MAT257 Assignment 1

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1. a) Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, we will prove that T is norm preserving if and only if it is inner product preserving.
First, for the "$\Rightarrow$" direction, suppose that T is norm preserving. Then, for all $x, y \in \mathbb{R}^n$, we obtain:

$$\begin{aligned}
\langle Tx, Ty\rangle &= \frac{|Tx+Ty|^2 - |Tx-Ty|^2}{4} && \text{(Applying polarization identity)}\\
&= \frac{\left|T(x+y)\right|^2 - \left|T(x-y)\right|^2}{4} && (T \text{ is linear})\\
&= \frac{|x+y|^2 - |x-y|^2}{4} && (T \text{ is norm preserving})\\
&= \langle x, y\rangle, && \text{(Applying polarization identity)}
\end{aligned}$$

which proves that T is inner product preserving.
Next, for the "$\Leftarrow$" direction, suppose that T is inner product preserving. Then, for all $x \in \mathbb{R}^n$, we obtain:

$$\begin{aligned}
|Tx| &= \sqrt{\langle Tx, Tx\rangle}\\
&= \sqrt{\langle x, x\rangle} && (T \text{ is inner product preserving})\\
&= |x|,
\end{aligned}$$

which proves that T is norm preserving.
Therefore, T is norm preserving if and only if it is inner product preserving, as required. □

b) Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which is norm preserving, we will prove that T is an isomorphism between vector spaces and that T^{-1} is also norm and inner product preserving.
We will first show that T is injective. For all nonzero $x \in \mathbb{R}^n$, we get $|x| > 0$ because the norm is positive definite. Since T is norm preserving, we obtain $|Tx| = |x| > 0$, and it follows that $Tx \neq 0$ for all nonzero $x \in \mathbb{R}^n$. Thus, T is injective. Since the domain and codomain of T have the same finite dimension, it follows (e.g., by the rank-nullity theorem) that T is also surjective. Thus, T is an isomorphism, as required.
Now, since T is an isomorphism, we can consider the linear map T^{-1}. Since T is norm preserving, we obtain for all $x \in \mathbb{R}^n$ that:

$$\left|T^{-1}x\right| = \left|T(T^{-1}x)\right| = |x|,$$

so T^{-1} is also norm preserving. Finally, it follows from part a) that T^{-1} is both norm preserving and inner product preserving, as required. □

2. Given any linear map $T : \mathbb{R}^m \to \mathbb{R}^n$, we will show that there exists a number M such that $|T(h)| \leq M|h|$ for all $h \in \mathbb{R}^m$.
First, let us denote the matrix of T by $A = (a_{ij})$, and let us denote h by the column vector $(h_1, \dots, h_m)^T$. Then, $T(h)$ is represented by the matrix product:

$$Ah = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m a_{1j}h_j \\ \vdots \\ \sum_{j=1}^m a_{nj}h_j \end{pmatrix}.$$

As a result, we obtain:

$$\begin{aligned} |T(h)| &= \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^m a_{ij}h_j\right)^2} \\ &= \sqrt{\sum_{i=1}^n \left|\sum_{j=1}^m a_{ij}h_j\right|^2} \\ &\leq \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^m |a_{ij}h_j|\right)^2} && \text{(Applying triangle inequality)} \\ &= \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^m |a_{ij}||h_j|\right)^2} \end{aligned}$$

Now, let us define the nonnegative real number:

$$a' := \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} |a_{ij}|.$$

Then, we obtain:

$$\begin{aligned} |T(h)| &\leq \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^m |a_{ij}||h_j|\right)^2} \\ &\leq \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^m a'|h_j|\right)^2} \\ &= a'\sqrt{\sum_{i=1}^n \left(\sum_{j=1}^m |h_j|\right)^2} \\ &= a'\sqrt{n\left(\sum_{j=1}^m |h_j|\right)^2} \end{aligned}$$

