MAT257 Assignment 5 (Inverse Function Theorem)

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1. We are given an open set $A \subseteq \mathbb{R}^n$ and a continuously differentiable 1-1 function $f : A \to \mathbb{R}^n$ such that $f'(x)$ is invertible for all $x \in A$. Then, we will show that $f(A)$ is also open and that $f^{-1} : f(A) \to A$ is differentiable. We will also show for all open subsets $B \subseteq A$ that $f(B)$ is also open.
First, for all $a \in A$, since f is continuously differentiable in the open neighbourhood A around a, and since $f'(a)$ is invertible, we can apply the Inverse Function Theorem on $f : A \to \mathbb{R}^n$. Then, we obtain an open neighbourhood $V_a \subseteq A$ around a and an open neighbourhood W_a around $f(a)$ such that $f : V_a \to W_a$ has a continuous and differentiable inverse $f^{-1} : W_a \to V_a$. (Technically, V_a is open in the topology of A, meaning that V_a can be expressed as $U \cap A$ for some open set $U \subseteq \mathbb{R}^n$. Since A is also open, we safely obtain that V_a is open in $\mathbb{R}^n$ as an intersection of two open sets. The same technical note is not needed for W_a since we treat $\mathbb{R}^n$ as the codomain of f.)
Now, since (the restriction of) f maps V_a to W_a, and since $V_a \subseteq A$, we obtain $W_a \subseteq f(A)$, so W_a is an open neighbourhood of $f(a)$ contained in $f(A)$. Moreover, for all $b \in f(A)$, there exists $a \in A$ such that $b = f(a)$, and then W_a is an open neighbourhood of $f(a) = b$ contained in $f(A)$. In other words, every point in $f(A)$ has an open neighbourhood around it that is contained in $f(A)$, so $f(A)$ is open, as required.
Next, for all open subsets $B \subseteq A$, we know that:

- B is open.
- f is continuously differentiable and 1-1 on B because it also has those features on A.
- $f'(x)$ is invertible for all $x \in B$ because $f'(x)$ is also invertible for all $x \in A$.

Therefore, we can use the same proof as above to show that $f(B)$ is open, as required.
It remains to show that $f^{-1} : f(A) \to A$ is differentiable. For all $a \in A$, we apply the Inverse Function Theorem as above to determine that $f^{-1} : W_a \to V_a$ is a differentiable inverse function of $f : V_a \to W_a$. What we mean by this is that we define the function $g_a : W_a \to V_a$ to be the inverse of the restriction $f|_{V_a}$ – defining g_a allows us to avoid confusion with the inverse $f^{-1} : f(A) \to A$ defined on the entire image $f(A)$. Using this distinction, we wish to show that g_a coincides with f^{-1} on W_a. Once we show this, we also know that g_a is differentiable at the point $f(a)$ for all $a \in A$, so we obtain that f^{-1} is differentiable at $f(a)$ for all $f(a) \in f(A)$.
To begin, since g_a is the inverse function of $f|_{V_a}$, the function $f|_{V_a} : V_a \to W_a$ must be surjective. In other words, for all $y \in W_a$, there exists $x \in V_a$ such that $f|_{V_a}(x) = y$. Then, by definition of inverse functions, we obtain:

$$x = g_a(f|_{V_a}(x)) = g_a(y),$$

as well as:

$$x = f^{-1}(f(x)) = f^{-1}(f|_{V_a}(x)) = f^{-1}(y).$$

This shows that $g_a(y) = f^{-1}(y)$ for all $y \in W_a$, as desired. Therefore, as discussed above, we are done proving that f^{-1} is differentiable, as required. □

2. Given any continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, we will prove that f is not 1-1.
First, in the special case when $D_1 f(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$, let us define the function $g_1 : \mathbb{R} \to \mathbb{R}$ by $g_1(x) = f(x, 0)$. Then, by plugging $y = 0$ into the condition $D_1 f(x, y) = 0$, we obtain $g_1'(x) = D_1 f(x, 0) = 0$ for all $x \in \mathbb{R}$. Now, consider $g_1(0)$ and $g_1(1)$. Since we found that $g_1'(x)$ exists and equals 0 for all $x \in \mathbb{R}$, g_1 is continuous and differentiable everywhere, so the Mean Value Theorem from MAT157 gives us some point $x' \in (0, 1)$ such that:

$$\frac{g_1(1) - g_1(0)}{1 - 0} = g_1'(x'),$$

or $g_1(1) - g_1(0) = g_1'(x')$. Since $g_1'(x') = 0$, this gives us $g_1(1) = g_1(0)$, which implies that $f(1, 0) = f(0, 0)$. Thus, f is not 1-1 if $D_1 f(x, y)$ is zero everywhere.
From now on, suppose to the contrary that $D_1 f(x_0, y_0) \neq 0$ at some point $(x_0, y_0) \in \mathbb{R}^2$. Then, let us define $g_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by $g_2(x, y) := (f(x, y), y) = (f(x, y), \pi_2(x, y))$. We can compute the differential $g_2'(x_0, y_0)$ as follows:

$$\begin{aligned} g_2'(x_0, y_0) &= \begin{pmatrix} D_1 f(x_0, y_0) & D_2 f(x_0, y_0) \\ D_1 \pi_2(x_0, y_0) & D_2 \pi_2(x_0, y_0) \end{pmatrix} && \text{(Applying Spivak's Theorem 2-7)} \\ &= \begin{pmatrix} D_1 f(x_0, y_0) & D_2 f(x_0, y_0) \\ 0 & 1 \end{pmatrix}. && \text{(Applying Spivak's Theorem 2-3(2))} \end{aligned}$$

This matrix has a determinant of $D_1 f(x_0, y_0) \cdot 1 - D_2 f(x_0, y_0) \cdot 0 = D_1 f(x_0, y_0)$, which we assumed to be nonzero. As a result, $g_2'(x_0, y_0)$ is invertible. Moreover, since f and π_2 are continuously differentiable, g_2 is also continuously differentiable, so we can apply the Inverse Function Theorem. Then, there exist an open set $V \subseteq \mathbb{R}^2$ containing (x_0, y_0) and an open set $W \subseteq \mathbb{R}^2$ containing $g_2(x_0, y_0) = (f(x_0, y_0), y_0)$ such that $g_2 : V \to W$ has an inverse $g_2^{-1} : W \to V$. Now, since W is an open neighbourhood around $(f(x_0, y_0), y_0)$, there exists an open ball with some radius $r > 0$ around $(f(x_0, y_0), y_0)$ that is contained in W. In particular, $(f(x_0, y_0), y_0 + \frac{r}{2})$ is in this open ball, so it is also inside W. As a result, we can apply g_2^{-1} to obtain the point $(x_1, y_1) := g_2^{-1}(f(x_0, y_0), y_0 + \frac{r}{2})$. Then, on the one hand:

$$g_2(x_1, y_1) = g_2(g_2^{-1}(f(x_0, y_0), y_0 + \frac{r}{2})) = (f(x_0, y_0), y_0 + \frac{r}{2}).$$

On the other hand, $g_2(x_1, y_1) = (f(x_1, y_1), y_1)$ by definition. As a result:

$$(f(x_1, y_1), y_1) = (f(x_0, y_0), y_0 + \frac{r}{2}). \qquad (*)$$

Comparing the second coordinates of $(*)$ yields $y_1 = y_0 + \frac{r}{2}$, which implies $(x_1, y_1) \neq (x_0, y_0)$. Comparing the first coordinates of $(*)$ also yields $f(x_1, y_1) = f(x_0, y_0)$. This, along with $(x_1, y_1) \neq (x_0, y_0)$, proves that f is not 1-1.
Therefore, no matter whether $D_1 f$ is zero everywhere or not, f is not 1-1, as required. $\square$

3. (a) First, given any differentiable $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f'(a) \neq 0$ for all $a \in \mathbb{R}$, we will show that f is 1-1 on $\mathbb{R}$.

Assume for contradiction f is not 1-1. Then, there exist distinct $x, y \in \mathbb{R}$ such that $f(x) = f(y)$; without loss of generality, $x < y$. Now, since f is continuous and differentiable everywhere, the Mean Value Theorem from MAT157 gives us some $x_0 \in [x, y]$ such that:

$$f'(x_0) = \frac{f(y) - f(x)}{y - x} = \frac{0}{y - x} = 0.$$

This contradicts the condition that $f'(a) \neq 0$ for all $a \in \mathbb{R}$. Thus, by contradiction, f is 1-1 on $\mathbb{R}$, as required. $\square$

(b) Now, given the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) := (e^x \cos y, e^x \sin y)$, we will show that $f'(x, y)$ is always invertible yet f is not 1-1.

First, consider the points $(x_1, y_1) := (0, 0)$ and $(x_2, y_2) := (0, 2\pi)$. Then, $(x_1, y_1) \neq (x_2, y_2)$, while:

$$\begin{aligned} f(x_1, y_1) &= (e^0 \cos(0), e^0 \sin(0)) \\ &= (1, 0) \end{aligned} \qquad \begin{aligned} f(x_2, y_2) &= (e^0 \cos(2\pi), e^0 \sin(2\pi)) \\ &= (1, 0) \\ &= f(x_1, y_1). \end{aligned}$$

Thus, there exist distinct $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that $f(x_1, y_1) = f(x_2, y_2)$, so f is not 1-1, as required.

