MAT257 Assignment 8 (Fubini's Theorem)

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1. Given a Jordan-measurable set A, and given any $\varepsilon > 0$, we will show that there is a compact Jordan-measurable set $C \subseteq A$ such that the volume of $A - C$ is less than ε.
Step 1: We will construct a candidate for C.
First, since A is Jordan-measurable, A is also bounded. We will use the fact that A is bounded to proved that $\mathrm{Bd}(A)$ is also bounded, where $\mathrm{Bd}(A)$ denotes the boundary of A.
Since A is bounded, there exists some radius $r > 0$ such that all $x \in A$ satisfy $|x| < r$. Then, we claim that all $y \in \mathbb{R}^n$ satisfying $|y| > r$ are in the exterior of A. Given any y such that $|y| > r$, consider the open ball $B_{|y|-r}(y)$ of radius $|y| - r > 0$ around y. Then, all $z \in B_{|y|-r}(y)$ satisfy:

$$\begin{aligned} |z| &\geq |y| - |y - z| && \text{(Triangle inequality)} \\ &> |y| - (|y| - r) && \text{(Since } z \in B_{|y|-r}(y)\text{)} \\ &= r. \end{aligned}$$

Thus, for all $z \in B_{|y|-r}(y)$, we get $|z| > r$, and it follows that $z \notin A$. In other words, there exists an open neighbourhood $B_{|y|-r}(y)$ around y that contains no points in A, so y is in the exterior of A, as desired. It follows that y is not in $\mathrm{Bd}(A)$ whenever $|y| > r$. Then, the contrapositive of this statement is also true: all points in $\mathrm{Bd}(A)$ must have magnitude at most r, so $\mathrm{Bd}(A)$ is bounded.
Next, $\mathrm{Bd}(A)$ is known to be closed because the interior and exterior of A are open, and their union forms the complement of $\mathrm{Bd}(A)$. Thus, $\mathrm{Bd}(A)$ is both closed and bounded, so by Spivak's Corollary 1-7, $\mathrm{Bd}(A)$ is compact.
Next, since A is Jordan-measurable, $\mathrm{Bd}(A)$ has measure zero, so there exists a countable collection $\{U_i\}_{i \in \mathbb{N}}$ of open rectangles of total volume less than ε which cover $\mathrm{Bd}(A)$. Since $\mathrm{Bd}(A)$ is compact, we can extract a finite subcover $\{U_i\}_{i \in I}$ of $\mathrm{Bd}(A)$, where $I \subseteq \mathbb{N}$ is a finite indexing set. This subcover is a subset of the initial rectangles, so its total volume remains less than ε.
Finally, let us define $C := A - \bigcup_{i \in I} U_i$. Note that, by construction, $C \subseteq A$.
Step 2: We will show that C also satisfies the following 3 properties.
Property 1: C is bounded. Indeed, C must be bounded as a subset of the bounded set A.
Property 2: C is closed. It suffices to show that $\mathbb{R}^n - C$ is open. There are two ways for some point x to be outside $C = A - \bigcup_{i \in I} U_i$: We can have $x \in \bigcup_{i \in I} U_i$ or $x \notin A$. This gives us the following casework:
Case 1: $x \in \bigcup_{i \in I} U_i$. Then, we have $x \in U_i$ for some $i \in I$. By construction, U_i is completely outside C, so U_i is an open neighbourhood of x contained in $\mathbb{R}^n - C$.
Case 2: $x \notin A$. Then, x cannot be in the interior of A, so it must be in the boundary or exterior of A. If x is in the boundary of A, then we obtain $x \in \bigcup_{i \in I} U_i$ because $\{U_i\}_{i \in I}$ is an open cover of $\mathrm{Bd}(A)$, and we proceed as in Case 1. Otherwise, if x is in the exterior of A, then the exterior of A is an open neighbourhood of x contained in $\mathbb{R}^n - A$. Since $C \subseteq A$, it follows that the exterior of A is an open neighbourhood of x contained in $\mathbb{R}^n - C$.
Overall, we proved for all $x \in \mathbb{R}^n - C$ that there exists an open neighbourhood of x contained in $\mathbb{R}^n - C$. As a result, $\mathbb{R}^n - C$ is open, so C is closed, as required.
Property 3: $\mathrm{Bd}(C)$ has measure 0. To prove this, we begin with two Lemmas:
Lemma 1: For any open rectangle $R = (a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n$, we will explain why $\mathrm{Bd}(R)$ has measure 0.
First, as explained in lecture, $\mathrm{Bd}(R)$ consists of the "faces" of R, which are the finitely many closed $(n-1)$-dimensional rectangles at the boundaries $a_1, b_1, \ldots, a_n, b_n$ of R. It was also explained in lecture that each $(n-1)$-dimensional rectangle has measure 0, since each such rectangle can be covered with an arbitrarily thin n-dimensional rectangle of arbitrarily small volume. Thus, $\mathrm{Bd}(R)$ is measure-0 as a finite union of measure-0 sets (the "faces" of R), as desired.

