Pensieve header: Perturbing the Heisenberg-algebra knot invariant. Continues “Solving to $k=3.nb” at pensieve://Projects/BabyDoPeGDO/.

𝔼[ω, Q, P_ϵSeries] represents $\omega\, e^{Q+P}$, where ω is a scalar, Q is an ϵ-free quadratic, and $P=\sum_{k=0}^{\$k} P[\![k]\!]\,\epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: 𝔼_[_] // 𝔼_[_] calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

Initialization, minor utilities, and “Define” Code

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\21-1350-KnotTheory\\Preps"];
Once[<< "../Common.m"];
PP_ = Identity;
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
$k=1;
```

In[∘]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ ℰ;
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | x | η | ξ)_, ∞] ⋃ {y | x | η | ξ}},
    Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] × (Times @@ vs^ps)]
   ];
(*CF[ℰ_] := PP_CF@CCF[ℰ];*)
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

In[∘]:=
```
ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
ϵSeries[0] := ϵSeries @@ Table[0, $k + 1];
ϵSeries /: S1_ϵSeries + S2_ϵSeries :=
  ϵSeries @@ Table[S1[[k]] + S2[[k]], {k, Min[Length@S1, Length@S2]}];
ϵSeries /: S1_ϵSeries * S2_ϵSeries := ϵSeries @@
   Table[Sum[S1[[j + 1]] * S2[[k - j + 1]], {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
ϵSeries /: c_ * S_ϵSeries := (c #) & /@ S;
ϵSeries /: ∂_vs___ S_ϵSeries := (s ↦ ∂_vs s) /@ S;
```

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_nisp,$k_Integer, Block[{i, j, k}, op_isp,$k = ℰ; op_nis,$k]];
      SD[op_isp, op_{is},$k]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

The Basic Tensors

```
Define[m_i,j→k = 𝔼_{i,j}→{k}[1, -ξ_i η_j + (η_i + η_j) y_k + (ξ_i + ξ_j) x_k, ϵSeries[0]]]
```

```
AllMonomials[{}, 0] = {1};
AllMonomials[{}, d_Integer] /; d > 0 := {};
AllMonomials[{v_, vs___}, d_Integer] :=
  Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
```

```
Basis[js_List, m_] := Flatten@Outer[Times,
    AllMonomials[Table[y_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 0, m}]
```

```
Basis[{i, j}, {2}]
```

Out: $\{1, x_i y_i, x_j y_i, x_i y_j, x_j y_j, x_i^2 y_i^2, x_i x_j y_i^2, x_j^2 y_i^2, x_i^2 y_i y_j, x_i x_j y_i y_j, x_j^2 y_i y_j, x_i^2 y_j^2, x_i x_j y_j^2, x_j^2 y_j^2\}$

```
GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}];
GenericCombination[bas_, c_k_] := bas.Table[c_k,j, {j, Length@bas}];
```

