MAT257 Assignment 7 (Intro to Integrals and Measure 0)

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November 19, 2021

1. We are given a rectangle A in $\mathbb{R}^n$, as well as two integrable functions $f, g : A \to \mathbb{R}$.
(a) First, for any partition P of A and any subrectangle $S \in P$, we will prove that:

$$m_S(f) + m_S(g) \leq m_S(f+g) \text{ and } M_S(f+g) \leq M_S(f) + M_S(g).$$

Then, we will conclude that:

$$L(f,P) + L(g,P) \leq L(f+g,P) \text{ and } U(f+g,P) \leq U(f,P) + U(g,P).$$

Step 1: We will show that $m_S(f) + m_S(g) \leq m_S(f+g)$.
For all $x' \in S$, we have $f(x') \geq \inf_{x \in S} f(x) = m_S(f)$ and $g(x') \geq \inf_{x \in S} g(x) = m_S(g)$ by definition. Adding these two inequalities, we obtain:

$$f(x') + g(x') \geq m_S(f) + m_S(g)$$

for all $x' \in S$. As a result, $m_S(f) + m_S(g)$ is a lower bound for $f(x') + g(x')$ over all $x' \in S$. Since $m_S(f+g) = \inf_{x' \in S}(f(x') + g(x'))$ is the greatest lower bound for $f(x') + g(x')$ over all $x' \in S$, all lower bounds for $f(x') + g(x')$ must be at most $m_S(f+g)$, so $m_S(f) + m_S(g) \leq m_S(f+g)$ as required.
Step 2: Using a similar argument, we will show that $M_S(f+g) \leq M_S(f) + M_S(g)$.
For all $x' \in S$, we have $f(x') \leq \sup_{x \in S} f(x) = M_S(f)$ and $g(x') \leq \sup_{x \in S} g(x) = M_S(g)$ by definition. Adding these two inequalities, we obtain:

$$f(x') + g(x') \leq M_S(f) + M_S(g)$$

for all $x' \in S$. As a result, $M_S(f) + M_S(g)$ is an upper bound for $f(x') + g(x')$ over all $x' \in S$. Since $M_S(f+g) = \sup_{x' \in S}(f(x') + g(x'))$ is the least upper bound for $f(x') + g(x')$ over all $x' \in S$, all upper bounds for $f(x') + g(x')$ must be at least $M_S(f+g)$, so $M_S(f+g) \leq M_S(f) + M_S(g)$, as required.
Step 3: We will prove that $L(f,P) + L(g,P) \leq L(f+g,P)$ using the following LS-RS proof:

$$\begin{aligned}
\text{LS} &= L(f,P) + L(g,P) \\
&= \sum_{S \in P} \operatorname{vol}(S) m_S(f) + \sum_{S \in P} \operatorname{vol}(S) m_S(g) \\
&= \sum_{S \in P} \operatorname{vol}(S)(m_S(f) + m_S(g)) \\
&\leq \sum_{S \in P} \operatorname{vol}(S) m_S(f+g) \qquad \text{(Applying Step 1)} \\
&= \text{RS}
\end{aligned}
\qquad
\begin{aligned}
\text{RS} &= L(f+g,P) \\
&= \sum_{S \in P} \operatorname{vol}(S) m_S(f+g)
\end{aligned}$$

Since LS $\leq$ RS, we conclude that $L(f,P) + L(g,P) \leq L(f+g,P)$, as required.
Step 4: Similarly, we will prove that $U(f,P) + U(g,P) \geq U(f+g,P)$ using the following LS-RS proof:

$$\begin{aligned}
\text{LS} &= U(f,P) + U(g,P) \\
&= \sum_{S \in P} \operatorname{vol}(S) M_S(f) + \sum_{S \in P} \operatorname{vol}(S) M_S(g) \\
&= \sum_{S \in P} \operatorname{vol}(S)(M_S(f) + M_S(g)) \\
&\geq \sum_{S \in P} \operatorname{vol}(S) M_S(f+g) \qquad \text{(Applying Step 2)} \\
&= \text{RS}
\end{aligned}
\qquad
\begin{aligned}
\text{RS} &= U(f+g,P) \\
&= \sum_{S \in P} \operatorname{vol}(S) M_S(f+g)
\end{aligned}$$

Since LS $\geq$ RS, we conclude that $U(f,P)+U(g,P) \geq U(f+g,P)$, as required. $\square$

(b) We will prove that $f+g$ is integrable and that $\int_A (f+g) = \int_A f + \int_A g$.

Step 1: We will prove that $L(f+g) \geq L(f)+L(g)$ by proving that $L(f+g) > L(f)+L(g)-\varepsilon$ for all $\varepsilon > 0$.

Let any $\varepsilon > 0$ be given. Then, since $L(f)$ is the least upper bound of $L(f,P)$ over all partitions P of A, $L(f) - \frac{\varepsilon}{2}$ cannot be an upper bound of $L(f,P)$, so there exists some partition P_1 of A such that $L(f,P_1) > L(f) - \frac{\varepsilon}{2}$. Similarly, there exists some partition P_2 of A such that $L(g,P_2) > L(g) - \frac{\varepsilon}{2}$. Then, let P be a partition that refines both P_1 and P_2 (as explained in class, such a partition P exists). We obtain:

$$\begin{aligned}
L(f+g) &\geq L(f+g,P) \\
&\geq L(f,P)+L(g,P) && \text{(Applying part (a))} \\
&\geq L(f,P_1)+L(g,P_2) && \text{(Applying Spivak's Lemma 3-1)} \\
&> (L(f)-\frac{\varepsilon}{2}) + (L(g)-\frac{\varepsilon}{2}) \\
&= L(f)+L(g)-\varepsilon.
\end{aligned}$$

Thus, $L(f+g) > L(f)+L(g)-\varepsilon$ for all $\varepsilon > 0$, so we conclude that $L(f+g) \geq L(f)+L(g)$, as desired.

Step 2: Similarly, we will prove $U(f+g) \leq U(f)+U(g)$ by proving $U(f+g) < U(f)+U(g)+\varepsilon$ for all $\varepsilon > 0$.

Let any $\varepsilon > 0$ be given. Then, since $U(f)$ is the greatest lower bound of $U(f,P)$ over all partitions P of A, $U(f)+\frac{\varepsilon}{2}$ cannot be a lower bound of $U(f,P)$, so there exists some partition P_1 of A such that $U(f,P_1) < U(f)+\frac{\varepsilon}{2}$. Similarly, there exists some partition P_2 of A such that $U(g,P_2) > U(g)+\frac{\varepsilon}{2}$. Then, let P be a partition that refines both P_1 and P_2. We obtain:

$$\begin{aligned}
U(f+g) &\leq U(f+g,P) \\
&\leq U(f,P)+U(g,P) && \text{(Applying part (a))} \\
&\leq U(f,P_1)+U(g,P_2) && \text{(Applying Spivak's Lemma 3-1)} \\
&< (U(f)+\frac{\varepsilon}{2}) + (U(g)+\frac{\varepsilon}{2}) \\
&= U(f)+U(g)+\varepsilon.
\end{aligned}$$

Thus, $U(f+g) < U(f)+U(g)+\varepsilon$ for all $\varepsilon > 0$, so we conclude that $U(f+g) \leq U(f)+U(g)$, as desired.

Step 3: Combining the first two Steps, we will conclude that $f+g$ is integrable and that $\int_A (f+g) = \int_A f + \int_A g$.

First, since f and g are integrable, we have $L(f) = U(f) = \int_A f$ and $L(g) = U(g) = \int_A g$. As a result, Step 1 implies that $L(f+g) \geq L(f)+L(g) = \int_A f + \int_A g$, and Step 2 implies that $U(f+g) \leq U(f)+U(g) = \int_A f + \int_A g$. This results in the following chain of inequalities:

$$\int_A f + \int_A g \leq L(f+g) \leq U(f+g) \leq \int_A f + \int_A g.$$

Since the leftmost and rightmost expressions are equal, we get $L(f+g) = U(f+g) = \int_A f + \int_A g$. Therefore, we conclude that $f+g$ is integrable and that $\int_A (f+g) = \int_A f + \int_A g$, as required. $\square$

(c) Given any constant $c \in \mathbb{R}$, we will show that cf is integrable and $\int_A cf = c\int_A f$.

