MAT257 Assignment 10 (Change of Variables)

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1. We are given the hemisphere $V := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < a^2, z > 0\}$ for some parameter $a > 0$. Then, we will use the spherical coordinate transformation to express $\int_V z$ as an integral over an appropriate set in $\mathbb{R}^3_{r,\phi,\theta}$.
First, let us define the coordinate transformation $g_0 : \mathbb{R}^3_{r,\phi,\theta} \to \mathbb{R}^3_{x,y,z}$ by:

$$(x,y,z) := g_0(r,\phi,\theta) := (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi).$$

Then, we will find condition on (r,ϕ,θ) such that $g_0(r,\phi,\theta) \in V$ and such that the restriction of g_0 is 1-1.
First, in the formula $(x,y,z) = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)$, the input r represents the distance from the origin, as shown in the following computation:

$$\begin{aligned}
\left|(x,y,z)\right| &= \sqrt{x^2+y^2+z^2} \\
&= \sqrt{(r\cos\phi\cos\theta)^2 + (r\cos\phi\sin\theta)^2 + (r\sin\phi)^2} \\
&= \sqrt{r^2(\cos^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta + \sin^2\phi)} \\
&= \sqrt{r^2(\cos^2\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi)} \\
&= \sqrt{r^2(\cos^2\phi\cdot 1 + \sin^2\phi)} \\
&= \sqrt{r^2\cdot 1} \\
&= |r|\,.
\end{aligned}$$

Then, since $x^2+y^2+z^2 < a^2$ for all $(x,y,z) \in V$, we need $r^2 < a^2$, so $-a < r < a$. Since r represents a distance, r should also be positive. Thus, let us select the bounds $0 < r < a$ for r.
Next, in the formula $(x,y,z) = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)$, the input θ represents the "longitude" around the hemisphere, so it is bounded as $0 < \theta < 2\pi$.
Next, the input ϕ represents the "latitude" above the xy-plane, so it is bounded as $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Additionally, all $(x,y,z) \in V$ must satisfy $z > 0$, so the lower half of the hemisphere V is missing. Thus, let us select the bounds $0 < \phi < \frac{\pi}{2}$ for ϕ.
Overall, we obtained the bounds $0 < r < a$, $0 < \phi < \frac{\pi}{2}$, and $0 < \theta < 2\pi$. Then, let us define the open set $A \subseteq \mathbb{R}^3_{r,\phi,\theta}$ as the open rectangle $(0,a)\times(0,\frac{\pi}{2})\times(0,2\pi)$, and let us define $g : A \to \mathbb{R}^3_{x,y,z}$ to be the restriction $g_0|_A$. With the bounds that we selected, g is 1-1, and $g(A)$ is approximately V.
Next, we can compute $g'(r,\phi,\theta)$ for all $(r,\phi,\theta) \in A$ as follows:

$$\begin{aligned}
g'(r,\phi,\theta) &= \begin{pmatrix} \frac{\partial}{\partial r} r\cos\phi\cos\theta & \frac{\partial}{\partial \phi} r\cos\phi\cos\theta & \frac{\partial}{\partial \theta} r\cos\phi\cos\theta \\ \frac{\partial}{\partial r} r\cos\phi\sin\theta & \frac{\partial}{\partial \phi} r\cos\phi\sin\theta & \frac{\partial}{\partial \theta} r\cos\phi\sin\theta \\ \frac{\partial}{\partial r} r\sin\phi & \frac{\partial}{\partial \phi} r\sin\phi & \frac{\partial}{\partial \theta} r\sin\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi\cos\theta & -r\sin\phi\cos\theta & -r\cos\phi\sin\theta \\ \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi & r\cos\phi & 0 \end{pmatrix}.
\end{aligned}$$

All entries of $g'(r,\phi,\theta)$ are continuous, so g is continuously differentiable. Additionally, we can compute the determinant of $g'(r, \phi, \theta)$ to be:

$$\begin{aligned}
&\det g'(r, \phi, \theta) \\
=\ &(\cos\phi\cos\theta)(-r\sin\phi\sin\theta)(0) - (\cos\phi\cos\theta)(r\cos\phi\cos\theta)(r\cos\phi) - (-r\sin\phi\cos\theta)(\cos\phi\sin\theta)(0) \\
&+ (-r\sin\phi\cos\theta)(r\cos\phi\cos\theta)(\sin\phi) + (-r\cos\phi\sin\theta)(\cos\phi\sin\theta)(r\cos\phi) \\
&- (-r\cos\phi\sin\theta)(-r\sin\phi\sin\theta)(\sin\phi) \\
=\ &-r^2\cos^3\phi\cos^2\theta - r^2\sin^2\phi\cos\phi\cos^2\theta - r^2\cos^3\phi\sin^2\theta - r^2\sin^2\phi\cos\phi\sin^2\theta \\
=\ &-r^2\cos\phi(\cos^2\phi + \sin^2\phi)(\cos^2\theta + \sin^2\theta) \\
=\ &-r^2\cos\phi.
\end{aligned}$$