Now, let us define $x \in \mathbb{R}^m$ such that the j^{th} coordinate x_j is 1 if h_j is nonnegative and x_j is -1 if h_j is negative. In other words, $x_j h_j = |h_j|$. Then, let us apply the Cauchy-Schwarz inequality to x and h to obtain:

$$\begin{aligned} \big|\langle x, h\rangle\big| &\leq |x||h| \\ \left|\sum_{j=1}^{m} x_j h_j\right| &\leq \sqrt{\sum_{j=1}^{m}(\pm 1)^2}|h| \\ \left|\sum_{j=1}^{m}\big|h_j\big|\right| &\leq m|h| \end{aligned}$$

As a result:

$$\begin{aligned} \big|T(h)\big| &\leq a' \sqrt{n\left(\sum_{j=1}^{m}\big|h_j\big|\right)^2} \\ &\leq a'\sqrt{n(m|h|)^2} \\ &= a' m\sqrt{n}|h| \end{aligned}$$

Therefore, the number $\boxed{M = a' m\sqrt{n}}$ has the property that $\big|T(h)\big| \leq M|h|$ for all $h \in \mathbb{R}^m$, as required. $\square$

3. Given the dual space $(\mathbb{R}^n)^*$ of the vector space $\mathbb{R}^n$, we define $T : \mathbb{R}^n \to (\mathbb{R}^n)^*$ by $T(x) := \varphi_x$, where φ_x is defined by $\varphi_x(y) = \langle x, y\rangle$. (We can verify that $\varphi_x \in (\mathbb{R}^n)^*$ because the inner product is bilinear.) Then, we will show that T is an isomorphism (i.e. a bijective linear map) and that every $\varphi \in (\mathbb{R}^n)^*$ corresponds to a unique $x \in \mathbb{R}^n$ such that $\varphi = \varphi_x$.
   First, let x, y be any points in $\mathbb{R}^n$ and λ be any real number. Then, for all $z \in \mathbb{R}^n$, we have:

$$\begin{aligned}(T(\lambda x + y))(z) &= \varphi_{\lambda x+y}(z) \\ &= \langle \lambda x + y, z\rangle \\ &= \lambda\langle x, z\rangle + \langle y, z\rangle && \text{(Inner product is bilinear)} \\ &= \lambda\varphi_x(z) + \varphi_y(z) \\ &= (\lambda\varphi_x + \varphi_y)(z) \\ &= (\lambda T(x) + T(y))(z).\end{aligned}$$

   Since this is true for all $z \in \mathbb{R}^n$, it follows that $T(\lambda x + y) = \lambda T(x) + T(y)$, and since this is true for all $x, y \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$, it follows that T is a linear map.
   Next, for all nonzero $x \in \mathbb{R}$, we have $\varphi_x(x) = \langle x, x\rangle > 0$ by positive definiteness, which implies that φ_x is not the zero function in $(\mathbb{R}^n)^*$. As a result, T is injective. By Axler 3.95, $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ have the same finite dimension of n, so it follows (e.g., by the rank-nullity theorem) that T is also surjective. Thus, T is an isomorphism, as required.
   Finally, for all $\varphi \in (\mathbb{R}^n)^*$, since T is an isomorphism, there exists exactly one $x \in \mathbb{R}^n$ for which $T(x) = \varphi$. In other words, there exists exactly one $x \in \mathbb{R}^n$ for which $\varphi_x = \varphi$, as required. $\square$

4. Given $x, y \in \mathbb{R}^n$ such that x and y are orthogonal (i.e. $\langle x, y\rangle = \langle y, x\rangle = 0$), we will show that $|x+y|^2 = |x|^2 + |y|^2$.
   First, Spivak's Theorem 1-2(4) implies that $|z|^2 = \langle z, z\rangle$ for all $z \in \mathbb{R}^n$. As a result:

$$
\begin{aligned}
|x+y|^2 &= \langle x+y, x+y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle && \text{(Inner product is bilinear)} \\
&= \langle x, x\rangle + \langle y, y\rangle && \text{(Since } x, y \text{ are orthogonal)} \\
&= |x|^2 + |y|^2,
\end{aligned}
$$