We will prove this statement using cases on the sign of c.
Case 0: $c = 0$. Then, the function cf is zero everywhere. According to Spivak's remark below Theorem 3-3, this means that cf is integrable and that $\int_A cf = \int_A 0 = 0\,\mathrm{vol}(A) = 0$. Moreover, $c\int_A f = 0 \cdot \int_A f = 0$. Since $\int_A cf$ and $c\int_A f$ are both equal to 0, we conclude that they are equal to each other when $c = 0$.
Case 1: $c > 0$. Then, we will prove $\int_A cf = c\int_A f$ using the following steps.
Step 1: Given any partition P of A and any subrectangle $S \in P$, we will prove $m_S(cf) \geq cm_S(f)$ and $M_S(cf) \leq cM_S(f)$.
First, $m_S(cf) = \inf_{x\in S}(cf(x))$ is the greatest lower bound for $cf(x)$ over all $x \in S$, so other lower bounds are at most $m_S(cf)$. Moreover, for all $x \in S$, since $c > 0$ and $f(x) \geq m_S(f)$, we obtain that $cf(x) \geq cm_S(f)$. Then, $cm_S(f)$ is another lower bound for $cf(x)$ over all $x \in S$, so this lower bound is at most $m_S(cf)$. Thus, $m_S(cf) \geq cm_S(f)$, as desired.
Similarly, $M_S(cf) = \sup_{x\in S}(cf(x))$ is the least upper bound for $cf(x)$ over all $x \in S$, so other upper bounds are at least $M_S(cf)$. Moreover, for all $x \in S$, since $c > 0$ and $f(x) \leq M_S(f)$, we obtain $cf(x) \leq cM_S(f)$. Then, $cM_S(f)$ is another upper bound for $cf(x)$ over all $x \in S$, so this upper bound is at least $M_S(cf)$. Thus, $M_S(cf) \leq cM_S(f)$, as desired.
Step 2: Given any partition P of A, we will prove $L(cf, P) \geq cL(f, P)$ and $U(cf, P) \leq cU(f, P)$.
We begin with the following LS-RS proof that $L(cf, P) \geq cL(f, P)$:

$$\begin{aligned}
\text{LS} &= L(cf, P) \\
&= \sum_{S\in P} \mathrm{vol}(S) m_S(cf) \\
&\geq \sum_{S\in P} \mathrm{vol}(S) c m_S(f) && \text{(Applying Step 1)} \\
&= c\sum_{S\in P} \mathrm{vol}(S) m_S(f) \\
&= \text{RS}.
\end{aligned}
\qquad
\begin{aligned}
\text{RS} &= cL(f, P) \\
&= c\sum_{S\in P} \mathrm{vol}(S) m_S(f)
\end{aligned}$$

Since LS $\geq$ RS, we obtain $L(cf, P) \geq cL(f, P)$, as desired.
Similarly, we present the following LS-RS proof that $U(cf, P) \leq cU(f, P)$:

$$\begin{aligned}
\text{LS} &= U(cf, P) \\
&= \sum_{S\in P} \mathrm{vol}(S) M_S(cf) \\
&\leq \sum_{S\in P} \mathrm{vol}(S) c M_S(f) && \text{(Applying Step 1)} \\
&= c\sum_{S\in P} \mathrm{vol}(S) M_S(f) \\
&= \text{RS}.
\end{aligned}
\qquad
\begin{aligned}
\text{RS} &= cU(f, P) \\
&= c\sum_{S\in P} \mathrm{vol}(S) M_S(f)
\end{aligned}$$

Since LS $\leq$ RS, we obtain $U(cf, P) \leq cU(f, P)$, as desired.
Step 3: We will prove that $\int_A cf = c\int_A f$.
First, for all partitions P of A, the first part of Step 2 implies that $L(cf) \geq L(cf, P) \geq cL(f, P)$, so $L(f, P) \leq \frac{1}{c}L(cf)$. Then, $\frac{1}{c}L(cf)$ is an upper bound for $L(f, P)$ over all partitions P of A, so it is greater than or equal to the least upper bound for $L(f, P)$, which is $L(f)$. In other words, $\frac{1}{c}L(cf) \geq L(f)$, so $L(cf) \geq cL(f)$. Finally, $L(f) = \int_A f$ because f is integrable, so we conclude that $L(cf) \geq c\int_A f$.

Next, for all partitions P of A, the second part of Step 2 implies $U(cf) \le U(cf, P) \le cU(f, P)$, so $U(f, P) \ge \frac{1}{c}U(cf)$. Then, $\frac{1}{c}U(cf)$ is a lower bound for $U(f, P)$ over all partitions P of A, so it is less than or equal to the greatest lower bound for $U(f, P)$, which is $U(f)$. In other words, $\frac{1}{c}U(cf) \le U(f)$, so $U(cf) \le cU(f)$. Finally, $U(f) = \int_A f$ because f is integrable, so we conclude that $U(cf) \le c\int_A f$.

Overall, we obtain the following chain of inequalities:

$$c\int_A f \le L(cf) \le U(cf) \le c\int_A f.$$

The leftmost and rightmost expressions in this chain are equal, so $L(cf) = U(cf) = c\int_A f$. Therefore, cf is integrable and $\int_A cf = c\int_A f$ when $c > 0$, as required.

Case 2: $c < 0$. Then, we will prove $\int_A cf = c\int_A f$ using the following steps.

Step 1: Given any partition P of A and any subrectangle $S \in P$, we will prove $m_S(cf) \ge cM_S(f)$ and $M_S(cf) \le cm_S(f)$.

First, $m_S(cf) = \inf_{x\in S}(cf(x))$ is the greatest lower bound for $cf(x)$ over all $x \in S$, so other lower bounds are at most $m_S(cf)$. Moreover, for all $x \in S$, since $c < 0$ and since $f(x) \le M_S(f)$, we obtain $cf(x) \ge cM_S(f)$. Then, $cM_S(f)$ is another lower bound for $cf(x)$ over all $x \in S$, so this lower bound is at most $m_S(cf)$. Thus, $m_S(cf) \ge cM_S(f)$, as desired.

Similarly, $M_S(cf) = \sup_{x\in S}(cf(x))$ is the least upper bound for $cf(x)$ over all $x \in S$, so other upper bounds are at least $M_S(cf)$. Moreover, for all $x \in S$, since $c < 0$ and since $f(x) \ge m_S(f)$, we obtain $cf(x) \le cm_S(f)$. Then, $cm_S(f)$ is another upper bound for $cf(x)$ over all $x \in S$, so this upper bound is at least $M_S(cf)$. Thus, $M_S(cf) \le cm_S(f)$, as desired.

Step 2: Given any partition P of A, we will prove $L(cf, P) \ge cU(f, P)$ and $U(cf, P) \le cL(f, P)$. We begin with the following LS-RS proof that $L(cf, P) \ge cU(f, P)$:

$$\begin{aligned}
\text{LS} &= L(cf, P) \\
&= \sum_{S\in P} \mathrm{vol}(S)m_S(cf) \\
&\ge \sum_{S\in P} \mathrm{vol}(S)cM_S(f) && \text{(Applying Step 1)} \\
&= c\sum_{S\in P} \mathrm{vol}(S)M_S(f) \\
&= \text{RS}.
\end{aligned}
\qquad
\begin{aligned}
\text{RS} &= cU(f, P) \\
&= c\sum_{S\in P} \mathrm{vol}(S)M_S(f)
\end{aligned}$$

Since LS $\ge$ RS, we obtain $L(cf, P) \ge cU(f, P)$, as desired.

