MAT257 Assignment 6 (Implicit Function Theorem)

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1. We are given a C^1 function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(3, -1, 2) = 0$ and such that:

$$f'(3, -1, 2) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(a) First, we will show that there is a function $g : A \to \mathbb{R}^2$ defined on an open set $A \subseteq \mathbb{R}$ such that $3 \in A$ and such that $g(3) = (-1, 2)$ and $f(x, g_1(x), g_2(x)) = 0$ for all $x \in A$.
We will begin by checking that $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the conditions of the Implicit Function Theorem at $(3, -1, 2) \in \mathbb{R} \times \mathbb{R}^2$:

- f is given to be C^1 on the open set $\mathbb{R}^3$ containing $(3, -1, 2)$.
- $f(3, -1, 2) = 0$.
- Consider the matrix:

$$M_2 := \begin{pmatrix} D_2 f^1(3, -1, 2) & D_2 f^2(3, -1, 2) \\ D_3 f^1(3, -1, 2) & D_3 f^2(3, -1, 2) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

The determinant of this matrix is $(2)(1) - (1)(-1) = 3 \neq 0$, so M_2 is invertible.

Therefore, f satisfies all conditions of the Implicit Function Theorem at $(3, -1, 2)$, so we can use this theorem. This theorem gives us an open set $A \subseteq \mathbb{R}^1$ containing 3, an open set $B \subseteq \mathbb{R}^2$ containing $(-1, 2)$, and a unique function $g : A \to B$ such that $f(x, g(x)) = 0$ for all $x \in A$. Then, A and g satisfy the problem statement because:

- Since $B \subseteq \mathbb{R}^2$, the function $g : A \to B$ can be treated as a function from A to $\mathbb{R}^2$.
- From the Implicit Function Theorem, A is open, A is contained in $\mathbb{R}$, and A contains 3.
- By construction, $f(x, g_1(x), g_2(x)) = f(x, g(x)) = 0$ for all $x \in A$.
- We know that the point $(-1, 2) \in B$ satisfies $f(3, -1, 2) = 0$. Thus, applying the uniqueness of the function $g : A \to B$, we must have $g(3) = (-1, 2)$.

Therefore, we found a function $g : A \to \mathbb{R}^2$ defined on an open set A in $\mathbb{R}$ such that $3 \in A$, $g(3) = (-1, 2)$, and $f(x, g_1(x), g_2(x)) = 0$ for all $x \in A$, as required. $\square$

(b) Next, we will find $g'(3)$.
Recall that, in part (a), we defined the matrix:

$$M_2 := \begin{pmatrix} D_2 f^1(3, -1, 2) & D_2 f^2(3, -1, 2) \\ D_3 f^1(3, -1, 2) & D_3 f^2(3, -1, 2) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

Now, we will find M_2^{-1} by attaching a bookkeeping matrix to M_2:

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right).$$

Here, we can multiply the first row by $\frac{1}{2}$, then add the first row to the second row, to obtain:

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 \end{array}\right).$$

Next, we can multiply the second row by $\frac{2}{3}$, then add $\frac{-1}{2}$ times the second row to the first row, to obtain:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right).$$

Thus, our bookkeeping matrix tells us that:

$$M_2^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Next, let us define the matrix:

$$M_1 := \begin{pmatrix} D_1 f^1(3, -1, 2) \\ D_2 f^1(3, -1, 2) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the Implicit Function Theorem (as stated in class) gives us:

$$g'(3) = -M_2^{-1} M_1 = -\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}.$$

□

(c) Finally, we will discuss the problem of solving $f(x, y_1, y_2) = 0$ for an arbitrary pair of the unknowns in terms of the third, near the point $(3, -1, 2)$.
First, from part (a), there exists an open neighbourhood $A \subseteq \mathbb{R}$ around 3, an open neighbourhood $B \subseteq \mathbb{R}^2$ around $(-1, 2)$, and a function $g : A \to B$ such that $f(x, g_1(x), g_2(x)) = 0$ for all $x \in A$. In other words, g solves the equation $f(x, y_1, y_2) = 0$ for y_1 and y_2 in terms of x near $(3, -1, 2)$, as required.
Next, we will attempt to solve for x and y_1 in terms of y_2. To do this, we will check whether or not f satisfies the conditions of the Implicit Function Theorem at $(3, -1, 2)$. In part (a), we verified that f is continuously differentiable and that $f(3, -1, 2) = 0$, so it suffices to check whether or not the following matrix is invertible:

$$\begin{pmatrix} \frac{\partial f^1(3,-1,2)}{\partial x} & \frac{\partial f^1(3,-1,2)}{\partial y_1} \\ \frac{\partial f^2(3,-1,2)}{\partial x} & \frac{\partial f^2(3,-1,2)}{\partial y_1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

