MAT257 Assignment 9 (Partitions of Unity)

(Author's name here)

December 3, 2021

1. We are given that a function $f : \mathbb{R} \to \mathbb{R}$ vanishes for $x < 1$ and on all intervals of the form $(n - \frac{1}{3}, n + \frac{1}{3})$ for all $n \in \mathbb{N}$, yet $\int_{n+1/3}^{n+2/3} f = \frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$.

(a) We will prove that such a function f exists.

Step 1: We will find a candidate for f.

First, we defined the following bump function in class, where $\varepsilon > 0$ is a parameter:

$$\beta_\varepsilon(x) := \begin{cases} e^{-\frac{1}{\varepsilon+x}} e^{-\frac{1}{\varepsilon-x}}, & -\varepsilon < x < \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

It was proven in class that this function has the following useful properties:

- It is C^∞.
- It vanishes outside the interval $(-\varepsilon, \varepsilon)$; in other words, $\beta_\varepsilon(x) = 0$ for all $x \notin (-\varepsilon, \varepsilon)$.
- It has a positive integral; in other words, $\int_{-\varepsilon}^{\varepsilon} \beta_\varepsilon > 0$.

In particular, we could plug in the parameter $\varepsilon := \frac{1}{6}$. This lets us define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{k} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}} \cdot \beta_{1/6}(x - k - \frac{1}{2}) \right), \qquad (*)$$

where $\frac{(-1)^k}{k} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}}$ is a scaling coefficient.

Note that $x - k - \frac{1}{2} \in (-\frac{1}{6}, \frac{1}{6})$ is a necessary condition for $\beta_{1/6}(x - k - \frac{1}{2})$ to be nonzero. Moreover, since the positive integers are 1 unit apart from each other, there will be at most one positive integer k such that $x - k - \frac{1}{2}$ falls in the interval $(-\frac{1}{6}, \frac{1}{6})$. As a result, the above summation is a finite sum (with all but finitely many terms equal to zero), so f is well-defined.

Step 2: We will show that $f(x) = 0$ when $x < 1$ and when $x \in (n - \frac{1}{3}, n + \frac{1}{3})$ for some $n \in \mathbb{N}$.

It suffices to show that the contrapositive is true: If $f(x) \neq 0$, then we will show that $x \geq 1$ and that $x \notin (n - \frac{1}{3}, n + \frac{1}{3})$ for all $n \in \mathbb{N}$. Indeed, if $f(x) \neq 0$, then at least one of the terms in the summation in $(*)$ must be nonzero, so we obtain:

$$\beta_{1/6}(x - k - \frac{1}{2}) \neq 0$$

for some $k \in \mathbb{N}$. This means that $-\frac{1}{6} < x - k - \frac{1}{2} < \frac{1}{6}$, giving us $k + \frac{1}{3} < x < k + \frac{2}{3}$. Then, we obtain the following bounds on x:

(i) First, $x > k + \frac{1}{3} > 1$.

(ii) Next, for all $n \in \mathbb{N}$ such that $n \leq k$, we get $x > k + \frac{1}{3} \geq n + \frac{1}{3}$, so $x \notin (n - \frac{1}{3}, n + \frac{1}{3})$.

(iii) Next, for all $n \in \mathbb{N}$ such that $n \geq k + 1$, we get $x < k + \frac{2}{3} \leq (n - 1) + \frac{2}{3} = n - \frac{1}{3}$, so $x \notin (n - \frac{1}{3}, n + \frac{1}{3})$.

Bound (i) gives us $x \geq 1$, and bounds (ii) and (iii) combined tell us that $x \notin (n - \frac{1}{3}, n + \frac{1}{3})$ for all $n \in \mathbb{N}$. Thus, if $f(x) \neq 0$, then $x \geq 1$ and $x \notin (n - \frac{1}{3}, n + \frac{1}{3})$ for all $n \in \mathbb{N}$. Then, the contrapositive is also true: If $x < 1$, or if $x \in (n - \frac{1}{3}, n + \frac{1}{3})$ for some $n \in \mathbb{N}$, then $f(x) = 0$, as required.

Step 3: We will show that $\int_{n+1/3}^{n+2/3} f = \frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$.

First, for all $x \in [n + \frac{1}{3}, n + \frac{2}{3}]$, we have from $(*)$ that $f(x)$ is a summation whose terms consist of a scaling coefficient multiplied by $\beta_{1/6}(x - k - \frac{1}{2})$. We will show that $\beta_{1/6}(x - k - \frac{1}{2}) = 0$ for all k not equal to n, using the following casework:

- If $k \leq n-1$, then $x-k-\frac{1}{2} \geq (n+\frac{1}{3})-(n-1)-\frac{1}{2} = \frac{5}{6} > \frac{1}{6}$, so $x-k-\frac{1}{2} \notin (-\frac{1}{6}, \frac{1}{6})$, so $\beta_{1/6}(x-k-\frac{1}{2}) = 0$.
- If $k \geq n+1$, then $x-k-\frac{1}{2} \leq (n+\frac{2}{3})-(n+1)-\frac{1}{2} = -\frac{5}{6} < -\frac{1}{6}$, so $x-k-\frac{1}{2} \notin (-\frac{1}{6}, \frac{1}{6})$, so $\beta_{1/6}(x-k-\frac{1}{2}) = 0$.