Similarly, we present the following LS-RS proof that $U(cf, P) \le cL(f, P)$:

$$\begin{aligned}
\text{LS} &= U(cf, P) \\
&= \sum_{S\in P} \mathrm{vol}(S)M_S(cf) \\
&\le \sum_{S\in P} \mathrm{vol}(S)cm_S(f) && \text{(Applying Step 1)} \\
&= c\sum_{S\in P} \mathrm{vol}(S)m_S(f) \\
&= \text{RS}.
\end{aligned}
\qquad
\begin{aligned}
\text{RS} &= cL(f, P) \\
&= c\sum_{S\in P} \mathrm{vol}(S)m_S(f)
\end{aligned}$$

Since LS $\le$ RS, we obtain $U(cf, P) \le cL(f, P)$, as desired.

Step 3: We will prove that $\int_A cf = c\int_A f$.

First, for all partitions P of A, the first part of Step 2 implies that $L(cf) \geq L(cf, P) \geq cU(f, P)$, so we get $U(f, P) \geq \frac{1}{c}L(cf)$ by dividing by $c < 0$. Then, $\frac{1}{c}L(cf)$ is a lower bound for $U(f, P)$ over all partitions P of A, so it is less than or equal to the greatest lower bound for $U(f, P)$, which is $U(f)$. In other words, $\frac{1}{c}L(cf) \leq U(f)$, so $L(cf) \geq cU(f)$. Finally, $U(f) = \int_A f$ because f is integrable, so we conclude that $L(cf) \geq c \int_A f$.
Similarly, for all partitions P of A, the second part of Step 2 implies $U(cf) \leq U(cf, P) \leq cL(f, P)$, so we get $L(f, P) \leq \frac{1}{c}U(cf)$ by dividing by $c < 0$. Then, $\frac{1}{c}U(cf)$ is an upper bound for $L(f, P)$ over all partitions P of A, so it is greater than or equal to the least upper bound for $L(f, P)$, which is $L(f)$. In other words, $\frac{1}{c}U(cf) \geq L(f)$, so $U(cf) \leq cL(f)$. Finally, $L(f) = \int_A f$ because f is integrable, so we conclude that $U(cf) \leq c \int_A f$.
Overall, we obtain the following chain of inequalities:

$$c \int_A f \leq L(cf) \leq U(cf) \leq c \int_A f.$$

The leftmost and rightmost expressions in this chain are equal, so $L(cf) = U(cf) = c \int_A f$. Therefore, cf is integrable and $\int_A cf = c \int_A f$ when $c < 0$, as required.
Overall, we proved that cf is integrable and that $\int_A cf = c \int_A f$ for all possible values of c, as required. $\square$

2. (Note: This solution was edited to correct a minor error after it was graded.)
We are given a function $f : A \to \mathbb{R}$ and a partition P of A. Then, we will show that f is integrable if and only if for each subrectangle $S \in P$ the restriction $f|_S$ is integrable. If f is integrable, we will also show that $\int_A f = \sum_{S\in P} \int_S f|_S$.
Step 1: If f is integrable, then we will apply Spivak's Theorem 3-3 to show for each subrectangle $S \in P$ that the restriction $f|_S$ is integrable.
Let any subrectangle $S \in P$ be given. Then, let any $\varepsilon > 0$ be given. Since f is integrable, by Theorem 3-3, there exists some partition P' of A such that $U(f, P') - L(f, P') < \varepsilon$. Then, let P'' be a partition that refines both P and P' (as explained in class, such a partition must exist). By Spivak's Lemma 3-1, this gives us $U(f, P'') - L(f, P'') \leq U(f, P') - L(f, P') < \varepsilon$.
Now, since P'' refines P, its cutpoints include the boundaries of S, so P'' can be restricted to S to form a partition Q of S. Then, we can bound $U(f|_S, Q) - L(f|_S, Q)$ as follows:

$$\begin{aligned}
U(f|_S, Q) - L(f|_S, Q) &= \sum_{R\in Q} \mathrm{vol}(R)M_R(f) - \sum_{R\in Q} \mathrm{vol}(R)m_R(f) \\
&= \sum_{R\in Q} \mathrm{vol}(R)(M_R(f) - m_R(f)) \\
&= \sum_{\substack{R\in P'' \\ R\subseteq S}} \mathrm{vol}(R)(M_R(f) - m_R(f)) \qquad \text{(Since } Q \text{ is restriction of } P'' \text{ onto } S\text{)} \\
&\leq \sum_{R\in P''} \mathrm{vol}(R)(M_R(f) - m_R(f)) \\
&= \sum_{R\in P''} \mathrm{vol}(R)M_R(f) - \sum_{R\in P''} \mathrm{vol}(R)m_R(f) \\
&= U(f, P'') - L(f, P'') \\
&< \varepsilon.
\end{aligned}$$

Therefore, for all $\varepsilon > 0$, there exists some partition Q of S such that $U(f|_S, Q) - L(f|_S, Q) < \varepsilon$, so by Theorem 3-3, the restriction $f|_S$ is integrable. This proof applies for all subrectangles $S \in P$, as required.
Step 2: If $f|_S$ is integrable for all subrectangles $S \in P$, then we will apply Theorem 3-3 to prove that f is integrable.
Let any $\varepsilon > 0$ be given. Then, let N be the number of subrectangles in P. Note that N is a finite positive integer.
Now, for all subrectangles $S \in P$, since $f|_S$ is integrable, we have $L(f|_S) = U(f|_S) = \int_S f|_S$. Next, $U(f|_S)$ is the greatest lower bound of $U(f|_S, Q)$ over all partitions Q of S, so $U(f|_S) + \frac{\varepsilon}{2N}$ cannot be a lower bound. Thus, we can pick a partition Q_S^1 of S such that:

$$U(f|_S, Q_S^1) < U(f|_S) + \frac{\varepsilon}{2N} = \int_S f|_S + \frac{\varepsilon}{2N}.$$

Similarly, we can pick a partition Q_S^2 of S such that:

$$L(f|_S, Q_S^2) > L(f|_S) - \frac{\varepsilon}{2N} = \int_S f|_S - \frac{\varepsilon}{2N}.$$

Next, let Q_S be a partition of S that refines both Q_S^1 and Q_S^2. Then, for each $S \in P$, we can form a corresponding partition P_S of A by combining the cutpoints of Q_S, the boundaries of S,

and the boundaries of A into the cutpoints of P_S. By construction, the restriction of P_S to S is well-defined and is a refinement of Q_S.
Next, let us define P' to be a partition that refines all P_S over all $S \in P$. (Since there are finitely many partitions P_S, we can "combine" two partitions at a time into refinements of those two partitions, and repeating this process eventually gives a refinement of all P_S.) For all $S \in P$, since P' is a refinement of P_S, the restriction of P' to S exists and is a refinement of Q_S; let us call this restriction Q'_S. Then, we can bound $U(f, P')$ from above as follows:

$$\begin{aligned}
U(f, P') &= \sum_{R \in P'} \mathrm{vol}(R) M_R(f) \\
&= \sum_{S \in P} \sum_{R \in Q'_S} \mathrm{vol}(R) M_R(f) \\
&= \sum_{S \in P} U(f|_S, Q'_S) \\
&\leq \sum_{S \in P} U(f|_S, Q_S) && \text{(Applying Lemma 3-1)} \\
&\leq \sum_{S \in P} U(f|_S, Q^1_S) && \text{(Applying Lemma 3-1 again)} \\
&< \sum_{S \in P} \left(\int_S f|_S + \frac{\varepsilon}{2N}\right) && \text{(By construction of } Q^1_S\text{)} \\
&= \sum_{S \in P} \int_S f|_S + \frac{\varepsilon}{2}. && \text{(Since } P \text{ has } N \text{ subrectangles)}
\end{aligned}$$

We will label this bound as $(*1)$. We can also bound $L(f, P')$ from below as follows:

$$\begin{aligned}
L(f, P') &= \sum_{R \in P'} \mathrm{vol}(R) m_R(f) \\
&= \sum_{S \in P} \sum_{R \in Q'_S} \mathrm{vol}(R) m_R(f) \\
&= \sum_{S \in P} L(f|_S, Q'_S) \\
&\geq \sum_{S \in P} L(f|_S, Q_S) && \text{(Applying Lemma 3-1)} \\
&\geq \sum_{S \in P} L(f|_S, Q^2_S) && \text{(Applying Lemma 3-1 again)} \\
&> \sum_{S \in P} \left(\int_S f|_S - \frac{\varepsilon}{2N}\right) && \text{(By construction of } Q^2_S\text{)} \\
&= \sum_{S \in P} \int_S f|_S - \frac{\varepsilon}{2}. && \text{(Since } P \text{ has } N \text{ subrectangles)}
\end{aligned}$$

We will label this bound as $(*2)$.