This matrix has a determinant of $(1)(-1) - (1)(2) = -3 \neq 0$, so it is invertible. Then, the Implicit Function Theorem gives us an open neighbourhood $A_2 \subseteq \mathbb{R}$ around 2, an open neighbourhood $B_2 \subseteq \mathbb{R}^2$ around $(3, -1)$, and a function $h : A_2 \to B_2$ such that $f(h_1(y_2), h_2(y_2), y_2) = 0$ for all $y_2 \in A_2$. In other words, h solves the equation $f(x, y_1, y_2) = 0$ for x and y_1 in terms of y_2 near $(3, -1, 2)$, as required.
Finally, we will attempt to solve for x and y_2 in terms of y_1. Similarly to above, it suffices to check whether or not the following matrix is invertible:

$$\begin{pmatrix} \frac{\partial f^1(3,-1,2)}{\partial x} & \frac{\partial f^1(3,-1,2)}{\partial y_2} \\ \frac{\partial f^2(3,-1,2)}{\partial x} & \frac{\partial f^2(3,-1,2)}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has a determinant of $(1)(1) - (1)(1) = 0$, so it is not invertible. Therefore, the Implicit Function Theorem does not apply, so it is not necessarily possible to solve for x and y_2 in terms of y_1 near $(3, -1, 2)$, as required. □

2. We are given a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(2,-1) = -1$, and we define $G, H : \mathbb{R}^3 \to \mathbb{R}$ by:
$$G(x,y,u) := f(x,y) + u^2, \qquad H(x,y,u) := ux + 3y^3 + u^3.$$
Finally, we are given that both equations $G(x,y,u) = 0$ and $H(x,y,u) = 0$ have the solution $(x,y,u) = (2,-1,1)$ (which can be verified with the above formulas).
(a) First, we will determine the conditions on f' which ensure that there are C^1 functions $x = g(y)$ and $u = h(y)$ defined on an open set in $\mathbb{R}$ that satisfy both equations, and such that $g(-1) = 2$ and $h(-1) = 1$.
Define $F : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ by $F(y,(x,u)) := (G(x,y,u), H(x,y,u)) = (f(x,y)+u^2, ux+3y^3+u^3)$. In other words, $F^1(y,(x,u)) = f(x,y) + u^2$, and $F^2(y,(x,u)) = ux + 3y^3 + u^3$. Then, let us check the conditions of the Implicit Function Theorem on F at $(y,(x,u)) = (-1,(2,1))$:

- The partial derivatives of F are:
$$\begin{pmatrix} D_1F^1(y,(x,u)) & D_2F^1(y,(x,u)) & D_3F^1(y,(x,u)) \\ D_1F^2(y,(x,u)) & D_2F^2(y,(x,u)) & D_3F^2(y,(x,u)) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial}{\partial y}(f(x,y)+u^2) & \frac{\partial}{\partial x}(f(x,y)+u^2) & \frac{\partial}{\partial u}(f(x,y)+u^2) \\ \frac{\partial}{\partial y}(ux+3y^3+u^3) & \frac{\partial}{\partial x}(ux+3y^3+u^3) & \frac{\partial}{\partial u}(ux+3y^3+u^3) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial f(x,y)}{\partial y} & \frac{\partial f(x,y)}{\partial x} & 2u \\ 9y^2 & u & x+3u^2 \end{pmatrix} \qquad (*),$$
which are continuous because f is given to be C^1. Thus, without requiring conditions on f', F is always continuously differentiable on the open neighbourhood $\mathbb{R}\times\mathbb{R}^2$ of $(-1,(2,1))$.
- We have $F(-1,(2,1)) = (G(2,-1,1), H(2,-1,1)) = (0,0) = 0$, so without requiring conditions on f', we always have $F(-1,(2,1)) = 0$.
- Finally, we require the following matrix to be invertible at $(-1,(2,1))$:
$$\begin{pmatrix} D_2F^1(y,(x,u)) & D_3F^1(y,(x,u)) \\ D_2F^2(y,(x,u)) & D_3F^2(y,(x,u)) \end{pmatrix} = \begin{pmatrix} \frac{\partial f(x,y)}{\partial x} & 2u \\ u & x+3u^2 \end{pmatrix}. \qquad \text{(Plugging in } (*))$$
Plugging in $(y,(x,u)) = (-1,(2,1))$, this matrix becomes:
$$\begin{pmatrix} \frac{\partial f(2,-1)}{\partial x} & 2\cdot 1 \\ 1 & 2+3\cdot 1^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial f(2,-1)}{\partial x} & 2 \\ 1 & 5 \end{pmatrix}.$$
This matrix has a determinant of $\frac{\partial f(2,-1)}{\partial x}\cdot 5 - 2\cdot 1 = 5\frac{\partial f(2,-1)}{\partial x} - 2$. This matrix is invertible if and only if this determinant is nonzero, so $5\frac{\partial f(2,-1)}{\partial x} - 2 \neq 0$, or $\frac{\partial f(2,-1)}{\partial x} \neq \frac{2}{5}$. By Spivak's Theorem 2-7, this is equivalent to $f'(2,-1)$ having a first coordinate not equal to $\frac{2}{5}$.

Overall, the conditions of the Implicit Function Theorem are satisfied if and only if $f'(2,-1)$ has a first coordinate not equal to $\frac{2}{5}$. Under this condition, we can apply the theorem to obtain an open neighbourhood $A \subseteq \mathbb{R}$ around -1, an open neighbourhood $B \subseteq \mathbb{R}^2$ around $(2,1)$, and a unique C^1 function $F_2 : A \to B$ such that $F(y, F_2(y)) = 0$ for all $y \in A$. Now, let us define the functions $g, h : A \to \mathbb{R}$ by $g(y) := F_2^1(y)$ and $h(y) := F_2^2(y)$; in other words, $F_2(y) = (g(y), h(y))$. Then, if we set $x = g(y)$ and $u = h(y)$, we obtain $F(y,(g(y),h(y))) = 0$. Moreover, g and h are the components of F_2, so the partial derivatives of g and h are the partial derivatives of F_2. Since F_2 is

C^1, its partial derivatives are continuous, so it follows that g and h are also C^1. Finally, since the point $(2,1) \in B$ satisfies $F(-1,(2,1)) = 0$, the uniqueness of F_2 implies that $F_2(-1) = (2,1)$, giving us $g(-1) = 2$ and $h(-1) = 1$.
Therefore, if $f'(2,-1)$ has a first coordinate not equal to $\frac{2}{5}$, then we can find C^1 functions $x = g(y)$ and $u = h(y)$ defined on an open set A in $\mathbb{R}$ that satisfy both equations $G(g(y), y, h(y)) = 0$ and $H(g(y), y, h(y)) = 0$ such that $g(-1) = 2$ and $h(-1) = 1$, as required. $\square$

(b) Now, given $f'(2,-1) = (1,-3)$. we will find $g'(-1)$ and $h'(-1)$. (Note that $f'(2,-1)$ has a first coordinate of $1 \neq \frac{2}{5}$, so our condition from part (a) is satisfied.)
First, applying Spivak's Theorem 2-7, the condition $f'(2,-1) = (1,-3)$ gives us $\frac{\partial f(2,-1)}{\partial x} = 1$ and $\frac{\partial f(2,-1)}{\partial y} = -3$. Then, let us define the matrix:

$$M_2 := \begin{pmatrix} D_2F^1(2,-1,1) & D_3F^1(2,-1,1) \\ D_2F^2(2,-1,1) & D_3F^2(2,-1,1) \end{pmatrix}.$$