Thus, the summation in $(*)$ is reduced to the single term corresponding to $k = n$ because all other terms are zero. As a result:

$$f(x) = \frac{(-1)^n}{n} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}} \cdot \beta_{1/6}(x-n-\frac{1}{2})$$

for all $x \in [n+\frac{1}{3}, n+\frac{2}{3}]$. Then, we can evaluate the following integral:

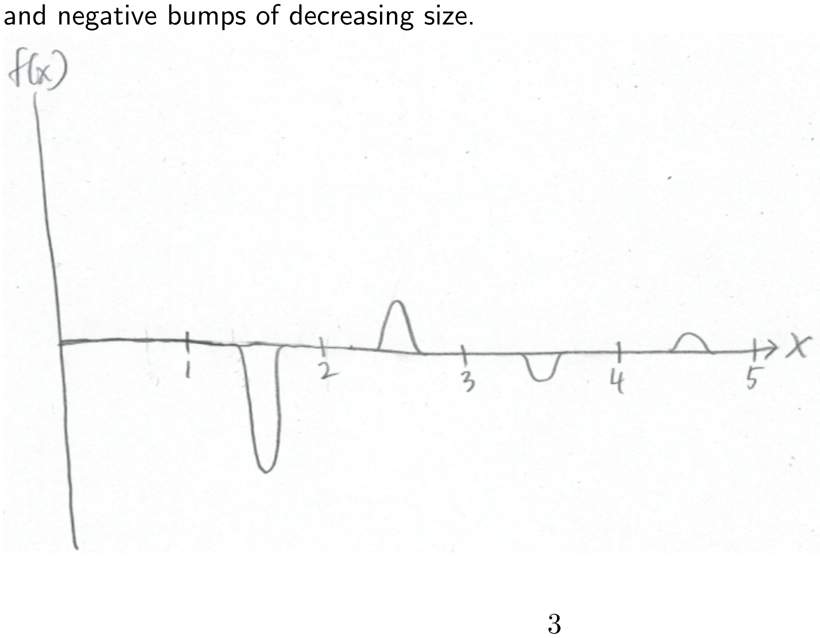
$$\begin{aligned}\int_{n+1/3}^{n+2/3} f &= \int_{n+1/3}^{n+2/3} \frac{(-1)^n}{n} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}} \cdot \beta_{1/6}(x-n-\frac{1}{2})dx \\ &= \frac{(-1)^n}{n} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}} \cdot \int_{n+1/3}^{n+2/3} \beta_{1/6}(x-n-\frac{1}{2})dx.\end{aligned}$$

Here, we use the u-substitution $u = x-n-\frac{1}{2}$, with $du = dx$. The bounds of the integral become $(n+\frac{1}{3})-n-\frac{1}{2} = -\frac{1}{6}$ and $(n+\frac{2}{3})-n-\frac{1}{2} = \frac{1}{6}$. Then, we obtain:

$$\begin{aligned}\int_{n+1/3}^{n+2/3} f &= \frac{(-1)^n}{n} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}} \cdot \int_{u=-1/6}^{u=1/6} \beta_{1/6}(u)du \\ &= \frac{(-1)^n}{n} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}} \cdot \int_{-1/6}^{1/6} \beta_{1/6} \\ &= \frac{(-1)^n}{n}.\end{aligned}$$

Therefore, we constructed a function $f : \mathbb{R} \to \mathbb{R}$ that vanishes for all $x < 1$ and on intervals of the form $(n-\frac{1}{3}, n+\frac{1}{3})$ for all $n \in \mathbb{N}$, yet $\int_{n+1/3}^{n+2/3} f = \frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$, as required.

Step 4: We will sketch a plot of our function f below. Note that the function consists of positive and negative bumps of decreasing size.



□

(b) We will show that f is not NT-integrable.
As explained in class, it suffices to find any open cover $\mathcal{U}$ of $\mathbb{R}$ (using bounded sets) and any partition of unity Φ of $\mathbb{R}$ such that f is not $(\mathcal{U}, \Phi)$-integrable.
Step 1: We will construct candidates for $\mathcal{U}$ and Φ.
First, consider the following open cover of $\mathbb{R}$ using bounded, open sets:

$$\mathcal{U} := \{(k - \frac{1}{3}, k + \frac{4}{3}) : k \in \mathbb{Z}\}.$$

(We will explain why $\mathcal{U}$ covers $\mathbb{R}$. First, every real number x is in between two consecutive integers, so there exists $k \in \mathbb{Z}$ such that $k \leq x \leq k+1$. Then, $k - \frac{1}{3} < k \leq x \leq k+1 < k + \frac{4}{3}$, so $x \in (k - \frac{1}{3}, k + \frac{4}{3})$. Thus, the intervals in $\mathcal{U}$ contain all $x \in \mathbb{R}$, so $\mathcal{U}$ covers $\mathbb{R}$, as desired.)
Next, for all $k \in \mathbb{Z}$, let us define $\phi_k^0 : \mathbb{R} \to [0, \infty)$ by $\phi_k^0(x) := \beta_{3/4}(x - k - \frac{1}{2})$ using the bump function defined above. Then, ϕ_k^0 takes positive values precisely when $-\frac{3}{4} < x - k - \frac{1}{2} < \frac{3}{4}$, which can be rewritten as $x \in (k - \frac{1}{4}, k + \frac{5}{4})$.
Next, to normalize these functions ϕ_k^0, let us define $g : \mathbb{R} \to \mathbb{R}$ by:

$$g(x) := \sum_{k \in \mathbb{Z}} \phi_k^0(x). \qquad (**)$$

For all $x \in \mathbb{R}$, we have $\phi_k^0(x) > 0$ if and only if $k - \frac{1}{4} < x < k + \frac{5}{4}$, or $x - \frac{5}{4} < k < x + \frac{1}{4}$. Since the interval $(x - \frac{5}{4}, x + \frac{1}{4})$ has a "length" of $(x + \frac{1}{4}) - (x - \frac{5}{4}) = \frac{3}{2} > 1$, and since all integers are 1 unit apart, there is a positive, finite number of indices $k \in \mathbb{Z}$ such that $\phi_k^0(x) > 0$. This shows that the summation in $(**)$ is both finite and positive, so g is both well-defined and positive. Moreover, note that g is smooth as a sum of smooth functions.
Next, for all $k \in \mathbb{Z}$, let us define $\phi_k : \mathbb{R} \to \mathbb{R}$ by $\phi_k := \frac{\phi_k^0}{g}$; this division is allowed since g is positive everywhere. Moreover, since $\phi_k^0(x)$ vanishes when x is outside $(k - \frac{1}{4}, k + \frac{5}{4})$, we find that ϕ_k also vanishes outside $(k - \frac{1}{4}, k + \frac{5}{4})$, so $\operatorname{supp} \phi_k \subseteq [k - \frac{1}{4}, k + \frac{5}{4}]$. Then, we will prove that the countable collection $\Phi := \{\phi_k\}_{k \in \mathbb{Z}}$ is a partition of unity for $\mathbb{R}$ subordinate to $\mathcal{U}$:

- First, we will show the local finiteness property at all $x \in \mathbb{R}$. To do this, we will show that finitely many supports $\operatorname{supp} \phi_k$ intersect the open neighbourhood $(x - \frac{1}{4}, x + \frac{1}{4})$ around x. Since $\operatorname{supp} \phi_k \subseteq [k - \frac{1}{4}, k + \frac{5}{4}]$, it suffices to show that $[k - \frac{1}{4}, k + \frac{5}{4}]$ intersects $(x - \frac{1}{4}, x + \frac{1}{4})$ for only finitely many k. This intersection happens only if both of the following inequalities hold at the same time:

$$k - \frac{1}{4} < x + \frac{1}{4}, \qquad k + \frac{5}{4} > x - \frac{1}{4}.$$

  These give us the bounds $k < x + \frac{1}{2}$ and $k > x - \frac{3}{2}$, respectively, so $k \in (x - \frac{3}{2}, x + \frac{1}{2})$. Since this is true for only finitely many integers k, this proves the local finiteness property.
- Next, we will show the "Sum = 1" property at all $x \in \mathbb{R}$:

$$\begin{aligned}\sum_{k \in \mathbb{Z}} \phi_k(x) &= \sum_{k \in \mathbb{Z}} \frac{\phi_k^0(x)}{g(x)} \\ &= \frac{1}{g(x)} \sum_{k \in \mathbb{Z}} \phi_k^0(x) \\ &= \frac{1}{g(x)} g(x) \\ &= 1.\end{aligned}$$

  This proves the "Sum = 1" property.

- Next, for all $x \in \mathbb{R}$, we will show that $\phi_k(x) \in [0,1]$ for all $k \in \mathbb{Z}$. Since $\phi_k(x) = \frac{\phi_k^0(x)}{g(x)}$ is a quotient of nonnegative quantities, it follows that $\phi_k(x)$ is nonnegative for all $k \in \mathbb{Z}$. Since we proved that $\sum_{k\in\mathbb{Z}} \phi_k(x) = 1$, it follows that each $\phi_k(x)$ is at most 1. Thus, $\phi_k(x) \in [0,1]$.
- Next, for all $k \in \mathbb{Z}$, we see that $\phi_k = \frac{\phi_k^0}{g}$ is smooth as a quotient of smooth functions.
- Next, we will show that Φ is subordinate to $\mathcal{U}$. For all $k \in \mathbb{Z}$, we have $k - \frac{1}{4} > k - \frac{1}{3}$ and $k + \frac{5}{4} < k + \frac{4}{3}$, so:
$$\operatorname{supp} \phi_k \subseteq [k - \frac{1}{4}, k + \frac{5}{4}] \subseteq (k - \frac{1}{3}, k + \frac{4}{3}).$$
Thus, $\operatorname{supp} \phi_k \subseteq (k - \frac{1}{3}, k + \frac{4}{3})$ for all $k \in \mathbb{Z}$, so Φ is subordinate to $\mathcal{U}$.