Finally, subtracting $(*2)$ from $(*1)$, we obtain:

$$\begin{aligned}
U(f,P') - L(f,P') &< \left(\sum_{S\in P}\int_S f|_S + \frac{\varepsilon}{2}\right) - \left(\sum_{S\in P}\int_S f|_S - \frac{\varepsilon}{2}\right) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, for all $\varepsilon > 0$, there exists a partition P' of A such that $U(f,P') - L(f,P') < \varepsilon$, so f is integrable, as required.

Step 3: Continuing from Step 2, we will prove that $\int_A f = \sum_{S\in P}\int_S f|_S$.

According to the bound $(*1)$, there exist partitions P' of A such that $U(f,P')$ becomes arbitrarily close to $\sum_{S\in P}\int_S f|_S$, so any number above $\sum_{S\in P}\int_S f|_S$ cannot be a lower bound for $U(f,P')$. Since $U(f)$ is a lower bound for $U(f,P')$, it follows that $U(f) \leq \sum_{S\in P}\int_S f|_S$.

Moreover, according to the bound $(*2)$, there exist partitions P' of A such that $L(f,P')$ becomes arbitrarily close to $\sum_{S\in P}\int_S f|_S$, so any number below $\sum_{S\in P}\int_S f|_S$ cannot be an upper bound for $L(f,P')$. Since $L(f)$ is an upper bound for $L(f,P')$, it follows that $L(f) \geq \sum_{S\in P}\int_S f|_S$.

Overall, we obtain the following chain of inequalities:

$$\sum_{S\in P}\int_S f|_S \leq L(f) \leq U(f) \leq \sum_{S\in P}\int_S f|_S.$$

The leftmost and rightmost expressions in this chain are equal, so $L(f) = U(f) = \sum_{S\in P}\int_S f|_S$. Therefore, $\int_A f = \sum_{S\in P}\int_S f|_S$, as required.

Combining these three Steps, we proved that f is integrable if and only if $f|_S$ is integrable for all subrectangles $S \in P$, and we also proved that $\int_A f = \sum_{S\in P}\int_S f|_S$ in this case, as required. $\square$

3. We are given integrable functions $f, g : A \to \mathbb{R}$ such that $f \leq g$. Then, we will prove that $\int_A f \leq \int_A g$.
Step 1: Given any partition P of A and any subrectangle $S \in P$, we will prove $m_S(f) \leq m_S(g)$. First, since $f \leq g$, we have for all $x' \in S$ that $g(x') \geq f(x') \geq \inf_{x \in S} f(x) = m_S(f)$. This means that $m_S(f)$ is a lower bound for $g(x')$ over all $x' \in S$. Since $m_S(g) = \inf_{x' \in S} g(x')$ is the greatest lower bound for $g(x')$ over all $x' \in S$, all lower bounds for $g(x')$ must be at most $m_S(g)$, so we conclude that $m_S(g) \geq m_S(f)$.
Step 2: Given any partition P of A, we will prove that $L(f, P) \leq L(g, P)$ using the following LS-RS proof:

$$\begin{aligned}\text{LS} &= L(f,P) \\ &= \sum_{S \in P} \operatorname{vol}(S) m_S(f) \\ &\leq \sum_{S \in P} \operatorname{vol}(S) m_S(g) \qquad \text{(Applying Step 1)} \\ &= \text{RS}\end{aligned}$$

$$\begin{aligned}\text{RS} &= L(g,P) \\ &= \sum_{S \in P} \operatorname{vol}(S) m_S(g)\end{aligned}$$

Since LS $\leq$ RS, we conclude that $L(f, P) \leq L(g, P)$.
Step 3: We will conclude that $\int_A f \leq \int_A g$.
First, by Step 2, we have for all partitions P' of A that:

$$L(f, P') \leq L(g, P') \leq \sup_{P \text{ is partition of } A} L(g, P) = L(g),$$

so $L(g)$ is an upper bound on $L(f, P')$ over all partitions P' of A. Since $L(f)$ is the supremum of $L(f, P')$ over all partitions P' of A, it is also the least upper bound for such $L(f, P')$, so other upper bounds for such $L(f, P')$ must be at least $L(f)$. This gives us $L(g) \geq L(f)$. Finally, since f, g are integrable, we conclude that:

$$\int_A f = L(f) \leq L(g) = \int_A g,$$

so $\int_A f \leq \int_A g$, as required. $\square$

4. Given an integrable function $F : A \to \mathbb{R}$, we will prove that $|F|$ is integrable, with $\left|\int_A F\right| \leq \int_A |F|$. First, we will need to apply the $c < 0$ case of Question 1 (c), so its proof is reproduced below for reference:

   Given any $c < 0$ and any integrable function $f : A \to \mathbb{R}$, we will prove $\int_A cf = c\int_A f$ using the following steps.
   Step 1: Given any partition P of A and any subrectangle $S \in P$, we will prove $m_S(cf) \geq cM_S(f)$ and $M_S(cf) \leq cm_S(f)$.
   First, $m_S(cf) = \inf_{x\in S}(cf(x))$ is the greatest lower bound for $cf(x)$ over all $x \in S$, so other lower bounds are at most $m_S(cf)$. Moreover, for all $x \in S$, since $c < 0$ and since $f(x) \leq M_S(f)$, we obtain $cf(x) \geq cM_S(f)$. Then, $cM_S(f)$ is another lower bound for $cf(x)$ over all $x \in S$, so this lower bound is at most $m_S(cf)$. Thus, $m_S(cf) \geq cM_S(f)$, as desired.
   Similarly, $M_S(cf) = \sup_{x\in S}(cf(x))$ is the least upper bound for $cf(x)$ over all $x \in S$, so other upper bounds are at least $M_S(cf)$. Moreover, for all $x \in S$, since $c < 0$ and since $f(x) \geq m_S(f)$, we obtain $cf(x) \leq cm_S(f)$. Then, $cm_S(f)$ is another upper bound for $cf(x)$ over all $x \in S$, so this upper bound is at least $M_S(cf)$. Thus, $M_S(cf) \leq cm_S(f)$, as desired.
   Step 2: Given any partition P of A, we will prove $L(cf, P) \geq cU(f, P)$ and $U(cf, P) \leq cL(f, P)$. We begin with the following LS-RS proof that $L(cf, P) \geq cU(f, P)$:

$$\begin{aligned}\text{LS} &= L(cf,P) \\ &= \sum_{S\in P} \mathrm{vol}(S) m_S(cf) \\ &\geq \sum_{S\in P} \mathrm{vol}(S) cM_S(f) \qquad \text{(Applying Step 1)} \\ &= c\sum_{S\in P} \mathrm{vol}(S) M_S(f) \\ &= \text{RS}.\end{aligned} \qquad \begin{aligned}\text{RS} &= cU(f,P) \\ &= c\sum_{S\in P} \mathrm{vol}(S) M_S(f)\end{aligned}$$

   Since LS $\geq$ RS, we obtain $L(cf, P) \geq cU(f, P)$, as desired.
   Similarly, we present the following LS-RS proof that $U(cf, P) \leq cL(f, P)$:

$$\begin{aligned}\text{LS} &= U(cf,P) \\ &= \sum_{S\in P} \mathrm{vol}(S) M_S(cf) \\ &\leq \sum_{S\in P} \mathrm{vol}(S) cm_S(f) \qquad \text{(Applying Step 1)} \\ &= c\sum_{S\in P} \mathrm{vol}(S) m_S(f) \\ &= \text{RS}.\end{aligned} \qquad \begin{aligned}\text{RS} &= cL(f,P) \\ &= c\sum_{S\in P} \mathrm{vol}(S) m_S(f)\end{aligned}$$