```
GenericCombination[Basis[{i, j}, {2}], c_1]
```

Out: $c_{1,1} + x_i y_i c_{1,2} + x_j y_i c_{1,3} + x_i y_j c_{1,4} + x_j y_j c_{1,5} + x_i^2 y_i^2 c_{1,6} + x_i x_j y_i^2 c_{1,7} + x_j^2 y_i^2 c_{1,8} + x_i^2 y_i y_j c_{1,9} + x_i x_j y_i y_j c_{1,10} + x_j^2 y_i y_j c_{1,11} + x_i^2 y_j^2 c_{1,12} + x_i x_j y_j^2 c_{1,13} + x_j^2 y_j^2 c_{1,14}$

In[∘]:=
```
R_{i_,j_} := 𝔼_{{}→{i,j}}[1, (-1 + T) x_j (y_i - y_j),
    ϵSeries @@ Prepend[0] @ Table[GenericCombination[Basis[{i, j}, {k + 1}], c_k], {k, $k}]];
R̄_{i_,j_} := 𝔼_{{}→{i,j}}[1, (-1 + 1/T) x_j (y_i - y_j),
    ϵSeries @@ Prepend[0] @ Table[GenericCombination[Basis[{i, j}, {k + 1}], d_k], {k, $k}]];
CC_{i_} := 𝔼_{{}→{i}}[√T, 0, ϵSeries @@ Prepend[0] @
      Table[GenericCombination[Basis[{i}, {k + 1}], e_k], {k, $k}]];
C̄C̄_{i_} := 𝔼_{{}→{i}}[1/√T, 0, ϵSeries @@ Prepend[0] @
      Table[GenericCombination[Basis[{i}, {k + 1}], f_k], {k, $k}]];
```

In[∘]:=
```
{R_{1,2}, R̄_{1,2}, CC_1, C̄C̄_1}
```

Out[∘]=

$$\left\{\mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ (-1+T)\,x_2\,(y_1-y_2),\ \epsilon\text{Series}\left[0,\ c_{1,1}+x_1 y_1 c_{1,2}+x_2 y_1 c_{1,3}+x_1 y_2 c_{1,4}+x_2 y_2 c_{1,5}+x_1^2 y_1^2 c_{1,6}+x_1 x_2 y_1^2 c_{1,7}+x_2^2 y_1^2 c_{1,8}+x_1^2 y_1 y_2 c_{1,9}+x_1 x_2 y_1 y_2 c_{1,10}+x_2^2 y_1 y_2 c_{1,11}+x_1^2 y_2^2 c_{1,12}+x_1 x_2 y_2^2 c_{1,13}+x_2^2 y_2^2 c_{1,14}\right]\right],\right.$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ \left(-1+\frac{1}{T}\right) x_2\,(y_1-y_2),\ \epsilon\text{Series}\left[0,\ d_{1,1}+x_1 y_1 d_{1,2}+x_2 y_1 d_{1,3}+x_1 y_2 d_{1,4}+x_2 y_2 d_{1,5}+x_1^2 y_1^2 d_{1,6}+x_1 x_2 y_1^2 d_{1,7}+x_2^2 y_1^2 d_{1,8}+x_1^2 y_1 y_2 d_{1,9}+x_1 x_2 y_1 y_2 d_{1,10}+x_2^2 y_1 y_2 d_{1,11}+x_1^2 y_2^2 d_{1,12}+x_1 x_2 y_2^2 d_{1,13}+x_2^2 y_2^2 d_{1,14}\right]\right],$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[\sqrt{T},\ 0,\ \epsilon\text{Series}\left[0,\ e_{1,1}+x_1 y_1 e_{1,2}+x_1^2 y_1^2 e_{1,3}\right]\right],$$

$$\left.\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{\sqrt{T}},\ 0,\ \epsilon\text{Series}\left[0,\ f_{1,1}+x_1 y_1 f_{1,2}+x_1^2 y_1^2 f_{1,3}\right]\right]\right\}$$

The Main Program

Variables and their duals:

In[∘]:=
```
{y*, x*, η*, ξ*} = {η, ξ, y, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_i_)* := (u*)_i;
```

𝔼 operations:

In[∘]:=
```
𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
𝔼 /: 𝔼[ω1_, Q1_, P1_] × 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] × 𝔼[ℰ2s]);
𝔼_{dr_}[ℰs___]_{$k_} := 𝔼_{dr} @@ 𝔼[ℰs]_{$k};
```

In[∘]:=
```
𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
  lvs = Flatten@Table[{x_{$@i}, y_{$@i}}, {i, is}];
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{lvs⋃lvs*}[lvs*.lvs, Times[
      𝔼[ℰ1s] /. Table[(v : x | y)_i → v_{$@i}, {i, is}],
      𝔼[ℰ2s] /. Table[(v : ξ | η)_i → v_{$@i}, {i, is}]
     ]])
 ]
```

$$[F:\mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad \text{and} \quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

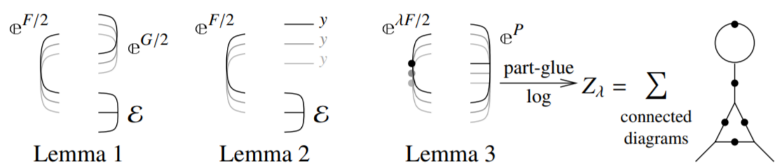
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\mathbb{e}^{\sum_{i\in B} y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Lemma 1 Lemma 2 Lemma 3

In[∘]:=
```
Zip_{vs_}[ℱ_, ℰ_] := ⟨ℱ, ℰ⟩ // Zip1_{vs} // Zip2_{vs} // Zip3_{vs}
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B$$