Overall, we proved that Φ is a partition of unity of $\mathbb{R}$ subordinate to $\mathcal{U}$, as desired.
Step 2: We will show that $\int \phi_k|f| \geq \frac{1}{k}$ for all $k \in \mathbb{N}$, and conclude that f is not $(\mathcal{U}, \Phi)$-integrable.
First, for each $k \in \mathbb{N}$, let any $x \in [k + \frac{1}{3}, k + \frac{2}{3}]$ be given. We have for all integers $j > k$ that $x \leq k + \frac{2}{3} \leq (j-1) + \frac{2}{3} < j - \frac{1}{4}$, so $x \notin (j - \frac{1}{4}, j + \frac{5}{4})$, giving us $\phi_j(x) = 0$. We also have for all integers $j < k$ that $x \geq k + \frac{1}{3} \geq (j+1) + \frac{1}{3} > j + \frac{5}{4}$, so $x \notin (j - \frac{1}{4}, j + \frac{5}{4})$, giving us $\phi_j(x) = 0$. Overall, $\phi_j(x) = 0$ for all integers $j \neq k$, so the "Sum = 1" property of Φ gives us $\phi_k(x) = 1$ for all $x \in [k + \frac{1}{3}, k + \frac{2}{3}]$. Then, we can bound $\int \phi_k|f|$ as follows:

$$\begin{aligned}
\int \phi_k|f| &= \int_{-\infty}^{k-\frac{1}{4}} \phi_k|f| + \int_{k-\frac{1}{4}}^{k+\frac{1}{3}} \phi_k|f| + \int_{k+\frac{1}{3}}^{k+\frac{2}{3}} \phi_k|f| + \int_{k+\frac{2}{3}}^{k+\frac{5}{4}} \phi_k|f| + \int_{k+\frac{5}{4}}^{\infty} \phi_k|f| \\
&= \int_{-\infty}^{k-\frac{1}{4}} 0 \cdot |f| + \int_{k-\frac{1}{4}}^{k+\frac{1}{3}} \phi_k \cdot 0 + \int_{k+\frac{1}{3}}^{k+\frac{2}{3}} 1 \cdot |f| + \int_{k+\frac{2}{3}}^{k+\frac{5}{4}} \phi_k \cdot 0 + \int_{k+\frac{5}{4}}^{\infty} 0 \cdot |f| \\
&= \int_{k+\frac{1}{3}}^{k+\frac{2}{3}} |f| \\
&\geq \left| \int_{k+\frac{1}{3}}^{k+\frac{2}{3}} f \right| \qquad \text{(Applying Assignment 7 Question 4)} \\
&= \left| \frac{(-1)^k}{k} \right| \\
&= \frac{1}{k}.
\end{aligned}$$

Thus, $\int \phi_k|f| \geq \frac{1}{k}$ for all $k \in \mathbb{N}$, as desired.
Finally, since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the infinite series $\sum_{k=1}^{\infty} \int \phi_k|f|$ also diverges, so f is not $(\mathcal{U}, \Phi)$-integrable. In other words, $\int_{\mathbb{R}} f$ does not exist, as required. □

(c) We will find two partitions of unity Φ and Ψ of $\mathbb{R}$ such that the sums $\sum_{\phi\in\Phi} \int \phi f$ and $\sum_{\psi\in\Psi} \int \psi f$ absolutely converge, yet to different values.
Step 1: We will construct Φ such that $\sum_{\phi\in\Phi} \left| \int \phi f \right| = \ln(2)$.
First, using the same procedure as in part (b), it is possible to construct a partition of unity $\Phi = \{\phi_k : \mathbb{R} \to [0,1]\}_{k\in\mathbb{Z}}$ such that $\operatorname{supp}(\phi_k) \subseteq [2k - \frac{5}{4}, 2k + \frac{5}{4}]$ for all $k \in \mathbb{Z}$. (There are no gaps between these intervals because $2k + \frac{5}{4} > 2(k+1) - \frac{5}{4}$.)
Next, for all positive integers k, we will show that $\int \phi_k f = -\frac{1}{2k-1} + \frac{1}{2k}$. For all $x \in [2k - \frac{2}{3}, 2k + \frac{2}{3}]$, we have for all integers $j > k$ that $x \leq 2k + \frac{2}{3} \leq 2(j-1) + \frac{2}{3} < 2j - \frac{5}{4}$, so $x \notin [2j - \frac{5}{4}, 2j + \frac{5}{4}]$,