Lemma 2: For any point $x \in \mathrm{Bd}(C)$, we will show that $x \in \mathrm{Bd}(U_i)$ for some $i \in I$.
We will prove this by proving the contrapositive: Given any $x \in \mathbb{R}^n$ such that $x \notin \mathrm{Bd}(U_i)$ for all $i \in I$, we will prove that $x \notin \mathrm{Bd}(C)$. Since x is in either the interior or exterior of U_i for all $i \in I$, there are several cases for such x:
Case 1: x is in the interior of U_i for some $i \in I$. Then, since C was defined to be $A - \bigcup_{i \in I} U_i$, we see that C contains no points in U_i. Thus, U_i is an open neighbourhood of x contained outside C, so x is in the exterior of C, not the boundary of C.
Case 2: x is in the exterior of U_i for all $i \in I$, and $x \notin A$. Then, since x is not contained in any U_i for $i \in I$, and since $\{U_i\}_{i \in I}$ covers $\mathrm{Bd}(A)$, it follows that $x \notin \mathrm{Bd}(A)$. Since x is also not inside A, it follows that x is in the exterior of A. Then, the exterior of A is an open neighbourhood of x contained outside A. Finally, since $C \subseteq A$ by construction, the exterior of A is also an open neighbourhood of x contained outside C. Thus, x is in the exterior of C, not the boundary of C.
Case 3: x is in the exterior of U_i for all $i \in I$, and $x \in A$. Then, similarly to Case 2, $x \notin \mathrm{Bd}(A)$. This time, since $x \in A$, it follows that x is inside the interior of A. Since we also assumed that x is in the exterior of U_i for all $i \in I$, it follows that the intersection:

$$U := (\text{interior of } A) \cap \bigcap_{i \in I} (\text{exterior of } U_i)$$

contains x. Here, since I is finite, U is open as a finite intersection of open sets. Moreover, every point in U is inside the interior of A and inside the exteriors of the rectangles in $\{U_i\}_{i \in \mathbb{N}}$, so every point in U is inside A and outside the rectangles in $\{U_i\}_{i \in \mathbb{N}}$. In other words, every point in U is also inside C, so $U \subseteq C$. Thus, U is an open neighbourhood of x contained in C, so x is in the interior of C, not the boundary of C.
Overall, we proved that if $x \notin \mathrm{Bd}(U_i)$ for all $i \in I$, then $x \notin \mathrm{Bd}(C)$. Therefore, the contrapositive is also true: If $x \in \mathrm{Bd}(C)$, then x must be in $\mathrm{Bd}(U_i)$ for some $i \in I$, as desired.
Now, to prove Property 3, we know from Lemma 1 that $\mathrm{Bd}(U_i)$ has measure 0 for all $i \in I$. Then, the union $\bigcup_{i \in I} \mathrm{Bd}(U_i)$ is also measure-0 as a finite union of measure-0 sets. Finally, Lemma 2 implies that $\mathrm{Bd}(C) \subseteq \bigcup_{i \in I} \mathrm{Bd}(U_i)$, so $\mathrm{Bd}(C)$ is also measure-0 as a subset of a measure-0 set, as required.
Next, using Spivak's Corollary 1-7, it follows from Property 1 (boundedness) and Property 2 (closedness) that C is compact. Additionally, it follows from Property 1 (boundedness) and Property 3 ($\mathrm{Bd}(C)$ has measure 0) that C is Jordan-measurable. Thus, C is both compact and Jordan-measurable, so it remains to prove that $A - C$ has a volume less than ε by proving that $\int_{A-C} 1$ exists and is less than ε.
Step 3: We will prove that $\int_{A-C} 1$ exists.
First, we claim that all points $x \in \mathrm{Bd}(A - C)$ satisfy $x \in \mathrm{Bd}(A)$ or $x \in \mathrm{Bd}(C)$. We will prove this by proving the contrapositive: If $x \notin \mathrm{Bd}(A), \mathrm{Bd}(C)$, then we will prove that $x \notin \mathrm{Bd}(A - C)$ using the following cases:
Case 1: x is in the interior of A and the interior of C. Then, we see that the interior of C is an open neighbourhood of x, and this open neighbourhood is contained outside $A - C$ because it is contained in C. As a result, x is in the exterior of $A - C$, not the boundary of $A - C$.
Case 2: x is in the interior of A and the exterior of C. Then, let us define U to be the intersection of the interior of A and the exterior of C. We see that U is open as a finite intersection of open sets. Moreover, U is both contained inside A and contained outside C. Thus, U is an open neighbourhood of x contained in $A - C$, so x is in the interior of $A - C$, not the boundary of $A - C$.
Case 3: x is in the exterior of A and the interior of C. Then, x would be both outside A and inside C, which is impossible since $C \subseteq A$. Thus, this case is impossible.

