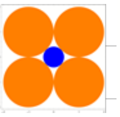
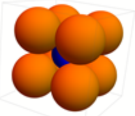
Hour 40 MAT257 Analysis II on January 17, 2022: Proof of the COV formula (4).
Read Along: Spivak 66-74.
TT2: Understand everything! And only thesn solve old problems. And good luck!





$$\lim_{n\to\infty}\frac{Vol(B_n)}{Vol(C_n)} = ?$$

COV: $\int_{g(A)} f = \int_A (f\circ g)|\det g'|$

Debts: ~~1. cov(g), cov(h) ⇒ cov(g∘h)~~

~~2. RCov(n-1) ⇒ RCov(n) Fubini!! for l.p. maps~~ 6

~~3. Pf that every g is a composition of l.p. maps (locally) IFT!!~~ 0

4 local ⇒ global   PO1

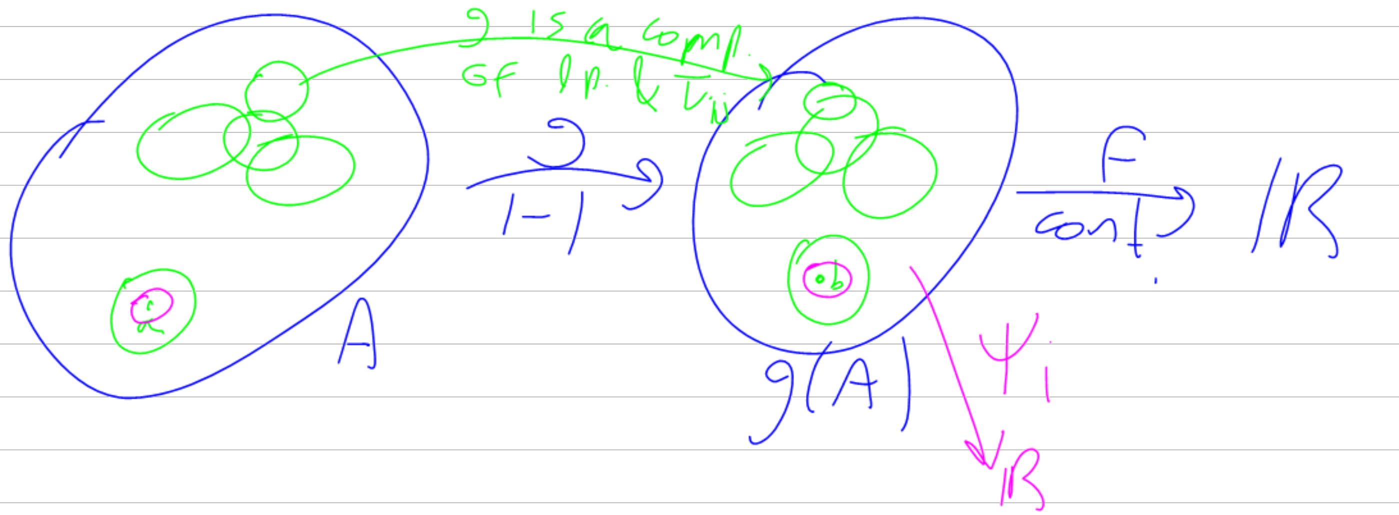
cov(small sets) ⇒ cov(large sets) as expected w/ tiny complication

5. cov(1D)   157   Dull.

6. RCOV ⇒ COV Tricky.

7. COV(T_{ij}) Dull.

Lemma 4 Local RCOV ⇒ global RCOV. (cont. f)



Let $\mathcal{V} = \{ V \underset{\text{open}}{\subset} g(A) : V$ bndd $g^{-1}(V)$ bndd on $g^{-1}(V)$ g is a comp. of l.p. & $T_{ij}\}$

$\mathcal{V}$ is a cover of g(A).