   Since LS $\leq$ RS, we obtain $U(cf, P) \leq cL(f, P)$, as desired.
   Step 3: We will prove that $\int_A cf = c\int_A f$.
   First, for all partitions P of A, the first part of Step 2 implies that $L(cf) \geq L(cf, P) \geq cU(f, P)$, so we get $U(f, P) \geq \frac{1}{c}L(cf)$ by dividing by $c < 0$. Then, $\frac{1}{c}L(cf)$ is a lower bound for $U(f, P)$ over all partitions P of A, so it is less than or equal to the greatest lower bound for $U(f, P)$, which is $U(f)$. In other words, $\frac{1}{c}L(cf) \leq U(f)$, so $L(cf) \geq cU(f)$. Finally, $U(f) = \int_A f$ because f is integrable, so we conclude that $L(cf) \geq c\int_A f$.

Similarly, for all partitions P of A, the second part of Step 2 implies $U(cf) \leq U(cf, P) \leq cL(f, P)$, so we get $L(f, P) \leq \frac{1}{c}U(cf)$ by dividing by $c < 0$. Then, $\frac{1}{c}U(cf)$ is an upper bound for $L(f, P)$ over all partitions P of A, so it is greater than or equal to the least upper bound for $L(f, P)$, which is $L(f)$. In other words, $\frac{1}{c}U(cf) \geq L(f)$, so $U(cf) \leq cL(f)$. Finally, $L(f) = \int_A f$ because f is integrable, so we conclude that $U(cf) \leq c\int_A f$.
Overall, we obtain the following chain of inequalities:

$$c\int_A f \leq L(cf) \leq U(cf) \leq c\int_A f.$$

The leftmost and rightmost expressions in this chain are equal, so $L(cf) = U(cf) = c\int_A f$. Therefore, cf is integrable and $\int_A cf = c\int_A f$ when $c < 0$, as required. □

Moreover, we will need to use the result of Question 3, so its proof is reproduced below for reference:

Given two integrable functions $f, g : A \to \mathbb{R}$ such that $f \leq g$, we will prove that $\int_A f \leq \int_A g$.
Step 1: Given any partition P of A and any subrectangle $S \in P$, we will prove $m_S(f) \leq m_S(g)$.
First, since $f \leq g$, we have for all $x' \in S$ that $g(x') \geq f(x') \geq \inf_{x \in S} f(x) = m_S(f)$. This means that $m_S(f)$ is a lower bound for $g(x')$ over all $x' \in S$. Since $m_S(g) = \inf_{x' \in S} g(x')$ is the greatest lower bound for $g(x')$ over all $x' \in S$, all lower bounds for $g(x')$ must be at most $m_S(g)$, so we conclude that $m_S(g) \geq m_S(f)$.
Step 2: Given any partition P of A, we will prove that $L(f, P) \leq L(g, P)$ using the following LS-RS proof:

$$\begin{aligned}
\text{LS} &= L(f, P) \\
&= \sum_{S \in P} \mathrm{vol}(S) m_S(f) \\
&\leq \sum_{S \in P} \mathrm{vol}(S) m_S(g) \qquad \text{(Applying Step 1)} \\
&= \text{RS}
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{RS} &= L(g, P) \\
&= \sum_{S \in P} \mathrm{vol}(S) m_S(g)
\end{aligned}$$

Since LS $\leq$ RS, we conclude that $L(f, P) \leq L(g, P)$.
Step 3: We will conclude that $\int_A f \leq \int_A g$.
First, by Step 2, we have for all partitions P' of A that:

$$L(f, P') \leq L(g, P') \leq \sup_{P \text{ is partition of } A} L(g, P) = L(g),$$

so $L(g)$ is an upper bound on $L(f, P')$ over all partitions P' of A. Since $L(f)$ is the supremum of $L(f, P')$ over all partitions P' of A, it is also the least upper bound for such $L(f, P')$, so other upper bounds for such $L(f, P')$ must be at least $L(f)$. This gives us $L(g) \geq L(f)$. Finally, since f, g are integrable, we conclude that:

$$\int_A f = L(f) \leq L(g) = \int_A g,$$

so $\int_A f \leq \int_A g$, as required. □

Now, we will solve Question 4 using the following steps.

Step 1: For any partition P of A, and for any subrectangle $S \in P$, we will prove that $M_S(|F|) - m_S(|F|) \leq M_S(F) - m_S(F)$.
Let us consider the following cases for F:
Case 1: $F(x) \geq 0$ for all $x \in S$. Then, $\left|F(x)\right| = F(x)$ for all $x \in S$, so $M_S(|F|) = M_S(F)$ and $m_S(|F|) = m_S(F)$. This directly results in $M_S(|F|) - m_S(|F|) = M_S(F) - m_S(F)$.
Case 2: $F(x) \leq 0$ for all $x \in S$. Then, $\left|F(x)\right| = -F(x)$ for all $x \in S$. By properties of infimums and supremums, we obtain $M_S(|F|) = \sup_{x\in S}(-F(x)) = -\inf_{x\in S} F(x) = -m_S(F)$, as well as $m_S(|F|) = \inf_{x\in S}(-F(x)) = -\sup_{x\in S} F(x) = -M_S(F)$. This results in:

$$M_S(|F|) - m_S(|F|) = (-m_S(F)) - (-M_S(F)) = M_S(F) - m_S(F).$$

Case 3: There exist $x_1, x_2 \in S$ for which $F(x_1) > 0$ and $F(x_2) < 0$. Then, $m_S(F) < 0$ and $M_S(F) > 0$. Moreover, since $|F|$ is nonnegative everywhere, we have $m_S(|F|) \geq 0$. Finally, we have:

$$\begin{aligned}
M_S(|F|) &= \sup_{x\in S}\left|F(x)\right| \\
&= \max\left(\sup_{\substack{x\in S\\F(x)\geq 0}}\left|F(x)\right|, \sup_{\substack{x\in S\\F(x)\leq 0}}\left|F(x)\right|\right) \\
&= \max\left(\sup_{\substack{x\in S\\F(x)\geq 0}} F(x), \sup_{\substack{x\in S\\F(x)\leq 0}} (-F(x))\right) \\
&= \max\left(\sup_{\substack{x\in S\\F(x)\geq 0}} F(x), -\inf_{\substack{x\in S\\F(x)\leq 0}} F(x)\right) \\
&= \max(M_S(F), -m_S(F)).
\end{aligned}$$

In the case when $M_S(|F|) = M_S(F)$, we obtain:

$$M_S(|F|) - m_S(|F|) \leq M_S(F) - 0 \leq M_S(F) - m_S(F).$$

Otherwise, in the case when $M_S(|F|) = -m_S(F)$, we obtain:

$$M_S(|F|) - m_S(|F|) \leq -m_S(F) - 0 \leq M_S(F) - m_S(F).$$

Overall, in all cases, we proved that $M_S(|F|) - m_S(|F|) \leq M_S(F) - m_S(F)$, as desired.
Step 2: We will apply Spivak's Theorem 3-3 to prove that $|F|$ is integrable.
Let any $\varepsilon > 0$ be given. Then, since F is integrable, Theorem 3-3 gives us some partition P of A such that $U(F, P) - L(F, P) < \varepsilon$. Next, we obtain the following bound on $U(|F|, P) - L(|F|, P)$:

$$\begin{aligned}
U(|F|, P) - L(|F|, P) &= \sum_{S\in P} \mathrm{vol}(S) M_S(|F|) - \sum_{S\in P} \mathrm{vol}(S) m_S(|F|) \\
&= \sum_{S\in P} \mathrm{vol}(S)(M_S(|F|) - m_S(|F|)) \\
&\leq \sum_{S\in P} \mathrm{vol}(S)(M_S(F) - m_S(F)) && \text{(Applying Step 1)} \\
&= \sum_{S\in P} \mathrm{vol}(S) M_S(F) - \sum_{S\in P} \mathrm{vol}(S) m_S(F) \\
&= U(F, P) - L(F, P) \\
&< \varepsilon.
\end{aligned}$$

Therefore, for all $\varepsilon > 0$, there exists some partition P of A such that $U(|F|, P) - L(|F|, P) < \varepsilon$. Then, by Theorem 3-3, $|F|$ is integrable, as required.