This determinant is nonzero for all $r > 0$ and all $0 < \phi < \frac{\pi}{2}$, so it is nonzero for all $(r, \phi, \theta) \in A$. This means that $g'(r, \phi, \theta)$ is invertible for all $(r, \phi, \theta) \in A$.
Finally, let us define $f : V \to \mathbb{R}$ by $f(x, y, z) = z$; note that f is integrable as a continuous function. Then, all conditions of the Change of Variables formula are satisfied, so we can apply this formula to obtain:

$$\begin{aligned}
\int_V z &= \int_V f \\
&= \int_{g(A)} f \\
&= \int_A f(g(r, \phi, \theta))\big|\det g'(r, \phi, \theta)\big| \\
&= \int_A f(r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)\Big|-r^2\cos\phi\Big| \\
&= \int_A r\sin\phi \cdot r^2\cos\phi \\
&= \int_A r^3\sin\phi\cos\phi.
\end{aligned}$$

Therefore, $\int_V z$ can be rewritten as $\boxed{\int_A r^3\sin\phi\cos\phi}$, where $A = (0, a)\times(0, \frac{\pi}{2})\times(0, 2\pi) \subseteq \mathbb{R}^3_{r,\phi,\theta}$, as required. $\square$

2. Let $f : \mathbb{R}^2_{x,y} \to \mathbb{R}$ be defined by $f(x,y) := \frac{1}{x^2+y^2}$. Then, we will determine if f is integrable over $U_1 := \{(x,y) \in \mathbb{R}^2_{x,y} : 0 < x^2 + y^2 < 1\}$ and over $U_2 := \{(x,y) \in \mathbb{R}^2_{x,y} : x^2 + y^2 > 1\}$.
Step 1: We will show that f is not integrable over U_1.
Assume for contradiction that f is integrable over U_1. First, let us define the polar coordinate transformation $g_0 : \mathbb{R}^2_{r,\theta} \to \mathbb{R}^2_{x,y}$ by $g_0(r,\theta) = (r\cos\theta, r\sin\theta)$. Then, we will find conditions on (r,θ) such that $g_0(r,\theta) \in U_1$ and such that the restriction of g_0 is 1-1.
In the formula $(x,y) = g_0(r,\theta) = (r\cos\theta, r\sin\theta)$, the input r represents the distance from the origin, as shown in the following computation:

$$\begin{aligned} \left|(x,y)\right| &= \sqrt{x^2+y^2} \\ &= \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} \\ &= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{r^2 \cdot 1} \\ &= |r|\,. \end{aligned}$$

Then, for (x,y) to be inside U_1, we need $x^2 + y^2 < 1$, so $r^2 < 1$, so $-1 < r < 1$. Since r represents a distance, r should also be positive. Thus, let us select the bounds $0 < r < 1$ for r.
Next, in the formula $(x,y) = (r\cos\theta, r\sin\theta)$, the input θ represents the angle, so it is bounded as $0 < \theta < 2\pi$.
Overall, we obtained the bounds $0 < r < 1$ and $0 < \theta < 2\pi$. Then, let us define the open set $A_1 \subseteq \mathbb{R}^2_{r,\theta}$ as the open rectangle $(0,1) \times (0,2\pi)$, and let us define $g_1 : A_1 \to \mathbb{R}^2_{x,y}$ to be the restriction $g_0|_{A_1}$. With the bounds that we selected, g_1 is 1-1, and $g_1(A_1)$ is approximately U_1.
Next, we can compute $g_1'(r,\theta)$ for all $(r,\theta) \in A$ as follows:

$$\begin{aligned} g_1'(r,\theta) &= \begin{pmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial\theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial\theta} r\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}. \end{aligned}$$

All entries of $g_1'(r,\theta)$ are continuous, so g_1 is continuously differentiable. Additionally, we can compute the determinant of $g_1'(r,\theta)$ to be:

$$\det g_1'(r,\theta) = (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r.$$

This determinant is always nonzero when $r > 0$, so this determinant is nonzero for all $(r,\theta) \in A_1$. This means that $g_1'(r,\theta)$ is invertible at all $(r,\theta) \in A_1$.
Finally, we assumed above that f is integrable over U_1. Thus, all conditions of the Change of