Find a PO1 ψ_i subordinate to $\mathcal{V}$

$\varphi_i = \psi_i \circ g$ is a POI for A subordinate to $\mathcal{U} = \{ g^{-1}(V) : V \in \mathcal{V} \}$

$$\int_{g(A)} f = \sum_i \int_{\cancel{g(A)}\ \text{Some } V \in \mathcal{V}} \psi_i f = \sum_i \int_{\cancel{A}\ \text{Some } U \in \mathcal{U}} (\psi_i \circ g)(f \circ g) \cdot |\det g'|$$

$$= \sum_i \int_{\text{Some } V} \varphi_i \, (f \circ g) \cdot |\det g'| = \int_A f \circ g \, |\det g'| \quad \square$$

<u>Lemma 5</u> COV holds if $n = 1$ $A = (a,b)$

$g : (a,b) \longrightarrow \mathbb{R}$ is 1-1 so it is monotone

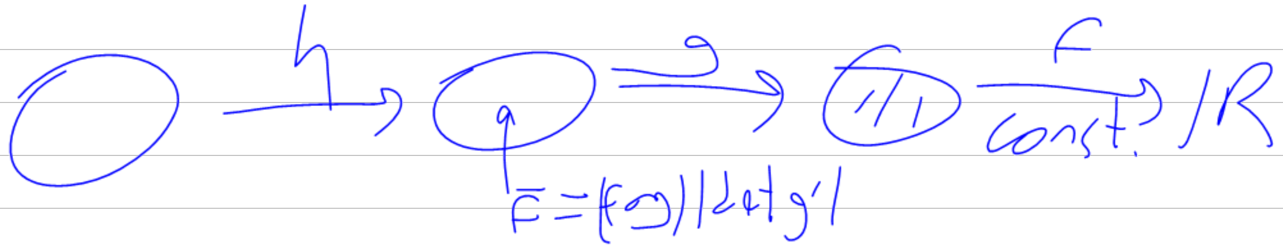
$$g(A) = g((a,b)) = \begin{cases} (g(a), g(b)) & g \text{ is increasing} \\ (g(b), g(a)) & g \text{ is decreasing} \end{cases}$$

Let's take the case where f is decreasing.