giving us $\phi_j(x) = 0$. We also have for all integers $j < k$ that $x \geq 2k - \frac{2}{3} \geq 2(j+1) - \frac{2}{3} > 2j + \frac{5}{4}$, so $x \notin [2j - \frac{5}{4}, 2j + \frac{5}{4}]$, giving us $\phi_j(x) = 0$. Overall, $\phi_j(x) = 0$ for all integers $j \neq k$, so the "Sum $= 1$" property of Φ gives us $\phi_k = 1$ for all $x \in [2k - \frac{2}{3}, 2k + \frac{2}{3}]$. Then, we can evaluate $\int \phi_k f$ as follows:

$$\begin{aligned}
&\int \phi_k f \\
&= \int_{-\infty}^{2k-\frac{5}{4}} \phi_k f + \int_{2k-\frac{5}{4}}^{2k-\frac{2}{3}} \phi_k f + \int_{2k-\frac{2}{3}}^{2k-\frac{1}{3}} \phi_k f + \int_{2k-\frac{1}{3}}^{2k+\frac{1}{3}} \phi_k f + \int_{2k+\frac{1}{3}}^{2k+\frac{5}{4}} \phi_k f + \int_{2k+\frac{5}{4}}^{\infty} \phi_k f \\
&= \int_{-\infty}^{2k-\frac{5}{4}} 0 \cdot f + \int_{2k-\frac{5}{4}}^{2k-\frac{2}{3}} \phi_k \cdot 0 + \int_{2k-\frac{2}{3}}^{2k-\frac{1}{3}} 1 \cdot f + \int_{2k-\frac{1}{3}}^{2k+\frac{1}{3}} \phi_k \cdot 0 + \int_{2k+\frac{1}{3}}^{2k+\frac{2}{3}} 1 \cdot f + \int_{2k+\frac{2}{3}}^{2k+\frac{5}{4}} \phi_k \cdot 0 + \int_{2k+\frac{5}{4}}^{\infty} 0 \cdot f \\
&= \int_{2k-\frac{2}{3}}^{2k-\frac{1}{3}} f + \int_{2k+\frac{1}{3}}^{2k+\frac{2}{3}} f \\
&= \frac{(-1)^{2k-1}}{2k-1} + \frac{(-1)^{2k}}{2k} \\
&= -\frac{1}{2k-1} + \frac{1}{2k}.
\end{aligned}$$

Next, for all non-positive integers k, we will show that $\int \phi_k f = 0$. Since $f(x)$ vanishes for $x < 1$ and for $x \in (\frac{2}{3}, \frac{4}{3})$, we find that $f(x)$ vanishes for $x < \frac{4}{3}$ overall. Since $k \leq 0$, we get $2k + \frac{5}{4} < \frac{4}{3}$, so $f(x)$ vanishes for $x \leq 2k + \frac{5}{4}$. Then, we can evaluate $\int \phi_k f$ as follows:

$$\begin{aligned}
\int \phi_k f &= \int_{-\infty}^{2k+\frac{5}{4}} \phi_k f + \int_{2k+\frac{5}{4}}^{\infty} \phi_k f \\
&= \int_{-\infty}^{2k+\frac{5}{4}} \phi_k \cdot 0 + \int_{2k+\frac{5}{4}}^{\infty} 0 \cdot f \\
&= 0.
\end{aligned}$$

Therefore, we can evaluate the sum $\sum_{k\in\mathbb{Z}} \left| \int \phi_k f \right|$ as follows:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \left| \int \phi_k f \right| &= \sum_{\substack{k\in\mathbb{Z} \\ k\leq 0}} \left| \int \phi_k f \right| + \sum_{k=1}^{\infty} \left| \int \phi_k f \right| \\
&= 0 + \sum_{k=1}^{\infty} \left| -\frac{1}{2k-1} + \frac{1}{2k} \right| \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k} \right) \\
&= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots .
\end{aligned}$$

Using Taylor series, it is known that the summation above converges to $\ln(2)$, as desired.

Step 2: We will similarly construct Ψ such that $\sum_{\psi\in\Psi} \left| \int \psi f \right| = 2 - \ln(2)$.

First, using the same procedure as in part (b), it is possible to construct a partition of unity $\Psi = \{\psi_k : \mathbb{R} \to [0, 1]\}_{k\in\mathbb{Z}}$ such that $\operatorname{supp}(\psi_k) \subseteq [2k - \frac{1}{4}, 2k + \frac{9}{4}]$ for all $k \in \mathbb{Z}$.