Variables formula are satisfied, so we can use this formula to obtain:

$$\begin{aligned}\int_{U_1} f &= \int_{g_1(A_1)} f \\ &= \int_{A_1} f(g_1(r,\theta))\big|\det g_1'(r,\theta)\big| \\ &= \int_{A_1} f(r\cos\theta, r\sin\theta)|r| \\ &= \int_{A_1} \frac{1}{(r\cos\theta)^2+(r\sin\theta)^2}\cdot r \\ &= \int_{A_1} \frac{1}{r^2(\cos^2\theta+\sin^2\theta)}\cdot r \\ &= \int_{A_1} \frac{1}{r}.\end{aligned}$$

Since $\frac{1}{r}$ is continuous whenever $r>0$, we can apply Fubini's theorem to obtain:

$$\begin{aligned}\int_{U_1} f &= \int_{A_1} \frac{1}{r} \\ &= \int_0^1 \int_0^{2\pi} \frac{1}{r} d\theta dr \\ &= \int_0^1 \frac{\theta}{r}\bigg|_{\theta=0}^{\theta=2\pi} dr \\ &= \int_0^1 \frac{2\pi}{r} dr \\ &= \lim_{t\to 0^+} \int_t^1 \frac{2\pi}{r} dr \\ &= \lim_{t\to 0^+} 2\pi \ln|r| \bigg|_{r=t}^{r=1} \\ &= \lim_{t\to 0^+} 2\pi(\ln 1 - \ln t) \\ &= -2\pi \lim_{t\to 0^+} \ln t.\end{aligned}$$

However, this limit does not exist because it diverges, so we obtain a contradiction. Thus, by contradiction, f is not integrable over U_1, as required.

Step 2: We will show that f is also not integrable over U_2.

Assume for contradiction that f is integrable over U_2. First, recall the coordinate transformation $g_0(r,\theta) = (r\cos\theta, r\sin\theta)$. We will find conditions on (r,θ) such that $g_0(r,\theta)\in U_2$ and such that the restriction of g_0 is 1-1.

Similarly to Step 1, r represents the distance $\big|(x,y)\big| = \sqrt{x^2+y^2}$. This time, for (x,y) to be inside U_2, we need $x^2+y^2>1$, so $r>1$. Thus, we pick the bounds $1<r<\infty$. Moreover, similarly to Step 1, the angle θ is bounded as $0<\theta<2\pi$.

Then, let us define the open set $A_2 \subseteq \mathbb{R}^2_{r,\theta}$ as the open "unbounded rectangle" $(1,\infty)\times(0,2\pi)$, and let us define $g_2 : A_2 \to \mathbb{R}^2_{x,y}$ to be the restriction $g_0|_{A_2}$. With the bounds that we selected, g_2 is 1-1, and $g_2(A_2)$ is approximately U_2.

Next, similarly to Step 1, we can compute the differential of g_2 to be $\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}$, and

we can compute $\det g_2'(r,\theta) = r > 0$. Then, g_2 is continuously differentiable, and $g_2'(r,\theta)$ is invertible at all $(r,\theta) \in A_2$.
Finally, we assumed that f is integrable over U_2. Thus, all conditions of the Change of Variables formula are satisfied, so we can apply this formula. Then, using the same calculations as in Step 1, we obtain $\int_{U_2} f = \int_{A_2} \frac{1}{r}$. We continue with:

$$\begin{aligned}\int_{U_2} f &= \int_{A_2} \frac{1}{r} \\ &= \int_{(1,\infty)\times(0,2\pi)} \frac{1}{r} \\ &= \lim_{t\to\infty} \int_{(1,t)\times(0,2\pi)} \frac{1}{r}.\end{aligned}$$

Since $\frac{1}{r}$ is continuous for $r > 1$, we can use Fubini's theorem to obtain:

$$\begin{aligned}\int_{U_2} f &= \lim_{t\to\infty} \int_1^t \int_0^{2\pi} \frac{1}{r} d\theta dr \\ &= \lim_{t\to\infty} \int_1^t \frac{\theta}{r}\Big|_{\theta=0}^{\theta=2\pi} dr \\ &= \lim_{t\to\infty} \int_1^t \frac{2\pi}{r} dr \\ &= \lim_{t\to\infty} 2\pi \ln|r| \Big|_{r=1}^{r=t} \\ &= \lim_{t\to\infty} 2\pi(\ln t - \ln 1) \\ &= 2\pi \lim_{t\to\infty} \ln t.\end{aligned}$$

However, this limit does not exist because it diverges, so we obtain a contradiction. Thus, by contradiction, f is also not integrable over U_2, as required. $\square$