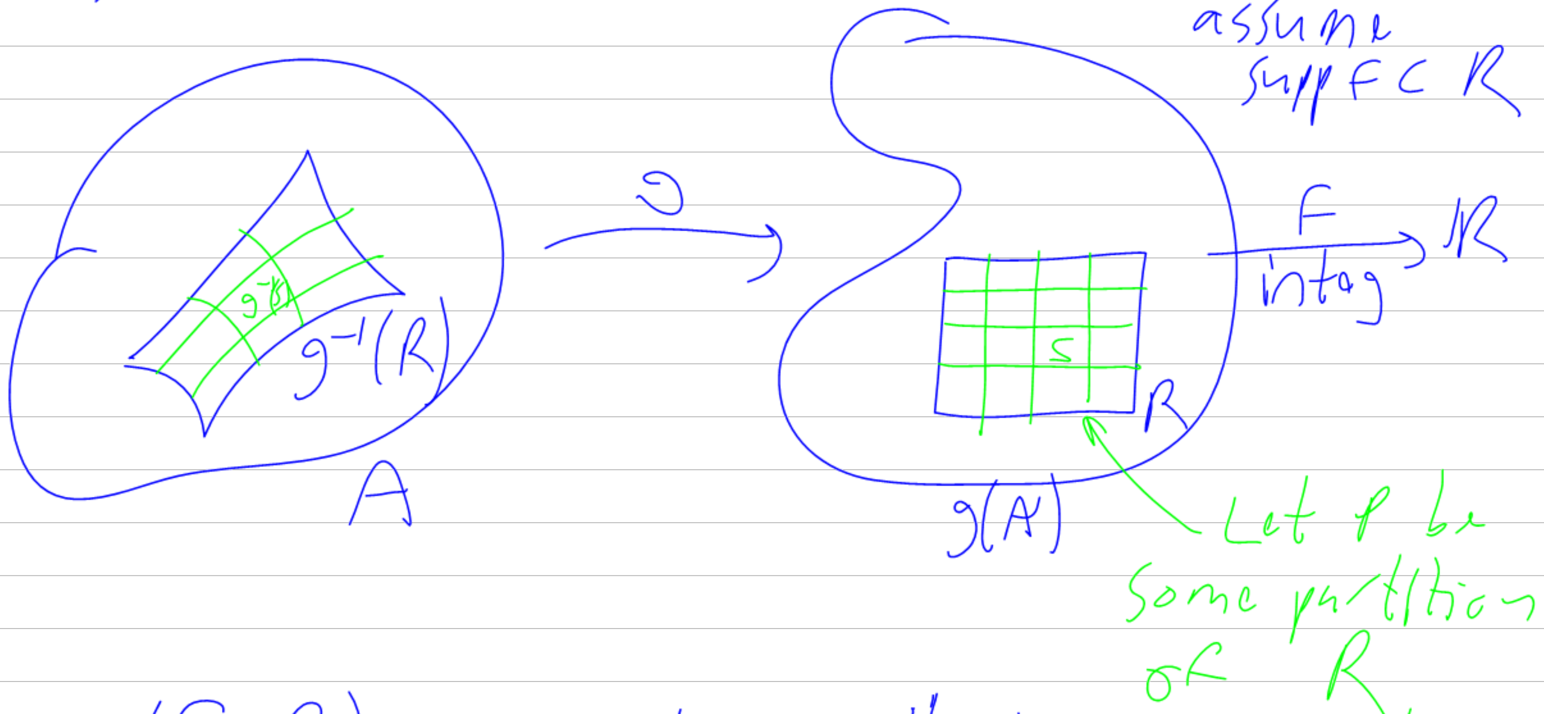
$$\int_{g(A)} f = \int_{g(b)}^{g(a)} f \overset{157}{=} \int_b^a (f \circ g) \underset{-|\det g'|}{\underset{\parallel}{g'}} = \cancel{-} \int_a^b (f \circ g)(\cancel{-} |\det g'|)$$

$$= \int_{a}^{b} (f\circ g)\,|\det g'| = \int_{A} (f\circ g)\,|\det g'| \quad \square$$

Lemma 6 Suppose COV holds for ~~cont.~~ const. f's, then it holds for any integrable f.



We'll prove a local version of COV for integrable function, which can be globalized as before.



$L(f,P)$ . . . $L(f\circ g)$ $U(f\circ g)$ $U(f,P)$

$$L(f,P) = \sum_{S\in P} v(S)\cdot m_S(f) = \sum_{S\in P} \int_S m_S(f)$$

$$\underline{\underline{\text{cov(const)}}}\ \sum_{S\in P} \underline{\int}_{g^{-1}(S)} m_S(f)\,|\det g'|$$