In[∘]:=
```
Zip1_{} = Identity;
Zip1_{vs_}@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PP_{Zip1}@Module[{I, F, G, u, v},
   I = IdentityMatrix@Length@vs;
   F = Table[∂_{u,v}ℱ, {u, vs*}, {v, vs*}];
   G = Table[∂_{u,v}Q, {u, vs}, {v, vs}];
   CF /@ ⟨vs*.F.Inverse[I - G.F].vs* / 2,
     𝔼[PowerExpand@Factor[ω Det[I - G.F]^{-1/2}], Q - vs.G.vs / 2, P]⟩
  ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}\,\mathbb{e}^{\sum_{i\in B} y_i z_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+F y_B}\right\rangle_B.$

In[∘]:=
```
Zip2{} = Identity;
Zip2vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PPZip2@Module[{F, Y, u, v},
   F = Table[∂u,v ℱ, {u, vs*}, {v, vs*}];
   Y = Table[∂v Q, {v, vs}];
   CF /@ ⟨ℱ, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
  ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

In[∘]:=
```
Zip3vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PPZip3@Module[{Z, u, v, m, j},
   Z[0] = P;
   For[m = 0, m < 2 $k, ++m,
    Z[m + 1] = CF[ 1/(2 (m + 1))
       Sum[∂u*,v* ℱ (∂u,v Z[m] + Sum[(∂u Z[j]) × (∂v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
   ];
   𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k}] /. Table[v → 0, {v, vs}]]]
  ]
```

Solving for R, CC, \$k = 1

In[∘]:=
```
$k = 1;
{R1,2, CC1}
unknowns = Cases[{R1,2, R̄1,2, CC1, C̄C̄1}, (c | d | e | f)$k,_, ∞] // Union
```

Out[∘]= $\{\mathbb{E}_{\{\}\to\{1,2\}}[1,\ (-1+T)\,x_2\,(y_1-y_2),$
$\epsilon\text{Series}[0,\ c_{1,1}+x_1 y_1 c_{1,2}+x_2 y_1 c_{1,3}+x_1 y_2 c_{1,4}+x_2 y_2 c_{1,5}+x_1^2 y_1^2 c_{1,6}+x_1 x_2 y_1^2 c_{1,7}+x_2^2 y_1^2 c_{1,8}+$
$x_1^2 y_1 y_2 c_{1,9}+x_1 x_2 y_1 y_2 c_{1,10}+x_2^2 y_1 y_2 c_{1,11}+x_1^2 y_2^2 c_{1,12}+x_1 x_2 y_2^2 c_{1,13}+x_2^2 y_2^2 c_{1,14}]],$
$\mathbb{E}_{\{\}\to\{1\}}[\sqrt{T},\ 0,\ \epsilon\text{Series}[0,\ e_{1,1}+x_1 y_1 e_{1,2}+x_1^2 y_1^2 e_{1,3}]]\}$

Out[∘]= $\{c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}, c_{1,8}, c_{1,9}, c_{1,10}, c_{1,11}, c_{1,12}, c_{1,13}, c_{1,14}, d_{1,1}, d_{1,2}, d_{1,3},$
$d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, e_{1,1}, e_{1,2}, e_{1,3}, f_{1,1}, f_{1,2}, f_{1,3}\}$