Case 4: x is in the exterior of A and the exterior of C. Then, we see that the exterior of A is an open neighbourhood of x, and this open neighbourhood is contained outside $A - C$ because it is contained outside A. As a result, x is in the exterior of $A - C$, not the boundary of $A - C$.
Overall, we proved in all cases that $x \notin \mathrm{Bd}(A - C)$ whenever $x \notin \mathrm{Bd}(A), \mathrm{Bd}(C)$. It follows that the contrapositive is true: All points $x \in \mathrm{Bd}(A - C)$ satisfy $x \in \mathrm{Bd}(A)$ or $x \in \mathrm{Bd}(C)$, so $\mathrm{Bd}(A - C) \subseteq \mathrm{Bd}(A) \cup \mathrm{Bd}(C)$. Now, recall that both A and C are Jordan-measurable, so $\mathrm{Bd}(A)$ and $\mathrm{Bd}(C)$ have measure 0. Then, since $\mathrm{Bd}(A) \cup \mathrm{Bd}(C)$ is measure-0 as a finite union of measure-0 sets, it follows that $\mathrm{Bd}(A - C)$ is measure-0 as a subset of a measure-0 set. Finally, $A - C$ is bounded because A is bounded. Therefore, $A - C$ is Jordan-measurable, so Spivak's Theorem 3-9 states that $\int_{A-C} 1$ is well-defined, as required.
Step 4: We will conclude that $A - C$ has a total volume less than ε.
Since $A - C$ is bounded, and since the finitely many rectangles in $\{U_i\}_{i \in I}$ are bounded, there exists some closed rectangle R containing $A - C$ and all rectangles in $\{U_i\}_{i \in I}$. Then, we can use the boundaries of the rectangles $\{U_i\}_{i \in I}$ to form cutpoints of a partition P of R. With this construction, since the rectangles $\{U_i\}_{i \in I}$ cover $A - C$, every subrectangle of P that intersects $A - C$ must be completely contained in some rectangle of the cover $\{U_i\}_{i \in I}$. Then, we bound the upper sum $U(\chi_{A-C}, P)$ as follows:

$$\begin{aligned}
U(\chi_{A-C}, P) &= \sum_{S \in P} M_S(\chi_{A-C}) \operatorname{vol}(S) \\
&= \sum_{\substack{S \in P \\ S \cap (A-C) = \emptyset}} M_S(\chi_{A-C}) \operatorname{vol}(S) + \sum_{\substack{S \in P \\ S \cap (A-C) \neq \emptyset}} M_S(\chi_{A-C}) \operatorname{vol}(S) \\
&= 0 + \sum_{\substack{S \in P \\ S \cap (A-C) \neq \emptyset}} 1 \cdot \operatorname{vol}(S) \\
&\leq \sum_{\substack{S \in P \\ S \subseteq \bigcup_{i \in I} U_i}} \operatorname{vol}(S) \\
&\leq \sum_{i \in I} \sum_{\substack{S \in P \\ S \subseteq U_i}} \operatorname{vol}(S) \\
&= \sum_{i \in I} \operatorname{vol}(U_i) \\
&< \varepsilon.
\end{aligned}$$

Therefore, we obtain $\int_{A-C} 1 = \int_R \chi_{A-C} \leq U(\chi_{A-C}, P) < \varepsilon$. (Again, the term $\int_R \chi_{A-C}$ is well-defined because $A - C$ is Jordan-measurable.)
Overall, given any $\varepsilon > 0$, we constructed a compact Jordan-measurable set $C \subseteq A$ such that the volume of $A - C$ is less than ε, as required. □

2. Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $D_{1,2}f$ and $D_{2,1}f$ exist and are continuous, we will prove that $D_{1,2}f = D_{2,1}f$.
Assume for contradiction that there exists some $a \in \mathbb{R}^n$ such that $D_{1,2}f(a) \neq D_{2,1}f(a)$, and let us fix a. Also assume without loss of generality that $D_{1,2}f(a) > D_{2,1}f(a)$, so $(D_{1,2}f - D_{2,1}f)(a) > 0$. Since $D_{1,2}f$ and $D_{2,1}f$ are continuous functions, their difference, $D_{1,2}f - D_{2,1}f$, is continuous. For convenience, let $D_{1,2}f - D_{2,1}f$ be denoted by the function g. Then, g is continuous, and we assumed above that $g(a) > 0$. Since g is continuous, and since the interval $(\frac{g(a)}{2}, \infty)$ is open, the preimage $g^{-1}((\frac{g(a)}{2}, \infty))$ is also open. Since $g(a) > 0$, we also have $g(a) > \frac{g(a)}{2}$, which means that a is in the open set $g^{-1}((\frac{g(a)}{2}, \infty))$. As a result, there exists a closed rectangle R of positive volume around a such that $R \subseteq g^{-1}((\frac{g(a)}{2}, \infty))$; in other words, $g(x) > \frac{g(a)}{2}$ for all $x \in R$. (We find R by first finding an open rectangle R' such that $a \in R' \subseteq g^{-1}((\frac{g(a)}{2}, \infty))$. Since R' does not contain its boundaries, R' must have positive volume to contain a, so we shrink its boundaries slightly to obtain the closed rectangle R with positive volume such that $a \in R \subseteq R' \subseteq g^{-1}((\frac{g(a)}{2}, \infty))$).
Now, we shall obtain a contradiction through the following two steps.
Step 1: We will prove that $\int_R g > 0$. (Note that g is integrable because g is continuous.)
Since $g(x) > \frac{g(a)}{2}$ for all $x \in R$, it follows that g is greater than the constant function $\frac{g(a)}{2}$ on R. Then, it follows from Assignment 7 Question 3 that:

$$\int_R g \geq \int_R \frac{g(a)}{2} = \frac{g(a)}{2}\,\mathrm{vol}(R) > 0,$$

so $\int_R g > 0$, as desired.
Step 2: We will prove, to the contrary, that $\int_R g = 0$.
First, let us split R into the Cartesian product $[a_1, b_1] \times [a_2, b_2] \times C \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$, where the rectangle $[a_1, b_1] \subseteq \mathbb{R}$ is assigned the x_1-direction, the rectangle $[a_2, b_2] \subseteq \mathbb{R}$ is assigned the x_2-direction, and the rectangle $C \subseteq \mathbb{R}^{n-2}$ is assigned the remaining directions. Correspondingly, we will write each point $x \in R$ using the coordinates $(x_1, x_2, x^*) \in [a_1, b_1] \times [a_2, b_2] \times C$. Then, since $D_{1,2}f$ is continuous, it follows from Spivak's Remark 2 below Theorem 3-10 that:

$$\begin{aligned}\int_R D_{1,2}f &= \int_C \left(\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} D_{1,2}f(x_1, x_2, x^*)dx_2\right)dx_1\right)dx^* \\ &= \int_C \left(\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} f(x_1, x_2, x^*)dx_2\right)dx_1\right)dx^*.\end{aligned}$$

By the Fundamental Theorem of Calculus, we have:

$$\begin{aligned}\int_{a_2}^{b_2} \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} f(x_1, x_2, x^*)dx_2 &= \frac{\partial}{\partial x_1} f(x_1, x_2, x^*)\Big|_{x_2=a_2}^{x_2=b_2} \\ &= \frac{\partial}{\partial x_1} f(x_1, b_2, x^*) - \frac{\partial}{\partial x_1} f(x_1, a_2, x^*).\end{aligned}$$

This allows us to simplify $\int_R D_{1,2}f$ as:

$$\int_R D_{1,2}f = \int_C \left(\int_{a_1}^{b_1} \left(\frac{\partial}{\partial x_1} f(x_1, b_2, x^*) - \frac{\partial}{\partial x_1} f(x_1, a_2, x^*)\right)dx_1\right)dx^*.$$