Exercise

$$\underline{\int} h_1 + \underline{\int} h_2 \leq \underline{\int}(h_1+h_2)$$

$$\leq \sum_{S\in P} \underline{\int}_{g^{-1}(S)} (f\circ g)\,|\det g'|$$

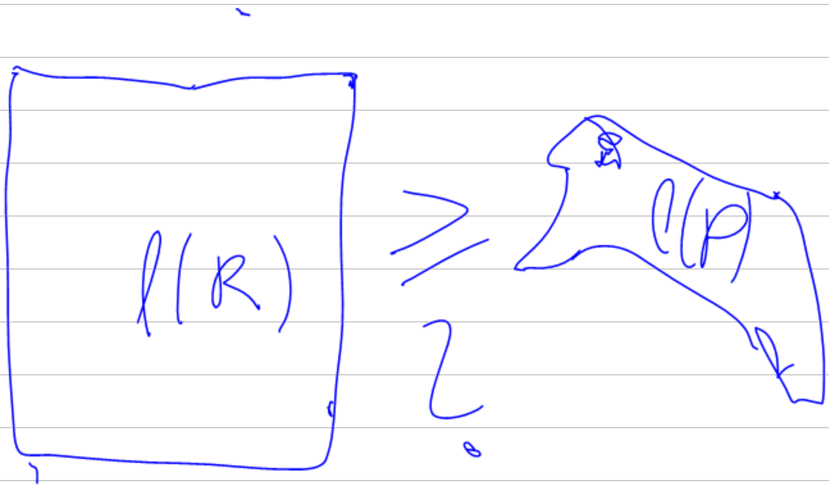
$$= \sum_{S\in P} \underline{\int}_{g^{-1}(R)} \chi_{g^{-1}(S)}\, f\circ g\,|\det g'|$$

$$\leq \underline{\int}_{g^{-1}(R)} \sum_S \chi_{g^{-1}(S)} (f\circ g)\,|\det g'|$$

$$= \boxed{\underline{\int}_{g^{-1}(R)} (f\circ g)\,|\det g'|} \leq \boxed{\overline{\int}_{g^{-1}(R)} (f\circ g)\,|\det g'|}$$

$$\leq \boxed{U(f,P)}$$

by integrability of f we can make L(f,P) & U(f,P) as close to each other as we want, so ~~as~~ all numbers above are close to each other, $\square$



$$\int_{x^2+y^2<R^2}$$

I hope TT2 went well!
Hour 41 MAT257 Analysis II on January 19, 2022:
Sketchy coordinate swaps and Sard's Theorem,
~~Detailed k-tensors.~~
Read Along: Spivak 66-74, ~~75-78.~~

$\int_M d\omega = \int_{\partial M} \omega$

Riddle Along: Let f be a distance-non-inreasing function from the plane to the plane (d(x,y) >= d(f(x),f(y)), for all x,y), and let R be a rectangle in the plane. Is it always true that the length of the boundary of R is greater or equal to the length of the boundary of f(R)?

COV: $\int_{g(A)} f = \int_A (f\circ g)|\det g'|$

Debt 7 COV(T_{ij}) So dull will skip.

Lemma 7 COV holds for coord swaps T_{ij}

PF NTS $\int_{T_{ij}(A)} f = \int_A f\circ T_{ij}$ E.g., $\int_{TA} f\circ T = \int_A f$ $\quad T(x,y)=(y,x)$

Q. How do you write the proof of something so disturbingly obvious?

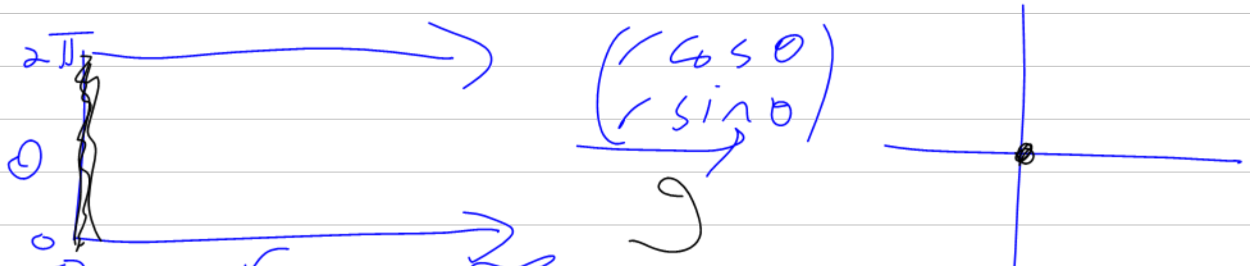
A. You go back to the defs.