In[∘]:=
```
Short[errors = {(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]] -
    (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]],
   (R1,2 R̄3,4 // m1,3→1 // m2,4→2)[[3, -1]],
   (CC1 C̄C̄2 // m1,2→1 )[[3, -1]],
   (CC3 R1,2 // m2,3→2 // m2,1→1)[[3, -1]] - (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)[[3, -1]]},
  10]
```

Out[∘]//Short= $\{-x_3 y_1 c_{1,3} - x_2 y_1 (c_{1,2} - T c_{1,2} + c_{1,3}) + x_1 y_2 c_{1,4} + x_1 y_3 c_{1,4} - T x_1 y_3 c_{1,4} + \ll 100 \gg + x_3^2 y_1 y_3 (T c_{1,11} + 2 T c_{1,14} - 2 T^2 c_{1,14}) - x_3^2 y_2 y_3 (T^2 c_{1,11} + 2 T c_{1,14} - 2 T^2 c_{1,14}) - x_3^2 y_2^2 (T^2 c_{1,8} + c_{1,14} - 2 T c_{1,14} + T^2 c_{1,14}) + x_3^2 y_1^2 (T^2 c_{1,6} - 2 T^3 c_{1,6} + T^4 c_{1,6} + T c_{1,7} - 2 T^2 c_{1,7} + T^3 c_{1,7} + 2 c_{1,8} - 4 T c_{1,8} + 3 T^2 c_{1,8} + c_{1,11} - 2 T c_{1,11} + T^2 c_{1,11} + c_{1,14} - 2 T c_{1,14} + T^2 c_{1,14}) + x_3^2 y_2^2 (T^2 c_{1,8} + T^2 c_{1,12} - 2 T^3 c_{1,12} + T^4 c_{1,12} + T c_{1,13} - 2 T^2 c_{1,13} + T^3 c_{1,13} + c_{1,14} - 2 T c_{1,14} + T^2 c_{1,14}), c_{\ll 1 \gg} + \ll 14 \gg, \ll 1 \gg, -c_{1,1} + \ll 12 \gg\}$

In[∘]:=
```
eqns = Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, y1, y2, y3}][[;; , 2]] & /@ errors)]
```