Next, for all positive integers k, we will show that $\int \psi_k f = \frac{1}{2k} - \frac{1}{2k+1}$. For all $x \in [2k+\frac{1}{3}, 2k+\frac{5}{3}]$, we have for all integers $j > k$ that $x \leq 2k + \frac{5}{3} \leq 2(j-1) + \frac{5}{3} < 2j - \frac{1}{4}$, so $x \notin [2j - \frac{1}{4}, 2j + \frac{9}{4}]$, giving us $\psi_j(x) = 0$. We also have for all integers $j < k$ that $x \geq 2k+\frac{1}{3} \geq 2(j+1)+\frac{1}{3} > 2j+\frac{9}{4}$, so $x \notin [2j - \frac{1}{4}, 2j + \frac{9}{4}]$, giving us $\psi_j(x) = 0$. Overall, $\psi_j(x) = 0$ for all integers $j \neq k$, so the "Sum = 1" property of Ψ gives us $\psi_k(x) = 1$ for all $x \in [2k + \frac{1}{3}, 2k + \frac{5}{3}]$. Then, we can evaluate $\int \psi_k f$ as follows:

$$\begin{aligned}
&\int \psi_k f \\
&= \int_{-\infty}^{2k-\frac{1}{4}} \psi_k f + \int_{2k-\frac{1}{4}}^{2k+\frac{1}{3}} \psi_k f + \int_{2k+\frac{1}{3}}^{2k+\frac{2}{3}} \psi_k f + \int_{2k+\frac{2}{3}}^{2k+\frac{4}{3}} \psi_k f + \int_{2k+\frac{4}{3}}^{2k+\frac{5}{3}} \psi_k f + \int_{2k+\frac{5}{3}}^{2k+\frac{9}{4}} \psi_k f + \int_{2k+\frac{9}{4}}^{\infty} \psi_k f \\
&= \int_{-\infty}^{2k-\frac{1}{4}} 0 \cdot f + \int_{2k-\frac{1}{4}}^{2k+\frac{1}{3}} \psi_k \cdot 0 + \int_{2k+\frac{1}{3}}^{2k+\frac{2}{3}} 1 \cdot f + \int_{2k+\frac{2}{3}}^{2k+\frac{4}{3}} \psi_k \cdot 0 + \int_{2k+\frac{4}{3}}^{2k+\frac{5}{3}} 1 \cdot f + \int_{2k+\frac{5}{3}}^{2k+\frac{9}{4}} \psi_k \cdot 0 + \int_{2k+\frac{9}{4}}^{\infty} 0 \cdot f \\
&= \int_{2k+\frac{1}{3}}^{2k+\frac{2}{3}} f + \int_{2k+\frac{4}{3}}^{2k+\frac{5}{3}} f \\
&= \frac{(-1)^{2k}}{2k} + \frac{(-1)^{2k+1}}{2k+1} \\
&= \frac{1}{2k} - \frac{1}{2k+1}.
\end{aligned}$$

Next, if $k = 0$, we will show that $\int \psi_k f = -1$. Similarly to the $k > 0$ case, we have $\psi_k(x) = 1$ for all $x \in [2k + \frac{1}{3}, 2k + \frac{5}{3}]$, or $x \in [\frac{1}{3}, \frac{5}{3}]$. Moreover, similarly to Step 1, $f(x)$ vanishes for $x < \frac{4}{3}$. Then, we can evaluate $\int \psi_k f$ as follows:

$$\begin{aligned}
\int \psi_k f &= \int_{-\infty}^{\frac{4}{3}} \psi_k f + \int_{\frac{4}{3}}^{\frac{5}{3}} \psi_k f + \int_{\frac{5}{3}}^{\frac{9}{4}} \psi_k f + \int_{\frac{9}{4}}^{\infty} \psi_k f \\
&= \int_{-\infty}^{\frac{4}{3}} \psi_k \cdot 0 + \int_{\frac{4}{3}}^{\frac{5}{3}} 1 \cdot f + \int_{\frac{5}{3}}^{\frac{9}{4}} \psi_k \cdot 0 + \int_{\frac{9}{4}}^{\infty} 0 \cdot f \\
&= \int_{\frac{4}{3}}^{\frac{5}{3}} f \\
&= \frac{(-1)^1}{1} \\
&= -1.
\end{aligned}$$

Next, for all negative integers k, we will show that $\int \psi_k f = 0$. We have $2k + \frac{9}{4} \leq 2(-1) + \frac{9}{4} < 1$, so $f(x)$ vanishes for $x \leq 2k + \frac{9}{4}$. Then, we can evaluate $\int \psi_k f$ as follows:

$$\begin{aligned}
\int \psi_k f &= \int_{-\infty}^{2k+\frac{9}{4}} \psi_k f + \int_{2k+\frac{9}{4}}^{\infty} \psi_k f \\
&= \int_{-\infty}^{2k+\frac{9}{4}} \psi_k \cdot 0 + \int_{2k+\frac{9}{4}} 0 \cdot f \\
&= 0.
\end{aligned}$$