From $(*)$, this matrix equals:

$$M_2 = \begin{pmatrix} \frac{\partial f(2,-1)}{\partial x} & 2u \\ u & x + 3u^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}.$$

We will find M_2^{-1} by attaching a bookkeeping matrix to M_2:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right).$$

Here, we can subtract Row 1 from Row 2, then divide Row 2 by 3, and finally subtract 2 times Row 2 from Row 1, to obtain:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{5}{3} & -\frac{2}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array}\right).$$

Thus, our bookkeeping matrix tells us that:

$$M_2^{-1} = \begin{pmatrix} \frac{5}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Next, let us define the matrix:

$$M_1 := \begin{pmatrix} D_1F^1(2,-1,1) \\ D_1F^2(2,-1,1) \end{pmatrix}.$$

From $(*)$, this matrix equals:

$$M_1 = \begin{pmatrix} \frac{\partial f(2,1)}{\partial y} \\ 9y^2 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}.$$

Therefore, the Implicit Function Theorem (as stated in class) gives us:

$$F_2'(-1) = -M_2^{-1}M_1 = -\begin{pmatrix} \frac{5}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -3 \\ 9 \end{pmatrix} = \begin{pmatrix} 11 \\ -4 \end{pmatrix}.$$

Finally, because $g = F^1$ and $h = F^2$ by definition, Spivak's Theorem 2-3(3) gives us that $F_2'(-1) = \begin{pmatrix} g'(-1) \\ h'(-1) \end{pmatrix} = \begin{pmatrix} 11 \\ -4 \end{pmatrix}$, so $\boxed{g'(-1) = 11}$ and $\boxed{h'(-1) = -4}$. $\square$

3. We are given C^1 functions $f, g : \mathbb{R}^3 \to \mathbb{R}$, and we are also given a point $p_0 = (x_0, y_0, z_0)$ that satisfies the two equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$. Finally, we are given that the matrix:

$$M := \frac{\partial(f, g)}{\partial(x, y, z)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$$

has rank 2 at p_0. Then, we will prove that we can solve these equations for two of x, y, z in terms of the third near p_0.

First, let us define the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x, y, z) := (f(x, y, z), g(x, y, z))$. In other words, $F^1(x, y, z) = f(x, y, z)$ and $F^2(x, y, z) = g(x, y, z)$. Then, we obtain:

$$M = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} D_1 F^1 & D_2 F^1 & D_3 F^1 \\ D_1 F^2 & D_2 F^2 & D_3 F^2 \end{pmatrix}.$$

Now, we are given that M has rank 2 at p_0, so M has 2 linearly independent columns at p_0. By symmetry, let us assume without loss of generality that the last two columns are linearly independent. Then, we will check that F satisfies the conditions of the Implicit Function Theorem at p_0 if we treat F as a function $\mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^2$, $(x, (y, z)) \mapsto F(x, (y, z))$:

- First, the partial derivatives of F, as listed in M, are continuous because f, g are given to be C^1. Therefore, F is continuously differentiable on the open neighbourhood $\mathbb{R}^3$ around p_0.
- Next, $F(p_0) = (f(p_0), g(p_0)) = (0, 0)$.
- Finally, we assumed that the last two columns of M, corresponding to variables y and z, are linearly independent at p_0, so the matrix $\begin{pmatrix} D_2 F^1 & D_3 F^1 \\ D_2 F^2 & D_3 F^2 \end{pmatrix}$ is invertible at p_0.

Therefore, F satisfies the conditions of the Implicit Function Theorem at p_0. Applying the Implicit Function Theorem, we obtain an open neighbourhood $A \subseteq \mathbb{R}$ around x_0, an open neighbourhood $B \subseteq \mathbb{R}^2$ around (y_0, z_0), and a unique function $G : A \to B$ such that $F(x, G(x)) = 0$ for all $x \in A$. In other words, $(f(x, G(x)), g(x, G(x))) = 0$, so $f(x, G(x)) = g(x, G(x)) = 0$. Therefore, $G(x)$ solves the equations $f(x, y, z) = g(x, y, z) = 0$ for y and z in terms of x in the open neighbourhood A around x.