Step 3: We will prove $\int_A |F| \geq \left|\int_A F\right|$ by proving that $\int_A |F| \geq \int_A F$ and that $\int_A |F| \geq -\int_A F$.

First, we know $\left|F(x)\right| \geq F(x)$ for all $x \in A$. Then, since $|F|$ and F are both integrable, it follows from Question 3 that $\int_A |F| \geq \int_A F$.

Next, since F is integrable, we can plug $c = -1$ into Question 1(c) to obtain that $-F$ is integrable and that $\int_A (-F) = -\int_A F$. Now, since $\left|F(x)\right| \geq -F(x)$ for all $x \in S$, and since $|F|$ and $-F$ are both integrable, it follows from Question 3 that $\int_A |F| \geq \int_A (-F) = -\int_A F$.

Finally, since $\int_A |F| \geq \int_A F$ and $\int_A |F| \geq -\int_A F$, we conclude that:

$$\int_A |F| \geq \max\left(\int_A F, -\int_A F\right) = \left|\int_A F\right|,$$

as required.

Combining these three Steps, we proved that $|F|$ is integrable and that $\left|\int_A F\right| \leq \int_A |F|$ whenever F is integrable, as required. $\square$

5. (a) We will show that an unbounded set cannot have content 0.
Assume for contradiction that there exists an unbounded set $S \subseteq \mathbb{R}^n$ of content 0. Then, let us pick $\varepsilon := 1 > 0$. By definition of content 0, there exist finitely many closed rectangles $A_1, \ldots, A_k$ of total volume less than ε that cover S. Over all indices $1 \le j \le k$, let us express each rectangle A_j as the product $\prod_{i=1}^n [a_{i,j}, b_{i,j}]$. Then, over all indices $1 \le i \le n$, we can define the bounds:

$$a_i' := \min_{1 \le j \le k} a_{i,j}, \qquad\qquad b_i' := \max_{1 \le j \le k} b_{i,j}.$$

Finally, we can define the rectangle $A' := \prod_{i=1}^n [a_i', b_i']$.
Now, let any $1 \le j \le k$ be given. By construction, we obtain for all $1 \le i \le n$ that $a_i' \le a_{i,j} \le b_{i,j} \le b_i'$, so $[a_{i,j}, b_{i,j}] \subseteq [a_i', b_i']$. Since this is true for all $1 \le i \le n$, it follows that $\prod_{i=1}^n [a_{i,j}, b_{i,j}] \subseteq \prod_{i=1}^n [a_i', b_i']$, so $A_j \subseteq A'$ for all $1 \le j \le k$.
Next, we assumed above that the rectangles $A_1, \ldots, A_k$ cover S. Since we just proved that the rectangle A' contains all of $A_1, \ldots, A_k$, it follows that A' contains S. However, this contradicts our assumption that S is unbounded.
Therefore, by contradiction, every unbounded set must not have content 0, as required. $\square$

(b) We will prove that $\mathbb{N}$, viewed as a subset of $\mathbb{R}$, is a closed set of measure 0 but not content 0.
Step 1: We will show that $\mathbb{N}$ is closed by showing that $\mathbb{R} - \mathbb{N}$ is open.
Let any $x \in \mathbb{R} - \mathbb{N}$ be given. Then, we will show that x has an open neighbourhood contained in $\mathbb{R} - \mathbb{N}$ using the following casework:
Case 1: $x < 1$. Then, the interval $(x-1, 1)$ is an open neighbourhood of x. This interval is also contained in $\mathbb{R} - \mathbb{N}$ because all elements of $\mathbb{N}$ are at least 1.
Case 2: $x \ge 1$. Then, since $x \notin \mathbb{N}$, we know that x is not an integer, so x must be between two consecutive integers. In other words, $x \in (n, n+1)$ for some $n \in \mathbb{Z}$. Since the interval $(n, n+1)$ cannot contain any positive integer, it must be an open neighbourhood of x contained in $\mathbb{R} - \mathbb{N}$.
In both cases, we showed that x has an open neighbourhood contained in $\mathbb{R} - \mathbb{N}$. Since this is true for all $x \in \mathbb{R} - \mathbb{N}$, we conclude that $\mathbb{R} - \mathbb{N}$ is open, so $\mathbb{N}$ is closed, as desired.
Step 2: We will show that $\mathbb{N}$ has measure 0.
Let any $\varepsilon > 0$ be given. Then, consider the sequence of closed rectangles $\{A_k\}_{k \in \mathbb{N}}$ defined by $A_k := [k - \frac{\varepsilon}{2^{k+2}}, k + \frac{\varepsilon}{2^{k+2}}]$. First, we have for all $k \in \mathbb{N}$ that $k \in A_k$, so the rectangles $\{A_k\}_{k \in \mathbb{N}}$ cover $\mathbb{N}$. Moreover, their total volume is:

$$\begin{aligned}
\sum_{k=1}^{\infty} \operatorname{vol}(A_k) &= \sum_{k=1}^{\infty} \left(\left(k + \frac{\varepsilon}{2^{k+2}} \right) - \left(k - \frac{\varepsilon}{2^{k+2}} \right) \right) \\
&= \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} \\
&= \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^n \\
&= \frac{\varepsilon}{2} \cdot \frac{\frac{1}{2}}{1 - \frac{1}{2}} && \text{(Infinite geometric series)} \\
&= \frac{\varepsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Therefore, for all $\varepsilon > 0$, we found countably many closed rectangles of total volume less than ε that cover $\mathbb{N}$, so $\mathbb{N}$ has measure 0, as desired.

Step 3: Since $\mathbb{N}$ is unbounded, part (a) implies that $\mathbb{N}$ does not have content 0.
Combining these three steps, we conclude that $\mathbb{N} \subseteq \mathbb{R}$ is a closed set of measure 0 which does not have content 0, as required. $\square$

6. Given a countable union $A = \bigcup_{i=1}^{\infty}(a_i, b_i)$ of open intervals such that $([0,1] \cap \mathbb{Q}) \subseteq A$ and such that $\sum_{i=1}^{\infty}(b_i - a_i) < 1$, we will show that the boundary of A is not of measure 0.
Step 1: Similarly to Assignment 1 Question 5, we will prove that the boundary of A contains $[0,1] - A$.
Let any $x \in [0,1] - A$ be given. Then, let (a,b) be any open interval around x. Since $\mathbb{Q}$ is dense, both intervals $(a,x]$ and $[x,b)$ must include rational numbers arbitrarily close to x. Then, since $x \in [0,1]$, either the rational numbers arbitrarily close to x from above or the rationals arbitrarily close to x from below (or both) must be in $[0,1]$. Such rational numbers would be inside $(a,x]$ or $[x,b)$, respectively. Since (a,b) contains both $(a,x]$ and $[x,b)$, this implies that (a,b) contains rational numbers in $[0,1]$ in either case. All such rational numbers are given to be in A, so (a,b) contains elements of A. Moreover, (a,b) also contains $x \notin A$, so (a,b) also contains an element outside A. Thus, every open interval (a,b) around x contains both an element inside A and an element outside A, so x is in the boundary of A. This proof applies for all $x \in [0,1] - A$, so we conclude that the boundary of A contains $[0,1] - A$, as required.
Step 2: We will prove that the boundary of A is not of measure 0.
First, since we are given $\sum_{i=1}^{\infty}(b_i - a_i) < 1$, we can define $\varepsilon := 1 - \sum_{i=1}^{\infty}(b_i - a_i) > 0$.
Next, let $\bigcup_{i=1}^{\infty}(c_i, d_i)$ be a countable union of open intervals that cover the boundary of A. Then, by Step 1, this union also covers $[0,1] - A$, so $[0,1] \subseteq \bigcup_{i=1}^{\infty}(c_i, d_i) \cup A$. As a result, the union:

$$\bigcup_{i=1}^{\infty}(c_i, d_i) \cup \bigcup_{i=1}^{\infty}(a_i, b_i)$$

is an open cover of $[0,1]$. Since $[0,1]$ is compact, we can extract a finite subcover:

$$\bigcup_{i \in I_1}(c_i, d_i) \cup \bigcup_{i \in I_2}(a_i, b_i)$$

of $[0,1]$, where I_1, I_2 are finite indexing sets. Now, let us define $k := |I_1| + |I_2|$, and let us order these intervals as $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ in increasing order of x_i. Then, we may assume without loss of generality that none of these intervals is completely contained inside another; otherwise, we could remove it from our cover of $[0,1]$. In other words, we may assume that $y_i < y_{i+1}$ for all indices $1 \leq i < k$, or else (x_i, y_i) would contain (x_{i+1}, y_{i+1}), and we could disregard (x_{i+1}, y_{i+1}). We may also assume without loss of generality that $y_1 > 0$ and that $x_k < 1$; otherwise, if $y_1 \leq 0$ or $x_k \geq 1$, then $[0,1]$ would not intersect (x_1, y_1) or (x_k, y_k), respectively, so (x_1, y_1) or (x_k, y_k) would not contribute to the cover of $[0,1]$. Overall, we are assuming without loss of generality that $x_1 < \cdots < x_k < 1$ and that $0 < y_1 < \cdots < y_k$.
Now, since the intervals (x_i, y_i) cover $[0,1]$, we must have $x_1 < 0$, or else the interval $[0, x_1]$ would remain uncovered (since (x_1, y_1) is the "leftmost interval"). We must also have $y_k > 1$, or else the interval $[y_k, 1]$ would remain uncovered (since (x_k, y_k) is the "rightmost interval"). This gives us $y_k - x_1 > 1 - 0 = 1$. Next, for all indices $1 \leq i < k$, we must have $y_i > x_{i+1}$, or else the interval $[y_i, x_{i+1}]$ would remain uncovered. Then, by telescoping series, we obtain:

$$\begin{aligned}
y_k - x_1 &= (y_1 - x_1) + (x_2 - y_1) + (y_2 - x_2) + (x_3 - y_2) + \cdots + (x_k - y_{k-1}) - (y_k - x_k) \\
&= \sum_{i=1}^{k}(y_i - x_i) + \sum_{i=1}^{k-1}(x_{i+1} - y_i) \\
&< \sum_{i=1}^{k}(y_i - x_i). \qquad\qquad \text{(Since } y_i > x_{i+1}\text{)}
\end{aligned}$$

This gives us the bound:

$$\sum_{i=1}^{k}(y_i - x_i) > y_k - x_1 > 1.$$

Now, the intervals $\{(x_i, y_i)\}_{1\leq i\leq k}$, consist of the intervals $\{(c_i, d_i)\}_{i\in I_1}$ and $\{(a_i, b_i)\}_{i\in I_2}$, so we obtain:

$$1 < \sum_{i=1}^{k}(y_i - x_i) = \sum_{i\in I_1}(d_i - c_i) + \sum_{i\in I_2}(b_i - a_i) \leq \sum_{i=1}^{\infty}(d_i - c_i) + \sum_{i=1}^{\infty}(b_i - a_i).$$

As a result, we obtain:

$$\sum_{i=1}^{\infty}\mathrm{vol}((c_i, d_i)) = \sum_{i=1}^{\infty}(d_i - c_i) > 1 - \sum_{i=1}^{\infty}(b_i - a_i) = \epsilon.$$

Therefore, we found $\varepsilon > 0$ such that all countable unions of open intervals that cover the boundary of A have a total volume of at least ε, so the boundary of A is not of measure 0, as required. $\square$

7. Given an increasing function $f : [a, b] \to \mathbb{R}$, let C be the set of discontinuities of f. Then, we will show that C is of measure 0.
Step 1: For all $n \in \mathbb{N}$, define $C_n := \{x \in [a, b] : O(f, x) > \frac{1}{n}\}$ (where O denotes oscillation). For convenience, let us also define $k_n := \lfloor n(f(b) - f(a)) \rfloor \in \mathbb{Z}$. Note that, since f is increasing, we have $f(b) > f(a)$, so $k_n \geq 0$. Also note that $k_n + 1 > n(f(b) - f(a))$ by construction. Then, given any $n \in \mathbb{N}$, we will prove that C_n contains at most $k_n + 2$ elements.
Assume for contradiction that C_n contains at least $k_n + 3$ elements. Then, C_n contains at least $k_n + 1$ elements not equal to a or b. Let us order those elements as $x_1 < x_2 < \cdots < x_{k_n+1}$, where $x_1 > a$ and $x_{k_n+1} < b$. Next, let us pick a sequence $a = t_0 < t_1 < \cdots < t_{k_n+1} = b$, where t_i is picked inside (x_i, x_{i+1}) for all $1 \leq i \leq k_n$. This gives us the chain of inequalities:

$$a = t_0 < x_1 < t_1 < x_2 < t_2 < \cdots < x_{k_n+1} < t_{k_n+1} = b.$$

Now, consider the differences $d_i := f(t_i) - f(t_{i-1})$ for all $1 \leq i \leq k_n + 1$. Then, applying telescopic series, the sum of the d_is is:

$$\sum_{i=1}^{k_n+1} d_i = \sum_{i=1}^{k_n+1} (f(t_i) - f(t_{i-1})) = f(t_{k_n+1}) - f(t_0) = f(b) - f(a).$$

As a result, the average of the d_is is:

$$\frac{1}{k_n + 1} \sum_{i=1}^{k_n+1} d_i = \frac{f(b) - f(a)}{k_n + 1} < \frac{f(b) - f(a)}{n(f(b) - f(a))} = \frac{1}{n},$$

so there exists some index $1 \leq j \leq k_n + 1$ such that $d_j < \frac{1}{n}$.
Now, let us define $\delta' := \min(t_j - x_j, x_j - t_{j-1})$. Since $t_{j-1} < x_j < t_j$, we know that δ' is positive. Then, given any $\delta > 0$ such that $\delta < \delta'$, let us consider the rectangle $R := (x_j - \delta, x_j + \delta)$. For all $x \in R$, we obtain $x > x_j - \delta > x_j - \delta' \geq x_j - (x_j - t_{j-1}) = t_{j-1}$, and we also obtain $x < x_j + \delta < x_j + \delta' \leq x_j + (t_j - x_j) = t_j$. Since f is increasing, it follows from $t_{j-1} < x < t_j$ that $f(t_{j-1}) < f(x) < f(t_j)$ for all $x \in R$. As a result, $\inf_{x \in R} f(x) \geq f(t_{j-1})$ and $\sup_{x \in R} f(x) \leq f(t_j)$, so:

$$O(f, R) = \sup_{x \in R} f(x) - \inf_{x \in R} f(x) \leq f(t_j) - f(t_{j-1}) = d_j.$$

Since this is true for all positive $\delta < \delta'$, this implies that $O(f, R)$ cannot approach anything above d_j as δ approaches 0, so $O(f, x_j) \leq d_j < \frac{1}{n}$. This contradicts our assumption that $x_j \in C_n$.
Therefore, by contradiction, we obtain for all $n \in \mathbb{N}$ that C_n contains at most $k_n + 2$ elements, as desired. In particular, every C_n is finite.
Step 2: We will conclude that C has measure zero.
First, recall that C is the set of discontinuities of f. Additionally, since f is increasing, f is bounded above by $f(b)$ and bounded below by $f(a)$. Then, by Spivak's Theorem 1-10, C is precisely the set $\{x \in [a, b] : O(f, x) > 0\}$. Next, we can rewrite this set as the countable union $C = \bigcup_{n=1}^{\infty} \{x \in [a, b] : O(f, x) > \frac{1}{n}\} = \bigcup_{n=1}^{\infty} C_n$. In Step 1, we proved that each C_n is finite, so it follows that C is countable. Therefore, C is measure-0 as a countable set, as required. $\square$

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted on Crowdmark.)
First, for the problems about integrals (Problems 1 to 4), there is a common multi-step process. First, we prove local statements for $m_S(f)$ and $M_S(f)$ using bounding arguments. Then, we sum those statements over all $S \in P$ to prove statements for $L(f, P)$ and $U(f, P)$ for individual partitions P. Finally, we use those statements to prove global statements about $L(f)$ and $U(f)$ with more bounding arguments. This is the same order that these concepts were introduced when defining integrals in lecture. As a result, this approach is an effective multi-step process for proving statements about integrals. We will see below how this approach applies to individual problems.

