will check that β is linearly independent. Assume for contradiction that β is linearly dependent. Then, there exist $c_{-1}, c_0, c_1 \in \mathbb{R}$, not all zero, such that $c_{-1}f_{-1} + c_0 f_0 + c_1 f_1 = 0$. As a result:

$$\begin{aligned} c_{-1} \cdot \frac{1}{2}x(x-1) + c_0 \cdot (-(x-1)(x+1)) + c_1 \cdot \frac{1}{2}x(x+1) &= 0 \\ c_{-1} \cdot (\frac{1}{2}x^2 - \frac{1}{2}x) + c_0 \cdot (-x^2+1) + c_1 \cdot (\frac{1}{2}x^2 + \frac{1}{2}x) &= 0 \\ (\frac{1}{2}c_{-1} - c_0 + \frac{1}{2}c_1)x^2 + (-\frac{1}{2}c_{-1} + \frac{1}{2}c_1)x + c_0 &= 0x^2 + 0x + 0 \end{aligned}$$

Comparing coefficients on both sides, we obtain the following system of equations:

$$\begin{aligned} \frac{1}{2}c_{-1} - c_0 + \frac{1}{2}c_1 &= 0 \qquad (1) \\ -\frac{1}{2}c_{-1} + \frac{1}{2}c_1 &= 0 \qquad (2) \\ c_0 &= 0 \qquad (3) \end{aligned}$$

Adding equations (1) and (2) together, we obtain $-c_0 + c_1 = 0$. Since equation (3) gives us $c_0 = 0$, we can plug this in to obtain $c_1 = 0$. Then, plugging $c_1 = 0$ into equation (2) yields $-\frac{1}{2}c_{-1} = 0$, so $c_{-1} = 0$. Overall, we obtain $c_{-1} = c_0 = c_1 = 0$, contradicting our condition that c_{-1}, c_0, c_1 are not all zero. Thus, by contradiction, β is linearly independent in V.
Next, since $(1, x, x^2)$ is a basis for V, we know that V has a dimension of 3. This is equal to the number of elements in β, so β is a basis for V, as required. Combining this with Step 1, we obtain that γ is the dual basis of β, as required. Finally, it follows from Axler 3.98 that γ is a basis of V^*, as required. $\square$

3. We are given an n-dimensional vector space V with a basis $(v_1, \ldots, v_n)$. We are also given $k \in \mathbb{N}$. Then, we define $B : \mathcal{T}^k(V) \times \mathcal{T}^k(V) \to \mathbb{R}$ as follows:

$$B(T_1, T_2) := \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_k}).$$

(a) We will show that $B \in \mathcal{T}^2(\mathcal{T}^k(V))$; in other words, we will show that B is a bilinear map on $\mathcal{T}^k(V)$.
Step 1: First, we will show that $B(\lambda S_1 + \mu T_1, T_2) = \lambda B(S_1, T_2) + \mu B(T_1, T_2)$ for all $\lambda, \mu \in \mathbb{R}$ and all $S_1, T_1, T_2 \in \mathcal{T}^k(V)$:

$$\begin{aligned}
B(\lambda S_1 + \mu T_1, T_2) &= \sum_{i_1,\ldots,i_k=1}^{n} (\lambda S_1 + \mu T_1)(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} (\lambda S_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_n}) + \mu T_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_n})) \\
&= \lambda \sum_{i_1,\ldots,i_k=1}^{n} S_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_n}) + \mu \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_k}) \\
&= \lambda B(S_1, T_2) + \mu B(T_1, T_2),
\end{aligned}$$

as desired.
Step 2: Next, we will show that $B(T_1, \lambda S_2 + \mu T_2) = \lambda B(T_1, S_2) + \mu B(T_1, T_2)$ for all $\lambda, \mu \in \mathbb{R}$ and all $T_1, S_2, T_2 \in \mathcal{T}^k(V)$:

$$\begin{aligned}
B(T_1, \lambda S_2 + \mu T_2) &= \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1}, \ldots, v_{i_k}) (\lambda S_2 + \mu T_2)(v_{i_1}, \ldots, v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} (\lambda T_1(v_{i_1}, \ldots, v_{i_k}) S_2(v_{i_1}, \ldots, v_{i_k}) + \mu T_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_k})) \\
&= \lambda \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1}, \ldots, v_{i_k}) S_2(v_{i_1}, \ldots, v_{i_k}) + \mu \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_k}) \\
&= \lambda B(T_1, S_2) + \mu B(T_1, T_2),
\end{aligned}$$

as desired.
From these two steps, we conclude that B is a bilinear map on $\mathcal{T}^k(V)$, as required. □
(b) We will show that B is an inner product.
Step 1: We will show that B is symmetric. In other words, for all $T_1, T_2 \in \mathcal{T}^k(V)$, we will show that $B(T_1, T_2) = B(T_2, T_1)$ as follows:

$$\begin{aligned}
B(T_1, T_2) &= \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} T_2(v_{i_1}, \ldots, v_{i_k}) T_1(v_{i_1}, \ldots, v_{i_k}) \qquad \text{(Multiplication of reals is commutative)} \\
&= B(T_2, T_1),
\end{aligned}$$

as desired.
Step 2: For all $T \in \mathcal{T}^k(V)$, we will verify that $B(T,T) \geq 0$ as follows:

$$\begin{aligned}
B(T,T) &= \sum_{i_1,\ldots,i_k=1}^{n} T(v_{i_1},\ldots,v_{i_k})T(v_{i_1},\ldots,v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} [T(v_{i_1},\ldots,v_{i_k})]^2 \\
&\geq \sum_{i_1,\ldots,i_k=1}^{n} 0 \\
&= 0,
\end{aligned}$$

as desired.
Step 3: For all $T \in \mathcal{T}^k(V)$, we will show that $B(T,T) = 0$ if and only if $T = 0$.
For the "$\Leftarrow$" direction, suppose that $T = 0$. Then, we obtain:

$$\begin{aligned}
B(T,T) &= \sum_{i_1,\ldots,i_k=1}^{n} T(v_{i_1},\ldots,v_{i_k})T(v_{i_1},\ldots,v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} 0 \cdot 0 \\
&= 0.
\end{aligned}$$