Applying the Fundamental Theorem of Calculus again, we obtain:

$$\begin{aligned}
&\int_{a_1}^{b_1}\left(\frac{\partial}{\partial x_1}f(x_1,b_2,x^*)-\frac{\partial}{\partial x_1}f(x_1,a_2,x^*)\right)dx_1\\
&=(f(x_1,b_2,x^*)-f(x_1,a_2,x^*))\Big|_{x_1=a_1}^{x_1=b_1}\\
&=f(b_1,b_2,x^*)-f(b_1,a_2,x^*)-f(a_1,b_2,x^*)+f(a_1,a_2,x^*).
\end{aligned}$$

This allows us to simplify $\int_R D_{1,2}f$ as:

$$\int_R D_{1,2}f=\int_C(f(b_1,b_2,x^*)-f(b_1,a_2,x^*)-f(a_1,b_2,x^*)+f(a_1,a_2,x^*))dx^*. \qquad (*)$$

Similarly, since $D_{2,1}f$ is continuous, it follows from Remark 2 below Theorem 3-10 that:

$$\begin{aligned}
\int_R D_{2,1}f&=\int_C\left(\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1}D_{2,1}f(x_1,x_2,x^*)dx_1\right)dx_2\right)dx^*\\
&=\int_C\left(\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}f(x_1,x_2,x^*)dx_1\right)dx_2\right)dx^*.
\end{aligned}$$

By the Fundamental Theorem of Calculus, we have:

$$\begin{aligned}
\int_{a_1}^{b_1}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}f(x_1,x_2,x^*)dx_1&=\frac{\partial}{\partial x_2}f(x_1,x_2,x^*)\Big|_{x_1=a_1}^{x_1=b_1}\\
&=\frac{\partial}{\partial x_2}f(b_1,x_2,x^*)-\frac{\partial}{\partial x_2}f(a_1,x_2,x^*).
\end{aligned}$$

This allows us to simplify $\int_R D_{2,1}f$ as:

$$\int_R D_{2,1}f=\int_C\left(\int_{a_2}^{b_2}\left(\frac{\partial}{\partial x_2}f(b_1,x_2,x^*)-\frac{\partial}{\partial x_2}f(a_1,x_2,x^*)\right)dx_2\right)dx^*.$$

Applying the Fundamental Theorem of Calculus again, we obtain:

$$\begin{aligned}
&\int_{a_2}^{b_2}\left(\frac{\partial}{\partial x_2}f(b_1,x_2,x^*)-\frac{\partial}{\partial x_2}f(a_1,x_2,x^*)\right)dx_2\\
&=(f(b_1,x_2,x^*)-f(a_1,x_2,x^*))\Big|_{x_2=a_2}^{x_2=b_2}\\
&=f(b_1,b_2,x^*)-f(b_1,a_2,x^*)-f(a_1,b_2,x^*)+f(a_1,a_2,x^*).
\end{aligned}$$

This allows us to simplify $\int_R D_{2,1}f$ as:

$$\int_R D_{2,1}f=\int_C(f(b_1,b_2,x^*)-f(b_1,a_2,x^*)-f(a_1,b_2,x^*)+f(a_1,a_2,x^*))dx^*. \qquad (**)$$

Comparing $(*)$ and $(**)$, we see that $\int_R D_{1,2}f=\int_R D_{2,1}f$, so it follows from Assignment 7 Question 1 that:

$$\int_R g=\int_R(D_{1,2}f-D_{2,1}f)=\int_R D_{1,2}f-\int_R D_{2,1}f=0.$$

Overall, we proved that $\int_R g>0$ and that $\int_R g=0$, which leads to a contradiction. Therefore, by contradiction, $D_{1,2}f(x)=D_{2,1}f(x)$ for all $x\in\mathbb{R}^n$, as required. $\square$

3. We are given a continuous function $f : [a,b]\times[c,d] \to \mathbb{R}$ such that $D_2 f$ is continuous. Also, let us define $F : [c,d] \to \mathbb{R}$ by $F(y) = \int_a^b f(x,y)dx$. Then, we will prove that $F'(y) = \int_a^b D_2 f(x,y)dx$.
Step 1: We will rewrite $F(y)$ as a more convenient expression.
First, by the MAT157 Fundamental Theorem of Calculus, we have for all $(x,y) \in \mathbb{R}^2$ that $\int_c^y D_2 f(x,t)dt = f(x,y) - f(x,c)$, which can be rewritten as $f(x,y) = \int_c^y D_2 f(x,t)dt + f(x,c)$.
Then, $F(y)$ can be rewritten as:

$$\begin{aligned} F(y) &= \int_a^b \left(\int_c^y D_2 f(x,t)dt + f(x,c) \right) dx \\ &= \int_a^b \left(\int_c^y D_2 f(x,t)dt \right) dx + \int_a^b f(x,c)dx. \qquad \text{(Applying Assignment 7 Question 1)} \end{aligned}$$