1. For this question, parts (a) and (b) are organized in a way that allows us to illustrate our multi-step process of $(m_S(f), M_S(f)) \to (L(f, P), U(f, P)) \to (L(f), U(f))$. First, the main challenge is to prove that $m_S(f) + m_S(g) \leq m_S(f + g)$ and $M_S(f + g) \leq M_S(f) + M_S(g)$; in other words, the sum of infimums is at most the infimum of the sums, and analogously for supremums. To solve this problem, recall that we proved a similar problem as an intermediate step when proving Fubini's Theorem in lecture. Afterward, the other steps of our multi-step process follow directly.
   For part (c), we use our multi-step process again to prove that $\int_A cf = c \int_A f$. If $c > 0$, then the analogous steps for our multi-step process are similar to those from part (a): Prove that $cm_S(f) \leq m_S(cf) \leq M_S(cf) \leq cM_S(f)$, then prove that $cL(f, P) \leq L(cf, P) \leq U(cf, P) \leq cU(f, P)$, then prove that $cL(f) \leq L(cf) \leq U(cf) \leq cU(f)$, which implies $L(cf) = U(cf)$ because $cL(f) = cU(f)$. For $c < 0$, there are some annoying details because infimums and supremums "swap places" when multiplying by a negative number. Afterward, the proof is not too much more difficult.

2. First, we will discuss the "$\Rightarrow$" direction: Proving that $f|_S$ is integrable over S if f is integrable over A. One possible approach is to apply Spivak's Theorem 3-3. Once we pick the partition P'' of A such that $U(f, P'') - L(f, P'')$ is small, the key idea is that only part of $U(f, P'') - L(f, P'')$ is contributed by S. This helps us to prove that $U(f|_S, Q) - L(f|_S, Q)$ is even smaller, where Q is the restriction of P'' to S. Then, this helps us to prove that $f|_S$ is integrable.
   Next, we will discuss the rest of the problem: Proving that f is integrable if all $f|_S$ are integrable, and proving that $\int_A f = \sum_{S \in P} \int_S f|_S$ in this case. Here, we want to find a partition P' such that $U(f, P') - \sum_{S \in P} \int_S f|_S$ and $\sum_{S \in P} \int_S f|_S - L(f, P')$ are small. Again, the key idea is that these differences can split up into contributions from individual subrectangles S. Each such contribution is $U(f|_S, Q) - \int_S f|_S$ for the upper sum and $\int_S f|_S - L(f|_S, Q)$ for the lower sum. By minimizing all contributions, we can minimize the entire differences $U(f, P') - \sum_{S \in P} \int_S f|_S$ and $\sum_{S \in P} \int_S f|_S - L(f, P')$, which helps us to solve the problem.

3. We want to prove that $\int_A f \leq \int_A g$ with our multi-step process. First, the analogous problems for each step are quite clear: Prove that $m_S(f) \leq m_S(g)$, then $L(f, P) \leq L(g, P)$, then $L(f) \leq L(g)$. (We could also do the same for $M_S(f) \leq M_S(g)$ and so on, but it turns out that we don't need that.) Afterward, the multi-step process can be applied directly.

4. For this question, since we need to show that $|F|$ is integrable, and we do not need to fully evaluate $\int_A |F|$, we can afford to prove that $|F|$ is integrable "non-constructively" using Spivak's Theorem 3-3. Additionally, it is somewhat intuitive that "$|F|$ varies less than F" because F can vary from negative to positive values while $|F|$ is constrained to being nonnegative. Combining

these two observations, we can construct a plan to prove that $|F|$ is integrable. First, prove that $M_S(|F|) - m_S(|F|) \leq M_S(F) - m_S(F)$ (i.e., $|F|$ varies less than F). Then, prove that $U(|F|, P) - L(|F|, P) \leq U(F, P) - L(F, P)$. Then, apply Spivak's Theorem 3-3 to prove that $|F|$ is integrable.
Finally, to show that $\left|\int_A F\right| \leq \int_A |F|$, the key idea is to realize that this inequality is a combination of two inequalities, $\int_A F \leq \int_A |F|$ and $\int_A (-F) \leq \int_A |F|$. Both inequalities are relatively simple to prove after solving Question 3.

5. First, the key idea for part (a) was to use proof by contradiction. This is because an unbounded set cannot be covered by finitely many rectangles, no matter their volume. We can formalize this by showing that the finitely many rectangles are themselves bounded by a larger rectangle surrounding them, and then the larger rectangle cannot cover any unbounded set.
Next, for part (b), we will discuss how to search for an example which satisfies the required conditions.

   i) First, for the set to not have content 0, part (a) hints that the set should be unbounded.
   ii) Moreover, for the set to be closed, we could try to define the set as a collection of single points that are far apart. This approach is similar to that of Assignment 2 Question 2(c).
   iii) Finally, for the set to be measure 0, there are several ways for this to happen, and one of the simplest ways is for the set to be countable. This is another reason that we want the set to consist of single points instead of closed intervals/rectangles.

   Now, we see that a good fit for these conditions is $\mathbb{N}$ (or $\mathbb{Z}$). Then, all that remains is some formalities to prove that $\mathbb{N}$ does satisfy the required conditions.

6. First, the condition $([0,1] \cap \mathbb{Q}) \subseteq A$ reminds us of Assignment 1 Question 5. Indeed, the key step is to prove that the boundary of A contains $[0,1] - A$, similarly to Assignment 1 Question 5. Then, if we were to cover the boundary of A with an open cover $\{U_i\}_{i \in \mathbb{N}}$ using open rectangles, then those rectangles, combined with the rectangles in $A = \bigcup_{i=1}^{\infty} (a_i, b_i)$, would have to "work together" to cover $[0,1]$. In fact, since $[0,1]$ is compact, finitely many such rectangles would have to cover $[0,1]$ together. Now, since $\sum (b_i - a_i) < 1$, we see that the rectangles in A do not cover enough of $[0,1]$, as expressed by Spivak's Theorem 3-5. Intuitively, the rectangles in $\{U_i\}_{i \in \mathbb{N}}$ would have to cover the entire remaining volume of $1 - \sum (b_i - a_i)$ – by proving this, we are done proving that the boundary of A does not have measure 0.
Remark: In hindsight, using Spivak's Theorem 3-5 would have simplified Step 2 of the solution greatly.

7. The key idea, as expressed in the hint, is to consider the constructions $C_n = \{x \in [a,b] : O(f,x) > \frac{1}{n}\}$. (One way to motivate this is that a similar construction was required to prove Spivak's Theorem 3-8.) It is intuitively clear why each C_n is finite: Every point inside C_n causes f to "jump up" by at least $\frac{1}{n}$, and it can only "jump up" finitely many times because it is bounded by $f(a)$ and $f(b)$. Indeed, our solution detects these "jumps" by measuring f at points $t_0, t_1, t_2, \ldots$ between points in C_n. After we argue that each C_n is finite, it easily follows that the set of discontinuities of f is countable and thus measure-0, as required.