Thus, $B(T,T) = 0$ if $T = 0$.
Next, for the "$\Rightarrow$" direction, suppose that $B(T,T) = 0$. Then, we obtain:

$$\begin{aligned}
B(T,T) &= 0 \\
\sum_{i_1,\ldots,i_k=1}^{n} T(v_{i_1},\ldots,v_{i_k})T(v_{i_1},\ldots,v_{i_k}) &= 0 \\
\sum_{i_1,\ldots,i_k=1}^{n} [T(v_{i_1},\ldots,v_{i_k})]^2 &= 0
\end{aligned}$$

Whenever a sum of squares equals zero, each individual square must equal zero. As a result, we obtain $T(v_{i_1},\ldots,v_{i_k}) = 0$ for all $1 \leq i_1,\ldots,i_k \leq n$.
Next, let $u_1,\ldots,u_k$ be picked arbitrarily from V. Then, using the basis $(v_1,\ldots,v_n)$ of V, we can write $u_1,\ldots,u_k$ as:

$$u_j = \sum_{i_j=1}^{n} c_{j,i_j} v_{i_j}, \qquad 1 \leq j \leq k.$$

Then, we can verify that $T(u_1, \ldots, u_k) = 0$ as follows:

$$\begin{aligned}
T(u_1, \ldots, u_k) &= T\left(\sum_{i_1=1}^{n} c_{1,i_1} v_{i_1}, \sum_{i_2=1}^{n} c_{2,i_2} v_{i_2}, \ldots, \sum_{i_k=1}^{n} c_{k,i_k} v_{i_k} \right) \\
&= \sum_{i_1=1}^{n} c_{1,i_1} T\left(v_{i_1}, \sum_{i_2=1}^{n} c_{2,i_2} v_{i_2}, \ldots, \sum_{i_k=1}^{n} c_{k,i_k} v_{i_k} \right) && \text{(Since } T \text{ is } k\text{-linear)} \\
&= \sum_{i_1=1}^{n} c_{1,i_1} \sum_{i_2=1}^{n} c_{2,i_2} T\left(v_{i_1}, v_{i_2}, \ldots, \sum_{i_k=1}^{n} c_{k,i_K} v_{i_k} \right) && \text{(Since } T \text{ is } k\text{-linear)} \\
&= \cdots \\
&= \sum_{i_1=1}^{n} c_{1,i_1} \sum_{i_2=1}^{n} c_{2,i_2} \cdots \sum_{i_k=1}^{n} c_{k,i_k} T(v_{i_1}, v_{i_2}, \ldots, v_{i_k}) \\
&= \sum_{i_1=1}^{n} c_{1,i_1} \sum_{i_2=1}^{n} c_{2,i_2} \cdots \sum_{i_k=1}^{n} c_{k,i_k} \cdot 0 \\
&= 0.
\end{aligned}$$

Since this is true for all $u_1, \ldots, u_k \in V$, we conclude that $T = 0$. This completes our proof that $B(T, T) = 0$ if and only if $T = 0$.

Overall, since we know from part (a) that B is bilinear, and since we proved the other required properties in Steps 1, 2, and 3, we conclude that B is an inner product, as required. ☐

Notes on intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. This problem can be solved by tracing definitions and using standard linear algebra techniques.

2. For this problem, the main challenge is to construct the basis $\beta = (f_{-1}, f_0, f_1)$ of V whose dual is $\gamma = (\phi_{-1}, \phi_0, \phi_1)$. For instance, to construct f_{-1}, we need $f_{-1}(-1) = \phi_{-1}(f_{-1}) = 1$, $f_{-1}(0) = \phi_0(f_{-1}) = 0$, and $f_{-1}(1) = \phi_1(f_{-1}) = 0$. Then, we know that f_{-1} has roots of 0 and 1. This motivates us to write $f_{-1}(x)$ in factored form as $f_{-1}(x) = cx(x-1)$, where c is a real constant. Finally, we plug in $x = -1$ to obtain $1 = f_{-1}(-1) = c(-1)(-1-1) = 2c$, so $c = \frac{1}{2}$. As a result, we have constructed the basis element $f_{-1}(x) = \frac{1}{2}x(x-1)$. This same procedure also allows us to construct f_0 and f_1. Afterwards, we can apply standard linear algebra techniques to finish the problem.

3. First, part (a) can be solved by tracing definitions.
   For part (b), we first perform basic computations to show that B is symmetric, that $B(T,T) \geq 0$ for all $T \in \mathcal{T}^k(V)$, and that $B(T,T) = 0$ if $T = 0$. Then, the main challenge is to show that $T = 0$ if $B(T,T) = 0$. First, plugging in the definition of $B(T,T)$ gives us that:

   $$\sum_{i_1,\ldots,i_k=1}^{n} [T(v_{i_1}, \ldots, v_{i_k})]^2 = 0,$$

   so each individual $T(v_{i_1}, \ldots, v_{i_k})$ is zero. Next, the key idea is that, for all $u_1, \ldots, u_k \in V$, $T(u_1, \ldots, u_k)$ can be evaluated in terms of individual $T(v_{i_1}, \ldots, v_{i_k})$ terms, which all equal zero. This helps us to prove that $T = 0$ whenever $B(T,T) = 0$, and then we are done.