Now, since $D_2 f$ is given to be continuous, it follows from Spivak's Remark 2 below Theorem 3-10 that:

$$\int_a^b \left(\int_c^y D_2 f(x,t)dt \right) dx = \int_{[a,b]\times[c,y]} D_2 f = \int_c^y \left(\int_a^b D_2 f(x,t)dx \right) dt.$$

Then, $F(y)$ can be rewritten as:

$$F(y) = \int_c^y \left(\int_a^b D_2 f(x,t)dx \right) dt + \int_a^b f(x,c)dx. \qquad (*)$$

Step 2: Now, we will compute $F'(y)$ by taking $\frac{\partial}{\partial y}$ of the right-hand side of $(*)$.
Since the term $\int_a^b f(x,c)dx$ is completely independent of y, we have:

$$\frac{\partial}{\partial y} \int_a^b f(x,c)dx = 0.$$

Moreover, the Fundamental Theorem of Calculus gives us:

$$\frac{\partial}{\partial y} \int_c^y \left(\int_a^b D_2 f(x,t)dx \right) dt = \int_a^b D_2 f(x,t)dx \bigg|_{t=y} = \int_a^b D_2 f(x,y)dx.$$

Adding these two results, we obtain from $(*)$ that:

$$F'(y) = \frac{\partial}{\partial y} \left(\int_c^y \left(\int_a^b D_2 f(x,t)dx \right) dt + \int_a^b f(x,c)dx \right) = \int_a^b D_2 f(x,y)dx.$$

Therefore, $F'(y) = \int_a^b D_2 f(x,y)dx$, as required. $\square$

4. We are given two continuously differentiable functions $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ such that $D_1 g_2 = D_2 g_1$. Let us define:

$$f(x,y) := \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

Then, we will prove that $D_1 f = g_1$. (Since g_1, g_2 are continuous, we already proved in Assignment 4 Question 3(a) that $D_2 f = g_2$.)
First, the result of Assignment 8 Question 3 (with x swapped with y) will be useful, so we will copy its solution (with x swapped with y) here for reference:

We are given a continuous function $f : [a,b] \times [c,d] \to \mathbb{R}$ such that $D_1 f$ is continuous. Also, let us define $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_c^d f(x,y)dy$. Then, we will prove that $F'(x) = \int_c^d D_1 f(x,y)dy$.
Step 1: We will rewrite $F(x)$ as a more convenient expression.
First, by the MAT157 Fundamental Theorem of Calculus, we have for all $(x,y) \in \mathbb{R}^2$ that $\int_a^x D_1 f(t,y)dt = f(x,y) - f(a,y)$, which can be rewritten as $f(x,y) = \int_a^x D_1 f(t,y)dt + f(a,y)$. Then, $F(x)$ can be rewritten as:

$$\begin{aligned} F(x) &= \int_c^d \left(\int_a^x D_1 f(t,y)dt + f(a,y) \right) dy \\ &= \int_c^d \left(\int_a^x D_1 f(t,y)dt \right) dy + \int_c^d f(a,y)dy. \qquad \text{(Applying Assignment 7 Question 1)} \end{aligned}$$

Now, since $D_1 f$ is given to be continuous, it follows from Spivak's Remark 2 below Theorem 3-10 that:

$$\int_c^d \left(\int_a^x D_1 f(t,y)dt \right) dy = \int_{[a,x]\times[c,d]} D_1 f = \int_a^x \left(\int_c^d D_1 f(t,y)dy \right) dt.$$

Then, $F(x)$ can be rewritten as:

$$F(x) = \int_a^x \left(\int_c^d D_1 f(t,y)dy \right) dt + \int_c^d f(a,y)dy. \qquad (*)$$

Step 2: Now, we will compute $F'(x)$ by taking $\frac{\partial}{\partial x}$ of the right-hand side of $(*)$.
Since the term $\int_c^d f(a,y)dy$ is completely independent of x, we have:

$$\frac{\partial}{\partial x} \int_c^d f(a,y)dy = 0.$$

Moreover, the Fundamental Theorem of Calculus gives us:

$$\frac{\partial}{\partial x} \int_a^x \left(\int_c^d D_1 f(t,y)dy \right) dt = \int_c^d D_1 f(t,y)dy \Big|_{t=x} = \int_c^d D_1 f(x,y)dy.$$