PF Given $A=[a_1,b_1]\times[a_2,b_2]$; $TA=[a_2,b_2]\times[a_1,b_1]$

$P=((a_1=t_{10},t_{11},\dots,t_{1n}=b_1),(a_2=t_{20},\dots,t_{2n}=b_2))$ $\quad TP=((a_2=t_{20},\dots,t_{2n}=b_2),\dots)$

$L(f\circ T,\underset{\text{of } TA}{P}) = \sum_{S\in P} v(S)\,m_S(f\circ T) = \sum_{S\in P} v(TS)\,m_{TS}(f) =$

$\sum_{TS\in TP} \_\_ = \sum_{S\in TP} \_\_ = L(f,TP)$ ...... $\boxed{7}$ $\boxed{COV}$



$|\det g'| = r$

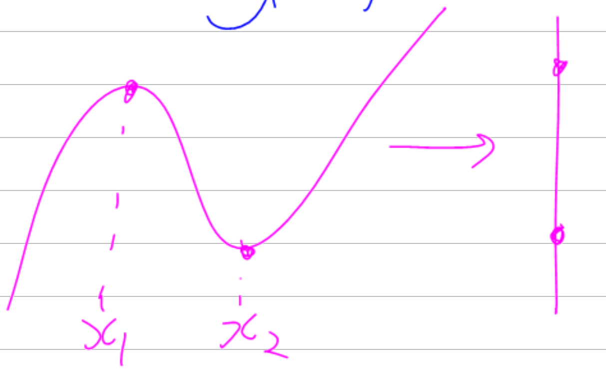
drop g' is 1-1

Baby Sard Theorem $A\subset\mathbb{R}^n$ open

$g: A\to\mathbb{R}^n$ cont. diff'ble

$$C := \text{"critical set of } g\text{"} = \{x \in A : \underbrace{\det g'(x)}_{h(x) \text{ is cont.}} = 0\}$$

then $g(C)$ is of meas-0.

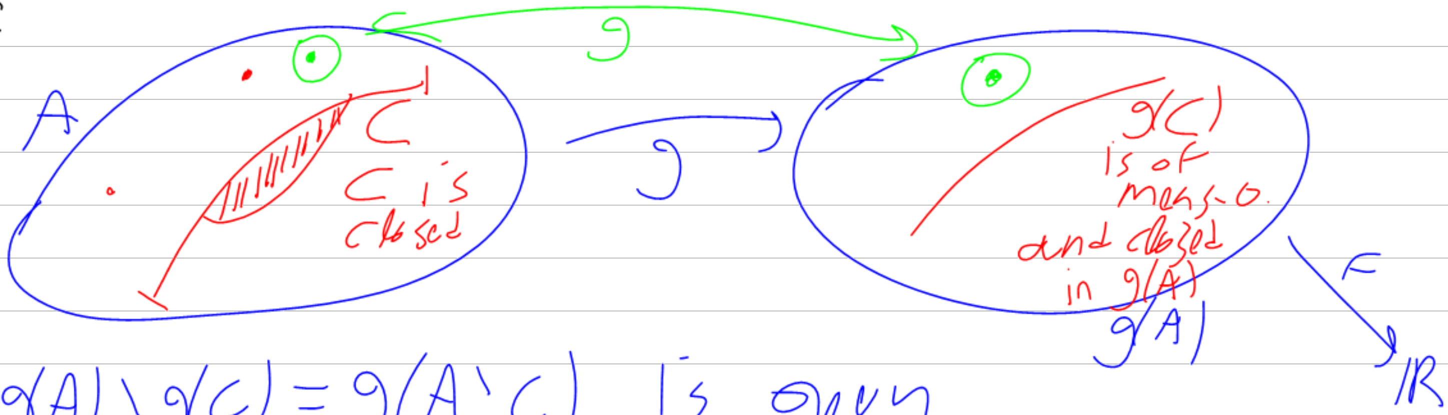


Claim C is closed in A. Pf $C = h^{-1}(\{0\})$

meaning $A \setminus C$ is open.