Note that we assumed that the columns of M corresponding to y, z are linearly independent, which allowed us to solve for y and z in terms of x. By symmetry, any two of the columns of M, corresponding to two variables, could be linearly independent. Then, analogous proofs allow us to solve for those two variables in terms of the third near p_0, as required. $\square$

4. We are given a C^1 function $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ such that $f(a) = 0$ and such that $f'(a)$ has rank n. Then, we will prove that if $c \in \mathbb{R}^n$ is sufficiently close to 0, then the equation $f(z) = c$ has a solution.

   First, since the $n \times (k+n)$ matrix $f'(a)$ has rank n, it has n linearly independent columns, corresponding to n of the $k+n$ variables in $\mathbb{R}^{k+n}$. By symmetry, we may assume without loss of generality that the n rightmost columns of $f'(a)$, corresponding to the n rightmost variables of $\mathbb{R}^{k+n}$, are linearly independent. In other words, we can express f as a function $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$, $(x, y) \mapsto f(x, y)$, and we can express $a \in \mathbb{R}^{k+n}$ as a point $(x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^n$. Under this viewpoint, we are given $f(x_0, y_0) = 0$, and we assumed that the matrix $(\frac{\partial}{\partial y_j} f^i(x_0, y_0))_{1 \le i,j \le n}$ is invertible.

   Now, let us define the function $g : (\mathbb{R}^n \times \mathbb{R}^k) \times \mathbb{R}^n \to \mathbb{R}^n$ by $g((c, x), y) := f(x, y) - c$. Then, we will check that g satisfies the conditions of the Implicit Function Theorem at $((0, x_0), y_0)$:

   - First, for all coordinates x_j or y_j, we have $\frac{\partial g^i((c,x),y)}{\partial x_j} = \frac{\partial}{\partial x_j}(f^i(x, y) - c_i) = \frac{\partial}{\partial x_j} f^i(x, y)$, and we also have $\frac{\partial g^i((0,x),y)}{\partial y_j} = \frac{\partial}{\partial y_j}(f^i(x, y) - c) = \frac{\partial}{\partial y_j} f^i(x, y)$. We are given that f is C^1, so all partial derivatives so far are continuous. Next, for all coordinates c_j, we have:

     $$\frac{\partial g^i((c, x), y)}{\partial c_j} = \frac{\partial}{\partial c_j}(f^i(x, y) - c_i) = \frac{\partial}{\partial c_j}(-c_i),$$

     which is -1 if $i = j$ and 0 otherwise. Either way, these partial derivatives are continuous because they are constant. Overall, all partial derivatives of g are continuous, so g is C^1 on the open neighbourhood $\mathbb{R}^{k+2n}$ around $((0, x_0), y_0)$.
   - Next, $g((0, x_0), y_0) = f(x_0, y_0) - 0 = 0 - 0 = 0$.
   - Finally, the matrix $(\frac{\partial g^i((0,x_0),y_0)}{\partial y_j})_{1 \le i,j \le n} = (\frac{\partial f^i(x_0,y_0)}{\partial y_j})_{1 \le i,j \le n}$ was assumed to be invertible.

   Therefore, g satisfies the conditions of the Implicit Function Theorem at $((0, x_0), y_0)$. Applying this theorem, we obtain an open set $A \subseteq \mathbb{R}^n \times \mathbb{R}^k$ around $(0, x_0)$, an open set $B \subseteq \mathbb{R}^n$ around y_0, and a function $h : A \to B$ such that $g((c, x), h(c, x)) = 0$ for all $(c, x) \in A$. Since A is an open neighbourhood around $(0, x_0)$, there exists an open ball $B_r(0, x_0)$ of some radius $r > 0$ around $(0, x_0)$ such that $B_r(0, x_0) \subseteq A$. In particular, all $c \in \mathbb{R}^n$ which satisfy $|c| < r$ also satisfy:

   $$\big|(c, x_0) - (0, x_0)\big| = \big|(c, 0)\big| = \sqrt{|c|^2 + |0|^2} < \sqrt{r^2} = r,$$

   so $(c, x_0) \in B_r(0, x_0) \subseteq A$. This gives us:

   $$\begin{aligned} g(c, x_0, h(c, x_0)) &= 0 \\ f(x_0, h(c, x_0)) - c &= 0 \\ f(x_0, h(c, x_0)) &= c \end{aligned}$$

   Therefore, for all points $c \in \mathbb{R}^n$ that are sufficiently close to 0 (i.e., a distance of less than r away), the point $z = (x_0, h(c, x_0))$ is a solution for $f(z) = c$, as required. □

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted on Crowdmark.)