Next, let us compute the partial derivatives of f as follows:

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= \frac{\partial}{\partial x} \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix} \\ &= \begin{pmatrix} \cos y \frac{\partial}{\partial x} e^x \\ \sin y \frac{\partial}{\partial x} e^x \end{pmatrix} && \text{(Treating } y \text{ as a constant)} \\ &= \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}. \\ \frac{\partial}{\partial y} f(x, y) &= \frac{\partial}{\partial y} \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix} \\ &= \begin{pmatrix} e^x \frac{\partial}{\partial y} \cos y \\ e^x \frac{\partial}{\partial y} \sin y \end{pmatrix} && \text{(Treating } x \text{ as a constant)} \\ &= \begin{pmatrix} -e^x \sin y \\ e^x \cos y \end{pmatrix}. \end{aligned}$$

Since both partial derivatives are continuous, Spivak's Theorem 2-8 tells us that $f'(x, y)$ exists, then Spivak's Theorem 2-7 tells us that:

$$f'(x, y) = (\frac{\partial}{\partial x} f(x, y), \frac{\partial}{\partial y} f(x, y)) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}.$$

Then, this differential has a determinant of:

$$\det f'(x, y) = e^x \cos y \cdot e^x \cos y - (-e^x \sin y) \cdot e^x \sin y = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} > 0.$$

This proves that $f'(x, y)$ is invertible at all (x, y). Therefore, the given function $f(x, y)$ is an example of a function which is not 1-1 and whose differential is always invertible, as required. $\square$

Notes on Intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted on Crowdmark.)

1. To solve the multiple parts of this problem, we must apply the Inverse Function Theorem in several different ways. First, we need to prove that $f(A)$ is open. Here, the key idea is that Inverse Function Theorem produces entire open neighbourhoods $W_a \subseteq f(A)$ around all $f(a) \in f(A)$, and these open neighbourhoods help to prove that $f(A)$ is open. Fortunately, the same trick works to prove that $f(B)$ is open for all open $B \subseteq A$. Next, when we prove that f^{-1} is differentiable, we first observe that the local inverses (labelled as g_a in the solution) produced by the Inverse Function Theorem are differentiable. This is useful if the global inverse f^{-1} coincides with the local inverses g_a, which is not too difficult to verify.
   Remark: For the submitted solution, it could have been better to begin with the proof that f^{-1} is differentiable. Then, f^{-1} would also be continuous. Then, since preimages of continuous functions preserve openness, this would have greatly simplified the proofs that $f(A) = (f^{-1})^{-1}(A)$ is open and that $f(B) = (f^{-1})^{-1}(B)$ is open for all open $B \subseteq A$.

2. This solution is strongly motivated by the textbook's hint, which tells us to consider the function $g(x, y) = (f(x, y), y)$. We would like to use the Inverse Function Theorem on g, which would allow us to find inputs (x, y) corresponding to an entire open set of outputs $g(x, y) = (f(x, y), y)$. Then, by adjusting the outputs slightly so that $f(x, y)$ remains constant while y changes, we can construct different points (x, y) such that $f(x, y)$ remains constant, proving that f is not 1-1.
   Our plan has one issue: We must check whether the Inverse Function Theorem can be applied. By computing g', we find that $g'(x, y)$ is invertible if and only if $D_1 f(x, y)$ is nonzero. Thus, this issue is only problematic if $D_1 f$ is zero everywhere. If this is the case, then f does not change when we move in the x-direction, which gives an alternative path to prove that f is not 1-1.
   Remark: In hindsight, this question can be thought of as an Implicit Function Theorem question. Since the domain of f (i.e., $\mathbb{R}^2$) is higher-dimensional than the codomain of f (i.e., $\mathbb{R}$), we could try using the Implicit Function Theorem to find x as a function of y to keep $f(x, y)$ constant. This approach explains the construction $g(x, y) = (f(x, y), y)$, since this construction was also used to prove the Implicit Function Theorem using the Inverse Function Theorem. Moreover, this approach requires a workaround if the matrix $(D_1 f(x, y))$ is not invertible, or if $D_1 f(x, y)$ is zero everywhere, similarly to above.

3. First, part (a) should be fairly standard MAT157 material. Since all tangent slopes must be nonzero, the Mean Value Theorem says that all secant slopes must also be nonzero, which directly implies that f is 1-1.
   Next, for part (b), a brute-force computation is enough to prove that $f'(x, y)$ is always invertible. Then, the main challenge is to prove that f is not 1-1. To do this, we notice that in the formula $f(x, y) = (e^x \cos y, e^x \sin y)$, the variable y is always plugged into trigonometric formulas. This implies that $f(x, y)$ is periodic in y, with a period of 2π. As a result, we can prove that f is not 1-1 by taking two inputs (x, y) with y-coordinates that are 2π apart, then verifying that f sends those two inputs to the same output.