   as required. □

5. (Note: This solution was submitted on Crowdmark but not marked.)
Given a subset A of $[0,1]$ which contains all of the rational numbers in $(0,1)$ and is the union of open intervals (a_i, b_i), let us denote the boundary of A by B. Then, we will show that $B = [0,1]\backslash A$.
First, we will show that $B \subseteq [0,1]\backslash A$. Let x be any point in A. Then, since A is a union of open intervals, x is inside one of those open intervals, so there exists an open interval around x which does not contain elements outside A. As a result, $x \notin B$ for all $x \in A$, so $B \subseteq \mathbb{R}\backslash A$. Moreover, for all $x > 1$, the open interval $(1, x+1)$ around x contains no points in $[0,1]$ and thus contains no points in A, and for all $x < 0$, the open interval $(x-1, 0)$ around x contains no points in $[0,1]$ and thus contains no points in A. As a result, $x \notin B$ for all x outside $[0,1]$. Overall, we obtain $B \subseteq [0,1]\backslash A$.
Next, we will show that $B \supseteq [0,1]\backslash A$. Let x be any point in $[0,1]\backslash A$, and let (a', b') be any open interval around x. Then, since $x \in (a', b')$, the interval (a', b') contains some point inside $\mathbb{R}\backslash A$.
Now, consider the two following cases:
Case 1: $x = 0$. Then, since $x = 0 \in (a', b')$, we have $0 < b'$, so we also have $0 < \min(b', 1)$. Now, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can pick a rational number r between 0 and $\min(b', 1)$. In other words, $0 < r < \min(b', 1)$, so we obtain $r \in (0,1)$, which implies that $r \in A$. We also obtain $a' < 0 < r < b'$, so $r \in (a', b')$. It follows that (a', b') contains some point in A.
Case 2: $x > 0$. Then, since $x \in (a', b')$, we also have $x > a'$, so $x > \max(0, a')$. Now, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can pick a rational number r in the interval $(\max(0, a'), x)$. In other words, $\max(0, a') < r < x$. Since $x < b'$, this gives us $a' < r < b'$, so $r \in (a', b')$. Moreover, since $x \in [0,1]\backslash A$, we have $x \leq 1$, which gives us $0 < r < x \leq 1$, so r is a rational number in $(0,1)$. This implies that $r \in A$, so (a', b') contains some point in A.
In both cases, we obtain that (a', b') contains some point in $\mathbb{R}\backslash A$ and some point in A for all open intervals (a', b') around x. As a result, $x \in B$ for all $x \in [0,1]\backslash A$, so $B \supseteq [0,1]\backslash A$. This, combined with $B \subseteq [0,1]\backslash A$, proves that $B = [0,1]\backslash A$, as required. $\square$

6. Given a closed set $A \subseteq \mathbb{R}$ that contains every rational number in $[0, 1]$, we will prove that $[0, 1] \subseteq A$.
   Let x be any point in $[0, 1]$. Then, assume for contradiction that $x \in \mathbb{R} \backslash A$. Since 0 and 1 are rational numbers in $[0, 1]$, we get $0, 1 \in A$, so $0 < x < 1$. Next, since A is closed, $\mathbb{R} \backslash A$ is open by definition, so there exists by definition an open rectangle (a, b) such that $x \in (a, b) \subseteq \mathbb{R} \backslash A$. In other words, $a < x < b$. Now, consider the open rectangle $R = (\max(a, 0), \min(b, 1))$. Since $0 < x < 1$ and $a < x < b$, we obtain $\max(a, 0) < x < \min(b, 1)$, so $x \in R$. Since $\max(a, 0) \geq a$ and $\min(b, 1) \leq b$, we also obtain $R \subseteq (a, b) \subseteq \mathbb{R} \backslash A$. Next, since $\mathbb{Q}$ is dense in $\mathbb{R}$, the open interval R contains some rational number r. Then, since $0 \leq \max(a, 0) < r < \min(b, 1) \leq 1$, we find that r is a rational number in $[0, 1]$, so $r \in A$, which contradicts $r \in R \subseteq \mathbb{R} \backslash A$. Therefore, by contradiction, $x \in A$ for all $x \in [0, 1]$, so $[0, 1] \subseteq A$, as required. $\square$

Some notes on intuition

Now, let's develop some intuition for how to approach these problems and come up with these solutions. (Note: This section was not submitted on Crowdmark.)