Cor (of Sard) In the COV thm, can drop the condition "g is 1-1"

Pf

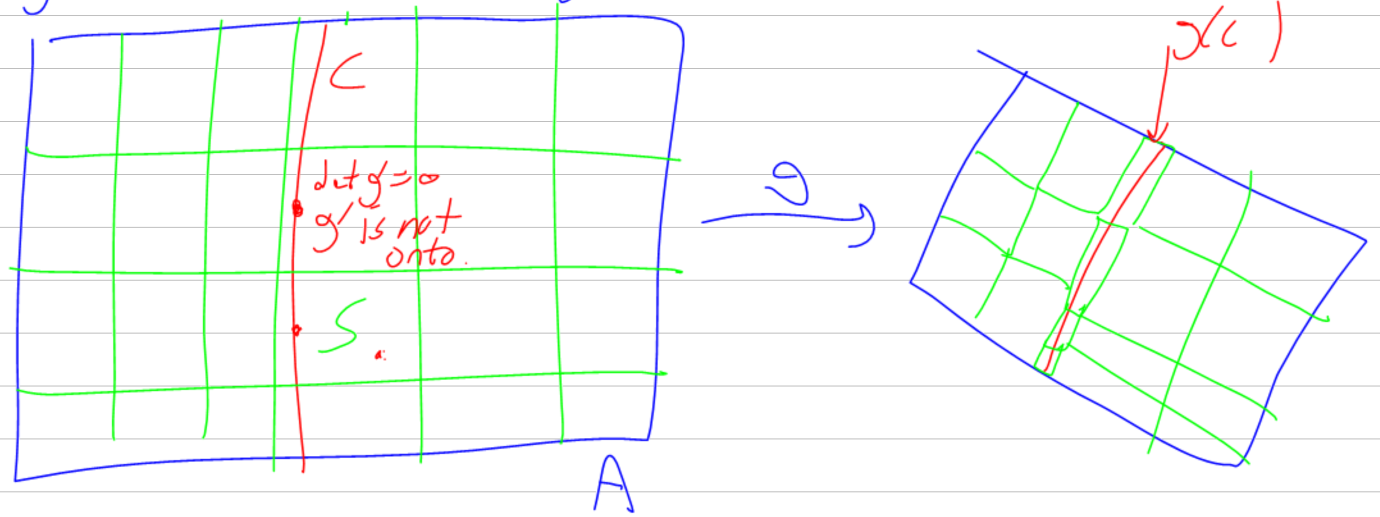


$$g(A) \setminus g(C) = g(\underbrace{A \setminus C}_{\text{open set on which } g' \text{ is invertible}}) \text{ is open}$$

$$\int_A (F \circ g) \cdot |\det g'| \overset{?}{=} \int_{g(A)} F$$

$$\overset{\parallel}{\int_{A\setminus C} (F\circ g)|\det g'|} \;\overset{\text{old}}{\underset{\text{COV}}{=}}\; \overset{\parallel}{\int_{g(A)-g(C)} F} \qquad \square.$$

~~Baby~~ Sketch. proof of baby Sard:



Adult Sard Thm $g : \underset{\text{open}}{A} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$C = \{x \in A : \operatorname{rank} g' < m\}$$

g is k times cont. diff'ble, $k = \max(1, n-m+1)$

Then $g(C)$ is meas-0. Way harder!