Adding these two results, we obtain from $(*)$ that:

$$F'(x) = \frac{\partial}{\partial x} \left(\int_a^x \left(\int_c^d D_1 f(t,y)dy \right) dt + \int_c^d f(a,y)dy \right) = \int_c^d D_1 f(x,y)dy.$$

Therefore, $F'(x) = \int_c^d D_1 f(x,y)dy$, as required. $\square$

Now, we are ready to prove Question 4. Recall that we defined:

$$f(x,y) := \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt,$$

and we want to prove that $D_1 f = g_1$.
Step 1: We will compute $\frac{\partial}{\partial x}\int_0^y g_2(x,t)dt$.
First, for all $y \in \mathbb{R}$, let us define the function $F_y : \mathbb{R} \to \mathbb{R}$ by:

$$F_y(x) := \int_0^y g_2(x,t)dt.$$

Then, since g_2 and $D_1 g_2$ are both given to be continuous, it follows from Question 3 that:

$$F_y'(x) = \int_0^y D_1 g_2(x,t)dt.$$

Since we are given $D_1 g_2 = D_2 g_1$, we obtain:

$$F_y'(x) = \int_0^y D_2 g_1(x,t)dt.$$

Next, by the Fundamental Theorem of Calculus, we obtain:

$$F_y'(x) = g_1(x,y) - g_1(x,0).$$

Therefore, we conclude that:

$$\frac{\partial}{\partial x}\int_0^y g_2(x,t)dt = F_y'(x) = g_1(x,y) - g_1(x,0).$$

Step 2: We will compute $\frac{\partial}{\partial x}\int_0^x g_1(t,0)dt$.
Applying the Fundamental Theorem of Calculus, we obtain:

$$\frac{\partial}{\partial x}\int_0^x g_1(t,0)dt = g_1(x,0).$$

Step 3: We will add these two results together to obtain:

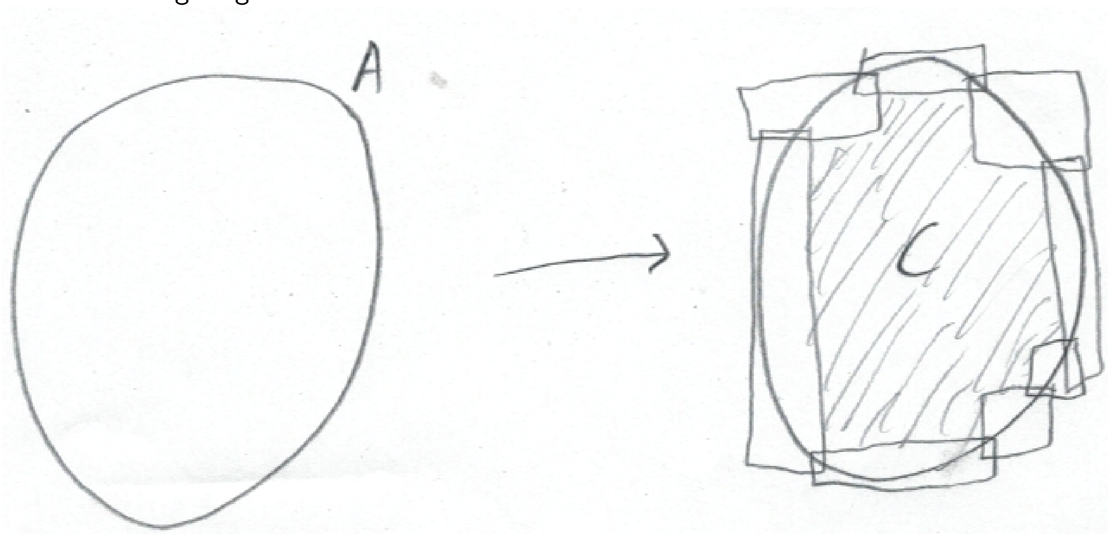
$$\begin{aligned} D_1 f(x,y) &= \frac{\partial}{\partial x}\int_0^x g_1(t,0)dt + \frac{\partial}{\partial x}\int_0^y g_2(x,t)dt \\ &= g_1(x,0) + (g_1(x,y) - g_1(x,0)) \\ &= g_1(x,y). \end{aligned}$$

Since this is true for all $(x,y) \in \mathbb{R}^2$, we conclude that $D_1 f = g_1$, as required. □

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted on Crowdmark.)