Out[∘]= $\{0 == c_{1,4} - T c_{1,4},\ 0 == -c_{1,4} + T c_{1,4},\ 0 == T c_{1,4} - T^2 c_{1,4},\ 0 == -c_{1,4} + 2 T c_{1,4} - T^2 c_{1,4},$
$0 == -T c_{1,4} + T^2 c_{1,4},\ 0 == T c_{1,2} - T^2 c_{1,2} + c_{1,3} - T c_{1,3} + c_{1,5} - T c_{1,5},$
$0 == -2 c_{1,6} + 2 T c_{1,6},\ 0 == 2 T c_{1,6} - 2 T^2 c_{1,6},\ 0 == c_{1,9} - T c_{1,9},$
$0 == -c_{1,9} + T c_{1,9},\ 0 == 2 T c_{1,9} - 2 T^2 c_{1,9},\ 0 == -2 c_{1,9} + 4 T c_{1,9} - 2 T^2 c_{1,9},$
$0 == -2 T c_{1,9} + 2 T^2 c_{1,9},\ 0 == 2 T c_{1,6} - 2 T^2 c_{1,6} - c_{1,9} + 4 T c_{1,9} - 4 T^2 c_{1,9} + T^3 c_{1,9},$
$0 == 2 T c_{1,8} - 2 T^2 c_{1,8} + T^2 c_{1,9} - 2 T^3 c_{1,9} + T^4 c_{1,9} + T c_{1,10} - 2 T^2 c_{1,10} + T^3 c_{1,10},$
$0 == 2 T c_{1,7} - 2 T^2 c_{1,7} - c_{1,10} + 4 T c_{1,10} - 3 T^2 c_{1,10} + 2 c_{1,11} - 2 T c_{1,11},$
$0 == T^2 c_{1,9} - T^3 c_{1,9} + 2 T c_{1,12} - 2 T^2 c_{1,12},\ 0 == c_{1,12} - T^2 c_{1,12},\ 0 == -c_{1,12} + 2 T c_{1,12} - T^2 c_{1,12},$
$0 == c_{1,9} - 2 T c_{1,9} + T^2 c_{1,9} + c_{1,12} - 2 T c_{1,12} + T^2 c_{1,12},\ 0 == -2 T c_{1,12} + 2 T^2 c_{1,12},$
$0 == -4 T c_{1,12} + 8 T^2 c_{1,12} - 4 T^3 c_{1,12},\ 0 == -2 c_{1,12} + 6 T c_{1,12} - 6 T^2 c_{1,12} + 2 T^3 c_{1,12},$
$0 == -2 T^2 c_{1,12} + 2 T^3 c_{1,12},\ 0 == -T^2 c_{1,12} + 2 T^3 c_{1,12} - T^4 c_{1,12},$
$0 == -c_{1,12} + 4 T c_{1,12} - 6 T^2 c_{1,12} + 4 T^3 c_{1,12} - T^4 c_{1,12},\ 0 == -2 T c_{1,12} + 6 T^2 c_{1,12} - 6 T^3 c_{1,12} + 2 T^4 c_{1,12},$
$0 == 2 T c_{1,13} - 2 T^2 c_{1,13},\ 0 == T c_{1,13} - T^2 c_{1,13},\ 0 == 2 T c_{1,12} - 2 T^2 c_{1,12} + T c_{1,13} - T^2 c_{1,13},$
$0 == 2 c_{1,8} - 2 T c_{1,8} + c_{1,10} - 2 T c_{1,10} + T^2 c_{1,10} + c_{1,13} - 2 T c_{1,13} + T^2 c_{1,13},$
$0 == -2 T c_{1,13} + 2 T^2 c_{1,13},\ 0 == -2 T c_{1,13} + 4 T^2 c_{1,13} - 2 T^3 c_{1,13},$
$0 == T^2 c_{1,12} - 2 T^3 c_{1,12} + T^4 c_{1,12} + T c_{1,13} - 2 T^2 c_{1,13} + T^3 c_{1,13},\ 0 == -T^2 c_{1,13} + T^3 c_{1,13},$
$0 == -c_{1,13} + 4 T c_{1,13} - 4 T^2 c_{1,13} + T^3 c_{1,13} + 2 c_{1,14} - 2 T c_{1,14},\ 0 == 2 T c_{1,14} - 2 T^2 c_{1,14},$
$0 == T^2 c_{1,6} - 2 T^3 c_{1,6} + T^4 c_{1,6} + T c_{1,7} - 2 T^2 c_{1,7} + T^3 c_{1,7} + c_{1,8} - 4 T c_{1,8} + 3 T^2 c_{1,8} + c_{1,11} - 2 T c_{1,11} + T^2 c_{1,11} + c_{1,14} - 2 T c_{1,14} + T^2 c_{1,14},\ 0 == -2 T c_{1,14} + 2 T^2 c_{1,14},\ 0 == c_{1,1} + d_{1,1},$
$0 == c_{1,2} + d_{1,2} + d_{1,4} - T d_{1,4},\ 0 == c_{1,4} + T d_{1,4},\ 0 == c_{1,2} - \frac{c_{1,2}}{T} + \frac{c_{1,3}}{T} + d_{1,3} + d_{1,5} - T d_{1,5},$
$0 == c_{1,4} - \frac{c_{1,4}}{T} + \frac{c_{1,5}}{T} + T d_{1,5},\ 0 == c_{1,9} + T d_{1,9} + 2 T d_{1,12} - 2 T^2 d_{1,12},$
$0 == c_{1,12} + T^2 d_{1,12},\ 0 == c_{1,6} + d_{1,6} + d_{1,9} - T d_{1,9} + d_{1,12} - 2 T d_{1,12} + T^2 d_{1,12},$
$0 == 2 c_{1,9} - \frac{2 c_{1,9}}{T} + \frac{c_{1,10}}{T} + T d_{1,10} + 2 T d_{1,13} - 2 T^2 d_{1,13},\ 0 == 2 c_{1,12} - \frac{2 c_{1,12}}{T} + \frac{c_{1,13}}{T} + T^2 d_{1,13},$