3. We are given the set $B = \{(x, y) \in \mathbb{R}^2_{x,y} : x > 0, y > 0, 1 < xy < 2, x < y < 4x\}$. Also, let us define $f : \mathbb{R}^2_{x,y} \to \mathbb{R}$ by $f(x, y) = x^2 y^3$. Then, we will compute $\int_B f$.
First, let us define the open set $A \subseteq \mathbb{R}^2_{u,v}$ as the open rectangle $(1, \sqrt{2}) \times (1, 2)$, and let us define $g : A \to \mathbb{R}^2_{x,y}$ by $g(u, v) = (\frac{u}{v}, uv)$. We can compute the differential of g to be:

$$g'(u, v) = \begin{pmatrix} \frac{\partial}{\partial u} \frac{u}{v} & \frac{\partial}{\partial v} \frac{u}{v} \\ \frac{\partial}{\partial u} uv & \frac{\partial}{\partial v} uv \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix}.$$

Here, all partial derivatives of g are continuous on A, so g is continuously differentiable. Moreover, the determinant of g' is $\det g' = (\frac{1}{v})(u) - (-\frac{u}{v^2})(v) = \frac{2u}{v}$. This determinant is always nonzero when $u \in (1, \sqrt{2})$, so $g'(u, v)$ is invertible for all $(u, v) \in A$.
Next, we will check that g is 1-1. Suppose that there exist two points $(u_1, v_1), (u_2, v_2) \in A$ such that $g(u_1, v_1) = g(u_2, v_2)$. Then, we get $(\frac{u_1}{v_1}, u_1 v_1) = (\frac{u_2}{v_2}, u_2 v_2)$, so:

$$\frac{u_1}{v_1} = \frac{u_2}{v_2}, \tag{1}$$

$$u_1 v_1 = u_2 v_2. \tag{2}$$

Multiplying equation (2) by equation (1), we obtain $u_1^2 = u_2^2$, so $\sqrt{u_1^2} = \sqrt{u_2^2}$. Since u_1, u_2 are both positive, it follows that $u_1 = u_2$.
Dividing equation (2) by equation (1), we obtain $v_1^2 = v_2^2$, so $\sqrt{v_1^2} = \sqrt{v_2^2}$. Since v_1, v_2 are both positive, it follows that $v_1 = v_2$.
Thus, we get $(u_1, v_1) = (u_2, v_2)$ whenever $g(u_1, v_1) = g(u_2, v_2)$, so g is 1-1, as required.
Next, we will check that $B \subseteq g(A)$. Let any $(x, y) \in B$ be given. Then, we have $x, y > 0$, so we can define $(u, v) := (\sqrt{xy}, \sqrt{\frac{y}{x}}) \in \mathbb{R}^2$. Since $(x, y) \in B$, we have $1 < xy < 2$, so $u = \sqrt{xy}$ satisfies $1 < u < \sqrt{2}$. Since $(x, y) \in B$, we also have $x < y < 4x$, so $1 < \frac{y}{x} < 4$, so $v = \sqrt{\frac{y}{x}}$ satisfies $1 < v < 2$. Thus, we have $1 < u < \sqrt{2}$ and $1 < v < 2$, so $(u, v) \in A$. Then, we can compute $g(u, v)$ to be $(\frac{u}{v}, uv) = (\frac{\sqrt{xy}}{\sqrt{\frac{y}{x}}}, \sqrt{xy} \cdot \sqrt{\frac{y}{x}}) = (x, y)$. Therefore, for all $(x, y) \in B$, we found $(u, v) \in A$ such that $g(u, v) = (x, y)$, so $B \subseteq g(A)$.
Next, we will check that $g(A) \subseteq B$. For all $(u, v) \in A$, we have $1 < u < \sqrt{2}$ and $1 < v < 2$. Then, let us define $(x, y) := g(u, v) = (\frac{u}{v}, uv)$. Since $u, v > 0$, we have $x = \frac{u}{v} > 0$ and $y = uv > 0$. Since $1 < u < \sqrt{2}$, we get that $xy = \frac{u}{v} \cdot uv = u^2$ satisfies $1 < xy < 2$. Since $1 < v < 2$, we also get that $\frac{y}{x} = \frac{uv}{\frac{u}{v}} = v^2$ satisfies $1 < \frac{y}{x} < 4$, so $x < y < 4x$. Thus, we have $x > 0$, $y > 0$, $1 < xy < 2$, and $x < y < 4x$, so $(x, y) \in B$. As a result, $g(u, v) \in B$ for all $(u, v) \in A$, so $g(A) \subseteq B$. This, combined with $B \subseteq g(A)$, gives us $g(A) = B$.
Finally, f is integrable as a continuous function. Thus, all of the conditions of the Change of