1. For this question, we are given that A is Jordan-measurable, so A is bounded, and the boundary of A is measure 0. With no other given information, a reasonable way to proceed is to cover the boundary of A with rectangles $\{U_i\}_{i\in I}$ with a combined volume that is arbitrarily small, as shown in the following diagram:



(We are allowed to use open or closed rectangles; later, we will see that open rectangles are more useful.) Since the rectangles are arbitrarily small, they will cover an arbitrarily small portion of A. This motivates us to pick the remaining uncovered portion of A to be C, since $A - C$ will be the portion of A covered by the rectangles, with an arbitrarily small volume.
Now, we need to prove that C is compact and Jordan-measurable. To be compact, C should be bounded and closed. First, C is automatically bounded as a subset of the bounded set A. Next, since C is surrounded by the rectangles $\{U_i\}_{i\in I}$ covering the boundary of A, we see that C is closed if we take $\{U_i\}_{i\in I}$ to be open rectangles. After these steps, we know that C is compact because C is bounded and closed. Now, for C to be Jordan-measurable, it should be bounded (which we have verified), and its boundary should be measure-0. From the diagram, we see that the boundary of C consists of some sides of the rectangles $\{U_i\}_{i\in I}$. Since these sides are lower-dimensional rectangles, they are each measure-0, so their union is also measure-0. As a result, the boundary of C is measure-0, and C is bounded, so C is Jordan-measurable.
Overall, after adding several technical details, these insights help us to prove that C is compact and Jordan-measurable and that $A - C$ has a small enough volume, as required.

2. Since we want to apply Fubini's theorem, we should begin by considering the integrals $\int_R D_{1,2}f$ and $\int_R D_{2,1}f$ along arbitrary rectangles R. When we apply Fubini's theorem, we can integrate along individual directions in $\mathbb{R}^n$ in a carefully chosen order. First, for $\int_R D_{1,2}f$, we have $D_{1,2}f = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}f$, where $\frac{\partial}{\partial x_2}$ comes in front of $\frac{\partial}{\partial x_1}$. Then, it makes sense to integrate in the x_2-direction, followed by the x_1-direction. Similarly, for $\int_R D_{2,1}f$, we integrate in the x_1-direction, followed by the x_2-direction. By following these steps, we can compute that $D_{1,2}f$ and $D_{2,1}f$ have the same integral along R.

Now, we want to use this fact to somehow obtain a contradiction if some point $a \in \mathbb{R}^n$ satisfies $D_{1,2}f(a) \neq D_{2,1}f(a)$; without loss of generality, $D_{1,2}f(a) > D_{2,1}f(a)$. Since single points do not affect integrals, it would be ideal to find a rectangle R around a such that $D_{1,2}f(x) > D_{2,1}f(x)$ for all $x \in R$. Fortunately, since $D_{1,2}f$ and $D_{2,1}f$ are continuous, this is possible (as stated by the textbook's hint). Then, $D_{1,2}f$ would have a larger integral than $D_{2,1}f$ along R, which contradicts our previous work, so we are done.

3. As stated in the textbook's hint, the key trick for this problem is to substitute $f(x, y)$ with $\int_c^y D_2 f(x,t)dt + f(x,c)$. A possible motivation to find this trick is that we want to produce some integral one of whose endpoints is y (i.e., $\int_c^y (\textit{stuff})dt$) so that we can use the Fundamental Theorem of Calculus to compute $F'(y)$. Since integration and differentiation are "opposites", we can force this integral to appear by integrating the derivative $D_2 f(x,t)$ along t, leading to the trick described above. This gives us:

$$F'(y) = \frac{\partial}{\partial y} \int_a^b \int_c^y D_2 f(x,t)dt + \frac{\partial}{\partial y} \int_a^b f(x,c).$$

The second term is clearly zero because $\int_a^b f(x,c)$ is independent of y. Then, to evaluate the first term, we must swap the $\int_a^b$ and the $\int_c^y$ to apply the Fundamental Theorem of Calculus. Fortunately, Fubini's theorem allows us to do so because $D_2 f$ is continuous, and we are done the problem.

4. As stated in the hint, this question requires us to apply Question 3. Then, this question consists of direct computations, where we apply Question 3, the Fundamental Theorem of Calculus, and $D_1 g_2 = D_2 g_1$ at appropriate points in the computations.