Therefore, we can evaluate the sum $\sum_{k\in\mathbb{Z}}\left|\int \psi_k f\right|$ as follows:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\left|\int \psi_k f\right| &= \sum_{\substack{k\in\mathbb{Z}\\k<0}}\left|\int \psi_k f\right| + \left|\int \psi_0 f\right| + \sum_{k=1}^{\infty}\left|\int \psi_k f\right| \\
&= 0 + |-1| + \sum_{k=1}^{\infty}\left|\frac{1}{2k} - \frac{1}{2k+1}\right| \\
&= 1 + \sum_{k=1}^{\infty}\left(\frac{1}{2k} - \frac{1}{2k+1}\right) \\
&= 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \\
&= 2 - \left(\frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{5}\right) + \cdots\right).
\end{aligned}$$

Using Taylor series, it is known that the summation above converges to $2 - \ln(2)$, as desired. Finally, since $2 - \ln(2) \neq \ln(2)$, we conclude from Steps 1 and 2 that $\sum_{k\in\mathbb{Z}}\left|\int \phi_k f\right|$ and $\sum_{k\in\mathbb{Z}}\left|\int \psi_k f\right|$ converge to different values. Thus, we constructed two partitions of unity Φ, Ψ of $\mathbb{R}$ such that $\sum_{\phi\in\Phi}\int \phi f$ and $\sum_{\psi\in\Psi}\int \psi f$ both absolutely converge, yet to different values, as required. $\square$

2. We are given an arbitrary subset $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}$ that is smooth at every $a \in A$. Then, we will prove that f can be extended to a smooth function g on some open set $V \supseteq A$.

   Step 1: We will use a certain partition of unity of A to construct a candidate for g and V.
   First, we are given that f is smooth at a for all $a \in A$. By definition, this means that there exist an open set U_a and a smooth function $g_a : U_a \to \mathbb{R}$ such that f and g_a agree on $A \cap U_a$. Then, since $a \in U_a$ for all $a \in A$, the sets $\{U_a\}_{a \in A}$ form an open cover of A; let us denote this open cover by $\mathcal{U}$.
   Now, let $\Phi = \{\phi_i : V \to [0, 1]\}_{i \in \mathbb{N}}$ be a partition of unity of A subordinate to $\mathcal{U}$, defined on some open neighbourhood $V \supseteq A$. (This is the same V on which we will extend the smooth function f.) Since Φ is subordinate to $\mathcal{U}$, it follows that for all $i \in \mathbb{N}$, there exists some point $a_i \in A$ such that $\operatorname{supp} \phi_i \subseteq U_{a_i}$.
   Now, let us define $g : V \to \mathbb{R}$ by $g(x) := \sum_{i=1}^{\infty} \phi_i(x) g_{a_i}(x)$. By the notation "$\phi_i(x) g_{a_i}(x)$", we mean the following function:

$$\begin{cases} \phi_i(x) g_{a_i}(x), & \text{if } \phi_i(x) > 0; \\ 0, & \text{if } \phi_i(x) = 0. \end{cases}$$

   Then, since $\operatorname{supp} \phi_i \subseteq U_{a_i}$, we have that $g_{a_i}(x)$ is well-defined whenever $\phi_i(x) > 0$, so $\phi_i(x) g_{a_i}(x)$ is well-defined everywhere (on V). Moreover, by the local finiteness property of Φ, we have $\phi_i(x) g_{a_i}(x) = 0$ for all but finitely many i, so $g(x) = \sum_{i=1}^{\infty} \phi_i(x) g_{a_i}(x)$ is a finite sum. Thus, g is well-defined.
   Step 2: We will prove that g satisfies the required properties.
   First, we will show that g is smooth as a sum of smooth functions. Formally, at all $a \in V$, by the local finiteness property of Φ, there exists an open neighbourhood $W \ni a$ contained in V such that $\operatorname{supp} \phi_i \cap W \neq \emptyset$ for only finitely many $i \in \mathbb{N}$. In other words, there exist only finitely many $i \in \mathbb{N}$ such that $\phi_i(x)$ is not flat zero over $x \in U_i$. Then, on W, the function $g(x) := \sum_{i=1}^{\infty} \phi_i(x) g_{a_i}(x)$ is reduced to a sum of finitely many smooth functions. As a result, all partial derivatives of g of all orders exist and are continuous at a. This is true for all $a \in V$, so g is smooth, as required.
   Next, we will prove that g agrees with f on A. For all $i \in \mathbb{N}$ and all $x \in A$, we claim that $\phi_i(x) g_{a_i}(x) = \phi_i(x) f(x)$. We have the following two cases for x:

   (i) If $\phi_i(x) = 0$, then LS = RS = 0, and we are done.
   (ii) If $\phi_i(x) \neq 0$, then $x \in \operatorname{supp} \phi_i \subseteq U_{a_i}$. We also assumed that $x \in A$, so $x \in A \cap U_{a_i}$. Since g_{a_i} agrees with f on $A \cap U_{a_i}$, it follows that $g_{a_i}(x) = f(x)$, so $\phi_i(x) g_{a_i}(x) = \phi_i(x) f(x)$, as desired.