Variables formula are satisfied, so we obtain:

$$\begin{aligned}
\int_B f &= \int_{g(A)} f \\
&= \int_A f(g(u,v))\big|\det g'(u,v)\big| \\
&= \int_A f(\frac{u}{v}, uv)\left|\frac{2u}{v}\right| \\
&= \int_A \left(\frac{u}{v}\right)^2 \cdot (uv)^3 \cdot \frac{2u}{v} \\
&= \int_A 2u^6.
\end{aligned}$$

Since $2u^6$ is continuous, we can apply Fubini's theorem to obtain:

$$\begin{aligned}
\int_B f &= \int_A 2u^6 \\
&= \int_1^{\sqrt{2}} \int_1^2 2u^6 dv du \\
&= \int_1^{\sqrt{2}} 2u^6 v\Big|_{v=1}^{v=2} du \\
&= \int_1^{\sqrt{2}} (4u^6 - 2u^6) du \\
&= \int_1^{\sqrt{2}} 2u^6 du \\
&= \frac{2}{7}u^7\Big|_{u=1}^{u=\sqrt{2}} \\
&= \frac{2}{7}((\sqrt{2})^7 - 1) \\
&= \frac{16}{7}\sqrt{2} - \frac{2}{7}.
\end{aligned}$$

Therefore, we obtain $\boxed{\int_B f = \frac{16}{7}\sqrt{2} - \frac{2}{7}}$, as required. $\square$

4. We are given a tetrahedron T in $\mathbb{R}^3_{x,y,z}$ with vertices $(0,0,0)$, $(1,2,3)$, $(0,1,2)$, and $(-1,1,1)$. We are also given the function $f(x,y,z) := x + 2y - z$. Then, we will compute $\int_T f$.
First, let us define the open set $A \subseteq \mathbb{R}^3_{u,v,w}$ to be the open tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Then, let us define $g : A \to \mathbb{R}^3_{x,y,z}$ to be the linear map represented by the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$. By Spivak's Theorem 2-3(2), we obtain that $Dg(u,v,w) = g$ for all $(u,v,w) \in A$, so:

$$g'(u,v,w) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

First, all entries of $g'(u,v,w)$ are continuous, so g is continuously differentiable. Next, the determinant of $g'(u,v,w)$ can be computed as:

$$\det g'(u,v,w) = (1)(1)(1) - (1)(2)(1) - (0)(2)(1) + (0)(1)(3) + (-1)(2)(2) - (-1)(1)(3) = -2.$$

This determinant is always nonzero, so $g'(u,v,w)$ is invertible at all $(u,v,w) \in A$. This computation also shows that $\det g \neq 0$, so g is an invertible linear map; in other words, g is 1-1.
Next, we will verify that $g(A) = T$. When we apply g to the vertices of the tetrahedron A, we obtain:

$$\begin{aligned}
g(0,0,0) &= (0,0,0)^T, \\
g(1,0,0) &= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1,2,3)^T, \\
g(0,1,0) &= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0,1,2)^T, \\
g(0,0,1) &= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (-1,1,1)^T.
\end{aligned}$$

Overall, g maps the vertices of A to the vertices of T, so $g(A) = T$, as desired.
Finally, f is integrable as a continuous function. Thus, all conditions of the Change of Variables formula are satisfied, so we can use this formula to obtain:

$$\begin{aligned}
\int_T f &= \int_{g(A)} f \\
&= \int_A f(g(u,v,w)) \left|\det g'(u,v,w)\right| \\
&= \int_A (g_1(u,v,w) + 2g_2(u,v,w) - g_3(u,v,w))|-2| \\
&= \int_A 2((u-w) + 2(2u+v+w) - (3u+2v+w)) \\
&= \int_A 4u.
\end{aligned}$$

Since points (u, v, w) inside the tetrahedron A are bounded under the plane $u + v + w = 1$, we have the bounds $0 < u < 1$, then we have the bounds $0 < v < 1 - u$ (since $u + v < 1$), then we have the bounds $0 < w < 1 - u - v$ (since $u + v + w < 1$). As a result:

$$\begin{aligned}
\int_T f &= \int_A 4u \\
&= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 4u dw dv du \\
&= \int_0^1 \int_0^{1-u} 4uw \Big|_{w=0}^{w=1-u-v} dv du \\
&= \int_0^1 \int_0^{1-u} 4u(1-u-v) dv du \\
&= \int_0^1 \int_0^{1-u} (4u - 4u^2 - 4uv) dv du \\
&= \int_0^1 (4uv - 4u^2 v - 2uv^2)\Big|_{v=0}^{v=1-u} du \\
&= \int_0^1 (4u(1-u) - 4u^2(1-u) - 2u(1-u)^2) du \\
&= \int_0^1 (2u^3 - 4u^2 + 2u) du \\
&= (\frac{1}{2}u^4 - \frac{4}{3}u^3 + u^2)\Big|_{u=0}^{u=1} \\
&= \frac{1}{2} - \frac{4}{3} + 1 \\
&= \frac{1}{6}.
\end{aligned}$$