$$0 == 2\,c_{1,6} - \frac{2\,c_{1,6}}{T} + \frac{c_{1,7}}{T} + d_{1,7} + d_{1,10} - T\,d_{1,10} + d_{1,13} - 2\,T\,d_{1,13} + T^2\,d_{1,13},$$

$$0 == c_{1,9} + \frac{c_{1,9}}{T^2} - \frac{2\,c_{1,9}}{T} - \frac{c_{1,10}}{T^2} + \frac{c_{1,10}}{T} + \frac{c_{1,11}}{T^2} + T\,d_{1,11} + 2\,T\,d_{1,14} - 2\,T^2\,d_{1,14},$$

$$0 == c_{1,12} + \frac{c_{1,12}}{T^2} - \frac{2\,c_{1,12}}{T} - \frac{c_{1,13}}{T^2} + \frac{c_{1,13}}{T} + \frac{c_{1,14}}{T^2} + T^2\,d_{1,14},$$

$$0 == c_{1,6} + \frac{c_{1,6}}{T^2} - \frac{2\,c_{1,6}}{T} - \frac{c_{1,7}}{T^2} + \frac{c_{1,7}}{T} + \frac{c_{1,8}}{T^2} + d_{1,8} + d_{1,11} - T\,d_{1,11} + d_{1,14} - 2\,T\,d_{1,14} + T^2\,d_{1,14},$$

$$0 == -\frac{c_{1,3}}{T} + c_{1,4} + \frac{2\,c_{1,8}}{T^2} - 2\,c_{1,12} + e_{1,1} + e_{1,2} - \frac{e_{1,2}}{T} + 2\,e_{1,3} + \frac{2\,e_{1,3}}{T^2} - \frac{4\,e_{1,3}}{T} - f_{1,1},$$

$$0 == e_{1,1} + f_{1,1},\ 0 == e_{1,2} + f_{1,2},\ 0 == c_{1,2} - T\,c_{1,2} - c_{1,3} + \frac{c_{1,3}}{T} + c_{1,4} - T\,c_{1,4} - c_{1,5} + \frac{c_{1,5}}{T} - \frac{2\,c_{1,7}}{T} - \frac{4\,c_{1,8}}{T^2} + 2\,T\,c_{1,9} + c_{1,10} - \frac{c_{1,10}}{T} - \frac{2\,c_{1,11}}{T^2} + 4\,T\,c_{1,12} + 2\,c_{1,13} + \frac{e_{1,2}}{T} - \frac{4\,e_{1,3}}{T^2} + \frac{4\,e_{1,3}}{T} - T\,f_{1,2},$$

$$0 == e_{1,3} + f_{1,3},\ 0 == c_{1,6} - T^2\,c_{1,6} + \frac{c_{1,7}}{T} - T\,c_{1,7} - c_{1,8} + \frac{c_{1,8}}{T^2} + c_{1,9} - T^2\,c_{1,9} + \frac{c_{1,10}}{T} - T\,c_{1,10} - c_{1,11} + \frac{c_{1,11}}{T^2} + c_{1,12} - T^2\,c_{1,12} + \frac{c_{1,13}}{T} - T\,c_{1,13} - c_{1,14} + \frac{c_{1,14}}{T^2} + \frac{e_{1,3}}{T^2} - T^2\,f_{1,3}\Big\}$$