   In either case, we showed that $\phi_i(x) g_{a_i}(x) = \phi_i(x) f(x)$, as desired. As a result, we obtain for all $x \in A$ that:

$$g(x) = \sum_{i=1}^{\infty} \phi_i(x) g_{a_i}(x) = \sum_{i=1}^{\infty} \phi_i(x) f(x) = f(x) \sum_{i=1}^{\infty} \phi_i(x).$$

   Finally, since Φ is a partition of unity of A, and since $x \in A$, we have $\sum_{i=1}^{\infty} \phi_i(x) = 1$, and it follows that $g(x) = f(x)$.
   Therefore, we constructed a smooth function $g : V \to \mathbb{R}$ on an open set $V \supseteq A$ such that g and f agree on A, as required. □

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted on Crowdmark.)

1. For part (a), we want to construct a function $f : \mathbb{R} \to \mathbb{R}$ such that it vanishes on certain intervals (intervals of the form $(n - \frac{1}{3}, n + \frac{1}{3})$) and such that it is nonzero on other intervals (intervals of the form $(n + \frac{1}{3}, n + \frac{2}{3})$). This motivates us to use the smooth "bump" function β_ε constructed in lecture, since the bump function is constructed to be positive in an interval $(-\varepsilon, \varepsilon)$ and to vanish elsewhere. Since we require f to be nonzero in multiple disjoint intervals, we have to add together multiple bumps to construct f. Finally, since we require each bump to have a certain integral, we must normalize each bump. This leads us to the desired formula:
$$f(x) := \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{k} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}} \cdot \beta_{1/6}(x - k - \frac{1}{2}) \right),$$
where $\beta_{1/6}(x - k - \frac{1}{2})$ gives us the required bumps centred at each $k + \frac{1}{2}$, and $\frac{(-1)^k}{k} \cdot \frac{1}{\int_{-1/6}^{1/6} \beta_{1/6}}$ normalizes the bump to have an integral of $\frac{(-1)^k}{k}$.
For part (b), it suffices to prove that f is not $(\mathcal{U}, \Phi)$-integrable for some convenient open cover $\mathcal{U}$ of $\mathbb{R}$ and some convenient partition of unity $\Phi = \{\phi_k\}_{k \in \mathbb{Z}}$ subordinate to $\mathcal{U}$. This requires us to prove that $\sum_k \int \phi_k |f|$ diverges to infinity. For convenience, we can make each function ϕ_k "contribute weight" on exactly one of the bumps. In other words, for all $k \in \mathbb{Z}$, we construct ϕ_k such that $\phi_k(x) = 1$ for all $x \in [k + \frac{1}{3}, k + \frac{2}{3}]$ and such that ϕ_k vanishes on all other intervals of the form $[n + \frac{1}{3}, n + \frac{2}{3}]$. Then, the resulting partition of unity is structured neatly so that we get $\int \phi_k |f| \geq \frac{1}{k}$ from the k^{th} bump. Finally, the series $\sum_{k=1}^{\infty} \int \phi_k |f| \geq \sum_{k=1}^{\infty} \frac{1}{k}$ diverges because the harmonic series diverges, as required. (Remark: The reason why I explicitly constructed the partition of unity Φ in my solution is to avoid potential complications from having multiple functions "contribute weight" on the same bump.)
For part (c), we want $\sum_{\phi \in \Phi} \int \phi f$ and $\sum_{\psi \in \Psi} \int \psi f$ to absolutely converge. To do this, the key idea is to neatly arrange partitions of unity so that each function ϕ_k "contributes weight" on *two* consecutive bumps. That way, integrals of consecutive bumps will cancel out because they have opposite signs, and we avoid the harmonic series that we obtained in part (b). There are two ways to organize this arrangement: We can have ϕ_k contribute to the $(2k-1)^{\text{th}}$ bump and the $(2k)^{\text{th}}$ bump (i.e., odd-indexed before even-indexed), and we can have ψ_k contribute to the $(2k)^{\text{th}}$ bump and the $(2k+1)^{\text{th}}$ bump (i.e., even-indexed before odd-indexed). This produces two different partitions of unity Φ and Ψ. Then, the series $\sum_{\phi \in \Phi} \int \phi f$ and $\sum_{\psi \in \Psi} \int \psi f$ happen to absolutely converge to different values when we compute them, so we are done.

2. We are given a local statement "f is smooth at all $a \in A$", and we want to prove a global statement "f can be extended to a smooth function g on some open set $V \supseteq A$". This question illustrates a major role of partitions of unity: Converting from local statements to global statements by "piecing together" the local statements. First, using the local statement "f is smooth at all $a \in A$", we can construct open sets U_a around each $a \in A$ on which f can be locally extended to smooth functions $g_a : U_a \to \mathbb{R}$. Then, we use the open cover $\mathcal{U} = \{U_a\}_{a \in A}$ to construct a partition of unity $\Phi = \{\phi_i\}_{i \in \mathbb{N}}$ subordinate to that open cover. Finally, we can combine the pieces g_a, using the ϕ_i functions as weights, to produce a function g that agrees with f on the entire set A. Since the sum of smooth functions is smooth, g is our required smooth function.