Thus, we obtain $\boxed{\int_T f = \frac{1}{6}}$, as required. $\square$

5. We are given parameters $0 < a < b$. Then, we construct the disk of radius a around the point $(b, 0, 0)$ in the xz-plane, and we construct the solid torus T by spinning this disk around the z-axis. Then, we will compute the volume of T.
Step 1: Consider the coordinate transformation $g_0 : \mathbb{R}^3_{r,\theta,z} \to \mathbb{R}^3_{x,y,z}$ defined by:

$$(x, y, z) := g_0(r, \theta, z) := (r\cos\theta, r\sin\theta, z).$$

Then, we will use this transformation to express $\mathrm{vol}(T)$ as an integral over an appropriate set in $\mathbb{R}^3_{r,\theta,z}$.
To begin, we will find conditions on (r, θ, z) such that $g_0(r, \theta, z) \in T$ and such that the restriction of g_0 is 1-1.
First, in the formula $(x, y, z) = (r\cos\theta, r\sin\theta, z)$, the input r represents the "horizontal distance" of (x, y) from the origin, as shown in the following computation:

$$\begin{aligned} \left|(x,y)\right| &= \sqrt{x^2+y^2} \\ &= \sqrt{(r\cos\theta)^2+(r\sin\theta)^2} \\ &= \sqrt{r^2(\cos^2\theta+\sin^2\theta)} \\ &= \sqrt{r^2\cdot 1} \\ &= |r|\,. \end{aligned}$$

Then, since T is constructed by spinning a disk of radius a whose centre is b units away from the origin, we see that $|r-b|$ represents the horizontal distance from the centre of a disk, and $|z|$ represents the vertical distance from the centre of a disk. Since the disk has radius a, we obtain the condition $(r-b)^2 + z^2 < a^2$.
Next, θ represents the "longitude" of (x, y, z), so θ is bounded as $0 < \theta < 2\pi$.
Overall, we obtain the conditions $(r-b)^2+z^2 < a^2$ and $0 < \theta < 2\pi$. The bound $(r-b)^2+z^2 < a^2$ represents an open ball in the rz-plane, and appending the bound $0 < \theta < 2\pi$ results in an open cylinder in $\mathbb{R}^3_{r,\theta,z}$. Then, let us define the open set $A \subseteq \mathbb{R}^3_{r,\theta,z}$ by:

$$A := \{(r, \theta, z) : (r-b)^2 + z^2 < a^2, 0 < \theta < 2\pi\},$$

and let us define $g : A \to \mathbb{R}^3_{x,y,z}$ to be the restriction $g_0|_A$. Under the conditions that we picked, g is 1-1, and $g(A)$ is approximately T.
Next, we will show that $r > 0$ for all $(r, \theta, z) \in A$. If $(r, \theta, z) \in A$, then $a^2 > (r-b)^2+z^2 \geq (r-b)^2$, so $-a < r-b < a$. In particular, $r-b > -a$, so $r > b-a$. Since $b > a$, it follows that $r > 0$, as desired.
Now, we can compute $g'(r, \theta, z)$ as follows:

$$\begin{aligned} g'(r,\theta,z) &= \begin{pmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta & \frac{\partial}{\partial z} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta & \frac{\partial}{\partial z} r\sin\theta \\ \frac{\partial}{\partial r} z & \frac{\partial}{\partial \theta} z & \frac{\partial}{\partial z} z \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Here, all entries of $g'(r, \theta, z)$ are continuous, so g is continuously differentiable. Moreover, the determinant of $g'(r, \theta, z)$ can be computed as:

$$\begin{aligned}
\det g'(r, \theta, z) &= (\cos\theta)(r\cos\theta)(1) - (\sin\theta)(-r\sin\theta)(1) \\
&= r(\cos^2\theta + \sin^2\theta) \\
&= r.
\end{aligned}$$

