# Homework Assignment 17

**Due:** Friday, March 25, 2022 11:59 pm (Eastern Daylight Time)

## Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

## Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

### Q1 (10 points)

**Spivak's 5-6** (modified).

(a) If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a function, the graph of $f$ is defined to be $\Gamma_f = \{(x, y) : y = f(x)\} \subset \mathbb{R}^{n+m}$. Prove that $\Gamma_f$ is a smooth $n$-manifold if $f$ is smooth.

(b) This isn't an "if and only if"! Find an example of a non-smooth $f$ for which nevertheless $\Gamma_f$ is a smooth manifold.

### Q2 (10 points)

**Spivak's 5-8(a)** (modified). Prove any 12-dimensional manifold $M$ in $\mathbb{R}^{22}$ is of measure 0 in $\mathbb{R}^{22}$.

## Q3 (10 points)

Show that $U(2) := \{A \in M_{2\times 2}(\mathbb{C}) : \bar{A}^T A = I\} \subset M_{2\times 2}(\mathbb{C}) = \mathbb{C}^4 = \mathbb{R}^8$ is a manifold. What is its dimension $\dim U(2)$?

If you have a problem visualizing $U(2)$, that's okay. So do I.

*Note.* $\bar{A}^T$ is the conjugate-transpose of $A$. Namely, you conjugate every entry of $A$ and then take the transpose.

*Hint.* It is best to present $U(2)$ as the zero set of some function $g$ and then show that the appropriate condition on the differential of $g$ is satisfied.

## Q4 (10 points)

Recall that the notion of "differentiable", and therefore also of "smooth", extends to functions $\phi\colon A \to \mathbb{R}^m$ defined on not-necessarily-open subsets $A \subset \mathbb{R}^n$. We simply say that $\phi$ is differentiable at a point $p \in A$ if $\phi$ has an extension $\bar{\phi}\colon U \to \mathbb{R}^m$ to some neighborhood $U$ of $p$ such that $\phi|_{U\cap A} = \bar{\phi}|_{U\cap A}$ and such that $\bar{\phi}$ is differentiable at $p$.

Show that if $U$ and $V$ are open subsets of $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k \colon x_k \geq 0\}$ (meaning, there exists $U'$ and $V'$ open in $\mathbb{R}^k$ such that $U = U' \cap \mathbb{R}^k$ and $V = V' \cap \mathbb{R}^k$), and if $\phi\colon U \to V$ is a diffeomorphism, then $\phi(U \cap \mathbb{R}^{k-1}) = V \cap \mathbb{R}^{k-1}$, where $\mathbb{R}^{k-1}$ is considered as a subset of $\mathbb{R}^k$ in the obvious manner.

(It follows that the notion of "the boundary of a manifold" is independent of the perspective).

## Q5 (10 points)

(a) Show that if $M^k \subset \mathbb{R}^m$ and $N^l \subset \mathbb{R}^n$ are manifolds without boundary, then $M \times N$ is a $(k+l)$-dimensional manifold without boundary in $\mathbb{R}^{m+n}$.

(b) Show that if $M^k \subset \mathbb{R}^m$ and $N^l \subset \mathbb{R}^n$ are manifolds with boundary, then $M \times N$ can be presented as the disjoint union of a $(k+l)$-manifold with boundary and a $(k+l-2)$-manifold without boundary.

*Hint.* Start by understanding the case of $M = N = [0,1]$.