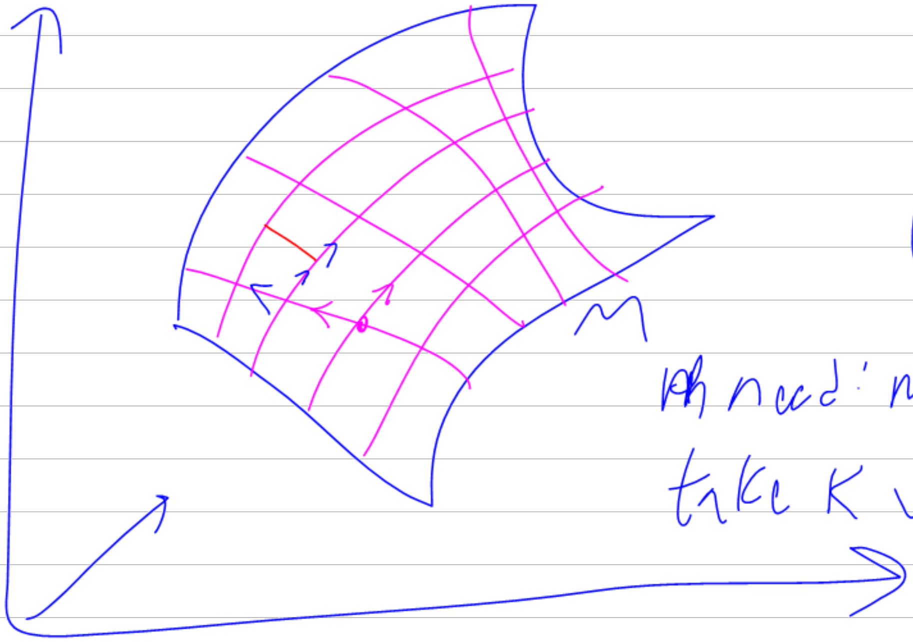
$A+B = \{a+b : \begin{smallmatrix} a\in A \\ b \in B\end{smallmatrix}\}$ $\quad S+S=S?$

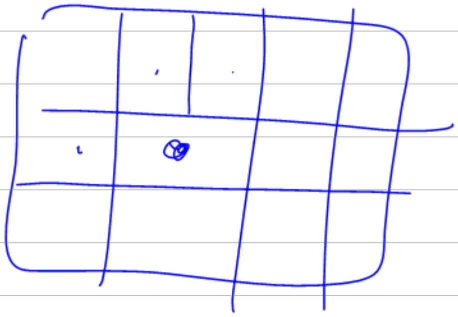
$F_1 + F_2$ = Finite $\quad$ Countable + Countable = countable

(meas-0) + (meas-0) = meas-0 ?

Hour 42 MAT257 Analysis II on January 21, 2022: k-tensors.
Read Along: Spivak 75-78.
HW11 on web by midnight! (Sorry, much of it will only make sense on Monday)
Riddle Along: The Cantor set C is of measure 0. Is the same true
for C+C={x+y: x,y in C}?

Remember, $\int_M d\omega = \int_{\partial M} \omega$





We need: machines that take k vects in $\mathbb{R}^n$ & out a number

<u>Definition</u> Let V be a v.s over $\mathbb{R}$ & $k \in \mathbb{N} = \mathbb{Z}_{\geq 0}$

$T : V^k \longrightarrow \mathbb{R}$ is called "multi-linear" or "k-linear" if

$$T(u_1 \dots, \alpha u_i' + \beta u_i'', \ u_k) =$$

$$\alpha T(u_1 \dots u_i', \dots u_k) + \beta T(u_1 \dots u_i'', \dots u_k).$$

<u>Example 2</u> An inner product is a 2-linear map:

$\langle u_1, u_2 \rangle \in \mathbb{R}$

On $\mathbb{R}^n$ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$T(x,y) = \sum x_i y_i$

Example n $\begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \sim (u_1, \ldots, u_n) \in (\mathbb{R}^n)^n$

$$\begin{tikzcd} (\mathbb{R}^n)^n \arrow[r, phantom, "="] \arrow[rrr, bend right, "\det"'] & \mathbb{R}^{n^2} \arrow[r, phantom, "="] & M_{n\times n}(\mathbb{R}) \arrow[r, "\det"] & \mathbb{R} \end{tikzcd}$$

is n-linear.