1. During lecture, Professor Bar-Natan commented that the formula $|x| = \sqrt{\langle x, x\rangle}$ is great for converting from inner products to norms, and that the polarization identity is great for converting from norms to inner products. Part a) of this problem exemplifies these relations very well. For our solution, our proof technique was to preserve Quantity A by writing it in terms of a preserved Quantity B, which is possible using the remark above.
   For part b), to see intuitively that T is injective, if we give it two distinct inputs, it must preserve the nonzero distance between the two inputs, so it cannot send them to the same output. Finally, we were able to prove that $T^{-1}x$ and x have the same norm by observing that they are "an iteration of T away from each other".

2. This was probably the hardest question in this problem set. Let's go through the key steps one-by-one:
   Expanding the matrix of T into (a_{ij}): The textbook's hint tells us to estimate $\left|T(h)\right|$ in terms of the matrix of T, which most likely requires considering the matrix's individual terms.
   Defining a' to be the maximum of the $\left|a_{ij}\right|$s: A quality-of-life substitution to let us factor out a' and isolate the terms $\left|h_j\right|$ inside the square root. This brings us closer to bounding the summation in terms of $|h|$.
   Applying the Cauchy-Schwarz inequality to prove $\left|\sum_{j=1}^m \left|h_j\right|\right| \leq m|h|$: This is a classic proof technique for various inequalities, and also the most difficult step of this problem. First, by expanding the Cauchy-Schwarz inequality as:

$$\left|\sum_{i=1}^n x_i y_i\right| \leq \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2},$$

   we see that this inequality is very powerful for proving inequalities of a summation-of-quadratics nature. In our case, we wanted to relate $\sum_{i=1}^n \left|h_j\right|$ with $|h| = \sqrt{\sum_{i=1}^n \left|h_j\right|^2}$. Since we have both a linear summation and a quadratic summation, we can plug in $y_i = 1$ to obtain linear terms on the left-hand side and quadratic terms on the right-hand side. (For more information, you can learn about the QM-AM Inequality, which states that:

$$\frac{x_1 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$$

   for nonnegative $x_1, \ldots, x_n$. This inequality is quite useful in math contests.)

3. Refer to MAT247, specifically our discussions of the Riesz Representation Theorem. For my solution, the line $\phi_x(x) = \langle x, x\rangle > 0 \Rightarrow \phi_x \neq 0$ is a classic proof technique in linear algebra, where we show that some quantity cannot always be zero because it is sometimes equal to an inner product $\langle x, x\rangle$. In fact, Axler's solution in his textbook also uses this proof technique.

4. Again, refer to MAT247. In this case, expanding the equation $|x+y|^2 = |x|^2 + |y|^2$ and cancelling out $\langle x, y\rangle$ and $\langle y, x\rangle$ terms was enough to solve the problem.

5. Let us develop a comprehensive picture of this problem. First, since A is a union of open intervals, this tells us directly that every point x in A is also in A's interior since x is inside one of the constituent open intervals of A. Moreover, since $A \subseteq [0,1]$, every point outside $[0,1]$ is "far away" from A and will be inside A's exterior. Finally, since every point outside $[0,1]\backslash A$ cannot be in the boundary of A, the core problem in Question 5 is to prove that every point inside $[0,1]\backslash A$ is in the boundary of A. To do this, since every point $x \in [0,1]\backslash A$ is outside A, we want to show that it is "very close" to A. The only potentially useful piece of information about A here is that it contains all rationals in $(0,1)$. Then, recalling that $\mathbb{Q}$ is dense, we see that x is surrounded by rationals very close to it. After formalizing this explanation, we are now done this problem.

6. This problem is very similar to problem 5, in that we again use the fact that $\mathbb{Q}$ is dense. In this case, we still know that every point x in $[0,1]\backslash A$ is surrounded by rational numbers in A, but we are also given the additional info that A is closed (i.e., A^c is open). As a result, we cannot construct an open rectangle around x without the rectangle containing rationals inside A, contradicting the openness of A^c. Thus, this additional info that A is closed tells us that A must contain the entire interval $[0,1]$.