This determinant is nonzero for all $r > 0$. Since we showed above that $r > 0$ for all $(r, \theta, z) \in A$, it follows that $g'(r, \theta, z)$ is invertible at all $(r, \theta, z) \in A$.
Finally, let us define $f_1 : \mathbb{R}^3_{x,y,z}$ by $f_1(x, y, z) = 1$; note that f_1 is integrable as a continuous function. Then, all conditions of the Change of Variables formula are satisfied, so we can use this formula to obtain:

$$\begin{aligned}
\mathrm{vol}(T) &= \int_T f_1 \\
&= \int_{g(A)} f_1 \\
&= \int_A f_1(g(r, \theta, z)) \big|\det g'(r, \theta, z)\big| \\
&= \int_A 1 \cdot |r| \\
&= \int_A r.
\end{aligned}$$

Step 2: Consider the coordinate transformation $h_0 : \mathbb{R}^3_{R,\theta,\phi} \to \mathbb{R}^3_{r,\theta,z}$ defined by:

$$(r, \theta, z) := h_0(R, \theta, \phi) := (R\cos\phi + b, \theta, R\sin\phi).$$

(Here, (R, ϕ) represent polar coordinates for (r, z), with r being shifted upward by b). Then, we will use this transformation to finish computing $\mathrm{vol}(T)$.
We will begin by finding conditions on (R, θ, ϕ) such that $h_0(R, \theta, \phi) \in A$ and such that the restriction of h_0 is 1-1.
First, for (r, θ, z) to be inside A, we have the condition:

$$\begin{aligned}
(r - b)^2 + z^2 &< a^2 \\
(R\cos\phi)^2 + (R\sin\phi)^2 &< a^2 \\
R^2(\cos^2\phi + \sin^2\phi) &< a^2 \\
R^2 &< a^2.
\end{aligned}$$

Since R is a distance, we have $R > 0$, so we conclude that $0 < R < a$. Moreover, θ inherits the condition $0 < \theta < 2\pi$ from A. Finally, ϕ also has the condition $0 < \phi < 2\pi$ because ϕ represents another angle.
Overall, we obtain the conditions $0 < R < a$, $0 < \theta < 2\pi$, and $0 < \phi < 2\pi$. Then, let us define the open set $B \subseteq \mathbb{R}^3_{R,\theta,\phi}$ as the open rectangle $(0, a) \times (0, 2\pi) \times (0, 2\pi)$, and let us define $h : B \to \mathbb{R}^3_{R,\theta,\phi}$ as the restriction $h_0|_B$. Using the bounds that we picked, h is 1-1, and $h(B)$ is approximately A.

Next, we can compute $h'(R,\theta,\phi)$ as follows:

$$
\begin{aligned}
h'(R,\theta,\phi) &= \begin{pmatrix} \frac{\partial}{\partial R}(R\cos\phi + b) & \frac{\partial}{\partial \theta}(R\cos\phi + b) & \frac{\partial}{\partial \phi}(R\cos\phi + b) \\ \frac{\partial}{\partial R}\theta & \frac{\partial}{\partial \theta}\theta & \frac{\partial}{\partial \phi}\theta \\ \frac{\partial}{\partial R}R\sin\phi & \frac{\partial}{\partial \theta}R\sin\phi & \frac{\partial}{\partial \phi}R\sin\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi & 0 & -R\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & R\cos\phi \end{pmatrix}.
\end{aligned}
$$

Here, all entries of $h'(R,\theta,\phi)$ are continuous, so h is continuously differentiable. Moreover, the determinant of $h'(R,\theta,\phi)$ can be computed as:

$$
\begin{aligned}
\det h'(R,\theta,\phi) &= (R\cos\phi)(1)(\cos\phi) - (-R\sin\phi)(1)(\sin\phi) \\
&= R(\cos^2\phi + \sin^2\phi) \\
&= R.
\end{aligned}
$$

This determinant is nonzero whenever $R > 0$, so it is nonzero for all $(R,\theta,\phi) \in B$. Thus, $h'(R,\theta,\phi)$ is invertible for all $(R,\theta,\phi) \in B$.