Example 1 1-linear map $\varphi: V \to \mathbb{R}$

linear.

"linear functional" $\varphi \in V^*$ $0! = 1$

Example 0 0-linear map: on V^0

$$w: V^0 = \{()\} \longrightarrow \mathbb{R}$$

$$\frac{22}{7} = w(()) \in \mathbb{R} \qquad \overset{0\text{-linear}}{\cong} \mathbb{R}.$$

Def $\mathcal{T}^k(V) = \left\{ \begin{matrix} k\text{-linear} \\ \text{maps on } V \end{matrix} \right\}$

Warning: many other sources call it $\mathcal{T}^k(V^*)$

$\langle , \rangle \in \mathcal{T}^2 V$ $\det \in \mathcal{T}^n V$

$\mathcal{T}^0 V \cong \mathbb{R}$

$\mathcal{T}^1 V = V^* =$ n-dim. $\quad$ dim $\mathcal{T}^k V$?

if V is n-dim

Suppose $V_1 \ldots V_n$ is a basis of V

$\exists !$ basis $\varphi_1 \ldots \varphi_n$ of $V^* = \mathcal{T}^1 V$

$$\text{s.t. } \varphi_i(V_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

$\{\varphi_i\}$ is called "the dual basis of $(V_1 \ldots V_n)$"

Example What is the dual basis of

$$\left\{ \underset{V_1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}, \underset{V_2}{\begin{pmatrix} 3 \\ 4 \end{pmatrix}} \right\} ? \qquad \mathbb{R}^2 = \left\{ \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\}$$

$$(\mathbb{R}^2)^* = \{ (\cdot \quad \cdot) \}$$

$$\varphi_1 = (\underline{\qquad\qquad})$$

$$\varphi_2 = (\underline{\qquad\qquad})$$

$$\boxed{\begin{array}{ll} \varphi_1(V_1) = 1 & \varphi_1(V_2) = 0 \\ \varphi_2(V_1) = 0 & \varphi_2(V_2) = 1 \end{array}}$$

$$\begin{pmatrix} - \varphi_1 - \\ - \varphi_2 - \end{pmatrix} \begin{pmatrix} | & | \\ V_1 & V_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 3/2 \\ 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} - & \varphi_1 & - \\ - & \varphi_2 & - \end{pmatrix}$$

$$\varphi_1 = \begin{pmatrix} -2 & 3/2 \end{pmatrix}$$

$$\varphi_2 = \begin{pmatrix} 1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}^{-1} = \begin{pmatrix} - & \varphi_1 & - \\ & \vdots & \\ - & \varphi_n & - \end{pmatrix}$$

Warning

Non-Example In $\mathbb{R}^n$ what is the dual of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$? ~~is $\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$~~

Claim $\mathcal{T}^k(V)$ is itself a v.s.

$T_1, T_2 \in \mathcal{T}^k$

$$(T_1 + T_2)(u_1, \dots u_k) = T_1(u_1 \dots u_k) + T_2(u_1 \dots u_k)$$

$$(\alpha T)(u_1 \dots u_k) = . \; - \; - \; .$$

$$0_{\mathcal{T}^k}(u_1 \ldots u_k) = 0 \qquad \square$$

"tensor multiplication"

Also a map $\otimes : \underset{\substack{\downarrow \\ T_1}}{\mathcal{T}^k} \times \underset{\substack{\downarrow \\ T_2}}{\mathcal{T}^l} \longrightarrow \mathcal{T}^{k+l}$

defined by

$$(T_1 \otimes T_2)(u_1 \ldots u_{k+l}) =$$

$$T_1(u_1 \ldots u_k) \cdot T_2(u_{k+1} \ldots u_{k+l})$$

<u>claim</u> $T_1 \otimes T_2 = T_1 T_2 \in \mathcal{T}^{k+l}$

$(\mathcal{T}^* V, +, \cdot, \otimes, \ldots) \ldots$ next time.