1. For part (a), we want to satisfy the equation $f(x, g_1(x), g_2(x)) = 0$, which motivates us to use the Implicit Function Theorem. Indeed, this subquestion can be directly solved using the Implicit Function Theorem. (This does require some formalities to check all conditions of the Implicit Function Theorem, and some formalities to check that the theorem solves all conditions of part (a)). Part (b) can be computed using the formula discussed in class for the derivative of an implicit function. Finally, part (c) is essentially the same as part (a), except with some variables swapping roles.

2. For this problem, we want to satisfy both equations $G(x, y, u) = 0$ and $H(x, y, u) = 0$ at the same time, but the Implicit Function Theorem only gives us a way to satisfy one equation at a time. To work around this, the key idea is to combine both equations into a single equation $(G(x, y, u), H(x, y, u)) = (0, 0)$. Then, we represent the left-hand side using the function $F(y, (x, u)) := (G(x, y, u), H(x, y, u))$. Note that we group together the variables x, u in $F(y, (x, u))$ to help ourselves solve for them in terms of y using the Implicit Function Theorem.
   The rest of the solution is merely computational. First, the Implicit Function Theorem requires a certain 2 by 2 matrix inside F' to be invertible, so we compute F' to determine when this condition is satisfied. This gives us conditions on f', which solves part (a). Finally, for part (b), we compute $g'(-1)$ and $h'(-1)$ using the formula we learned in lecture for the derivative of an implicit function.

3. Again, the key idea is to combine both equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$ into a single equation $(f(x, y, z), g(x, y, z)) = (0, 0)$. Then, we represent the left-hand side using the function $F(x, y, z) := (f(x, y, z), g(x, y, z))$ to try to use the Implicit Function Theorem on $F(x, y, z)$. This would allow us to write two variables as a function of the remaining variable to solve $F(x, y, z) = 0$. Now, only one issue remains: Although we are given that the matrix $\frac{\partial(f,g)}{\partial(x,y,z)}$ contains a 2 by 2 invertible matrix (since it has rank 2), we do not know which of its entries form the 2 by 2 invertible matrix. In other words, the conditions of the Implicit Function Theorem require one specific 2 by 2 matrix to be invertible (depending on which variables are being solved in terms of the others), but a different 2 by 2 matrix could be invertible instead. This issue is resolved by using casework on where the 2 by 2 invertible matrix is located; in different cases, we solve for different variables in terms of the remaining variable.

4. At first glance, this question feels like an Inverse Function Theorem question because we want to find inputs (x, y) corresponding to outputs $f(x, y) = c$. However, the domain of f is higher-dimensional than the codomain of f, so we should try to use the Implicit Function Theorem instead. First, the Implicit Function Theorem is designed to solve equations of the form *expression* $= 0$, so we must first convert the equation $f(x, y) = c$ into $f(x, y) - c = 0$. Then, we define the function $g((c, x), y) = f(x, y) - c$. After verifying the conditions of the Implicit Function Theorem on g, we can use the theorem to solve for y in terms of x and c. We have extra freedom here, so we can simply keep x constant while c varies, and then we can solve for y in terms of c. In other words, we solve for y such that $f(x, y) - c = 0$, or $f(x, y) = c$, which solves the problem.
   Remark: In fact, it may also be possible to use an Inverse Function Theorem approach to solve

this problem. First, we "balance" the dimensions of f's domain and codomain by defining the function $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n$ such that $F(x, y) := (x, f(x, y))$. Then, after verifying the conditions of the Inverse Function Theorem, we would be able to find inputs (x, y) such that the outputs $F(x, y) = (x, f(x, y))$ are certain values. In particular, we can control the outputs to keep x constant while setting $f(x, y)$ to equal c, then find corresponding inputs (x, y). Similarly to above, we again have some extra freedom to pick x.