Now, let us define $f_2 : \mathbb{R}^3_{r,\theta,z}$ by $f_2(r,\theta,z) = r$; note that f_2 is integrable as a continuous function. Then, all conditions of the Change of Variables formula are satisfied, so we can use this formula to obtain:

$$
\begin{aligned}
\mathrm{vol}(T) &= \int_A r \\
&= \int_{h(B)} f_2 \\
&= \int_B f_2(h(R,\theta,\phi))\big|\det h'(R,\theta,\phi)\big| \\
&= \int_B f_2(R\cos\phi + b, \theta, R\sin\phi)|R| \\
&= \int_B (R\cos\phi + b)\cdot R \\
&= \int_B (R^2\cos\phi + bR).
\end{aligned}
$$

Since $R^2 \cos\phi + bR$ is a continuous function, we can apply Fubini's theorem to obtain:

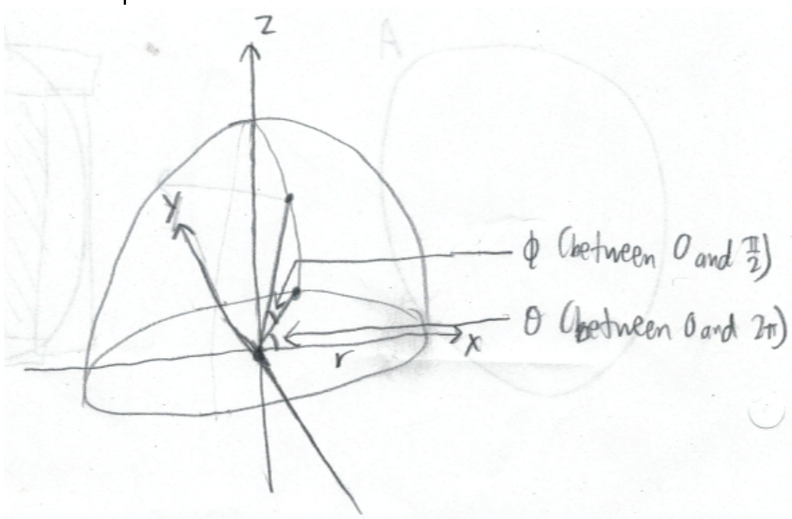
$$\begin{aligned}
\mathrm{vol}(T) &= \int_0^a \int_0^{2\pi} \int_0^{2\pi} (R^2 \cos\phi + bR) d\phi d\theta dR \\
&= \int_0^a \int_0^{2\pi} (R^2 \sin\phi + bR\phi)\Big|_{\phi=0}^{\phi=2\pi} d\theta dR \\
&= \int_0^a \int_0^{2\pi} ((0R^2 + 2\pi bR) - (0R^2 + 0bR)) d\theta dR \\
&= \int_0^a \int_0^{2\pi} 2\pi bR d\theta dR \\
&= \int_0^a 2\pi bR\theta\Big|_{\theta=0}^{\theta=2\pi} dR \\
&= \int_0^a 4\pi^2 bR dR \\
&= 2\pi^2 bR^2\Big|_{R=0}^{R=a} \\
&= 2\pi^2 ba^2.
\end{aligned}$$

Thus, the volume of T is $\boxed{2\pi^2 ba^2}$. $\square$

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted on Crowdmark.)
Due to the computational nature of this assignment, I will include less notes than usual.

1. In case it helps, here is a diagram to help you visualize the spherical coordinate transformation for the hemisphere:

   This question illustrates that spherical coordinates are useful for integrating over balls in $\mathbb{R}^3$. This is because spherical coordinates reduce the complicated bound $x^2 + y^2 + z^2 < a^2$ into the simple bound $r < a$. The angles ϕ and θ also have simple bounds (which are $0 < \phi < \frac{\pi}{2}$ and $0 < \theta < 2\pi$). Thus, we can use the Change of Variables formula to reduce the original problem into an integral over a rectangle, which we solve using Fubini's theorem.

2. We are trying to integrate over two sets U_1 and U_2, where U_1 is a "ball" (with its centre missing) and U_2 is the entire plane with a ball carved out. Similarly to Question 1, polar coordinate transformations are useful for integrating over balls in $\mathbb{R}^2$.

3. Similarly to Questions 1 and 2, we want to integrate along a rectangle after applying Change of Variables to switch to new coordinates (u, v). For this question, we can try to use the bounds $1 < xy < 2$ and $1 < \frac{y}{x} < 4$. To obtain a rectangle to integrate on, we need a coordinate transformation such that u is in terms of xy and v is in terms of $\frac{y}{x}$. As suggested in the hint, the transformation $x = \frac{u}{v}$ and $y = uv$ helps us to accomplish this.

4. Unlike Questions 1 to 3, it is difficult to transform a tetrahedron into a rectangle. Instead, our next-best option is to transform the tetrahedron into another tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, for which its bounds of integration will be relatively easy to find. We can do this using a transformation that maps vertices of one tetrahedron to vertices of the other tetrahedron.

5. This question was more complex because it required two separate "polar transformations" (at least in my solution). Recall that the torus in the problem was constructed by revolving a disk around an axis. Then, the first polar transformation accounts for the round trajectory of the disk, and the second polar transformation accounts for the round disk itself.