In[∘]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
$$\Big\{\Big\{c_{1,4} \to 0,\ c_{1,5} \to -T\,c_{1,2} - c_{1,3},\ c_{1,6} \to 0,\ c_{1,8} \to -\frac{1}{2} \times (1 - T)\,c_{1,10},\ c_{1,9} \to 0,\ c_{1,11} \to -T\,c_{1,7} - \frac{1}{2} \times (-1 + 3\,T)\,c_{1,10},\ c_{1,12} \to 0,\ c_{1,13} \to 0,\ c_{1,14} \to 0,\ d_{1,1} \to -c_{1,1},\ d_{1,2} \to -c_{1,2},\ d_{1,3} \to -\frac{c_{1,3}}{T^2},\ d_{1,4} \to 0,\ d_{1,5} \to \frac{c_{1,2}}{T} + \frac{c_{1,3}}{T^2},\ d_{1,6} \to 0,\ d_{1,7} \to -\frac{c_{1,7}}{T} - \frac{(-1 + T)\,c_{1,10}}{T^2},\ d_{1,8} \to -\frac{(1 - T)\,c_{1,10}}{2\,T^3},\ d_{1,9} \to 0,\ d_{1,10} \to -\frac{c_{1,10}}{T^2},\ d_{1,11} \to \frac{c_{1,7}}{T^2} - \frac{(-1 - T)\,c_{1,10}}{2\,T^3},\ d_{1,12} \to 0,\ d_{1,13} \to 0,\ d_{1,14} \to 0,\ e_{1,1} \to \frac{c_{1,3}}{2\,T},\ e_{1,2} \to -\frac{c_{1,10}}{T},\ e_{1,3} \to 0,\ f_{1,1} \to -\frac{c_{1,3}}{2\,T},\ f_{1,2} \to \frac{c_{1,10}}{T},\ f_{1,3} \to 0\Big\}\Big\}$$

In[∘]:=
```
sol /. (a_ → b_) ⧴ (a = b)
```

Out[∘]=
$$\Big\{0,\ -T\,c_{1,2} - c_{1,3},\ 0,\ -\frac{1}{2} \times (1 - T)\,c_{1,10},\ 0,\ -T\,c_{1,7} - \frac{1}{2} \times (-1 + 3\,T)\,c_{1,10},\ 0,\ 0,\ 0,\ -c_{1,1},\ -c_{1,2},\ -\frac{c_{1,3}}{T^2},\ 0,\ \frac{c_{1,2}}{T} + \frac{c_{1,3}}{T^2},\ 0,\ -\frac{c_{1,7}}{T} - \frac{(-1 + T)\,c_{1,10}}{T^2},\ -\frac{(1 - T)\,c_{1,10}}{2\,T^3},\ 0,\ -\frac{c_{1,10}}{T^2},\ \frac{c_{1,7}}{T^2} - \frac{(-1 - T)\,c_{1,10}}{2\,T^3},\ 0,\ 0,\ 0,\ \frac{c_{1,3}}{2\,T},\ -\frac{c_{1,10}}{T},\ 0,\ -\frac{c_{1,3}}{2\,T},\ \frac{c_{1,10}}{T},\ 0\Big\}$$

In[∘]:= $\{R_{1,2}, \overline{R}_{1,2}, CC_1, \overline{CC}_1\}$

Out[∘]= $\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ (-1+T)\ x_2\ (y_1-y_2),$
$\epsilon\text{Series}\Big[0,\ c_{1,1}+x_1 y_1 c_{1,2}+x_2 y_2 (-T c_{1,2}-c_{1,3})+x_2 y_1 c_{1,3}+x_1 x_2 y_1^2 c_{1,7}-\frac{1}{2}\times(1-T)\ x_2^2 y_1^2 c_{1,10}+$
$x_1 x_2 y_1 y_2 c_{1,10}+x_2^2 y_1 y_2 \left(-T c_{1,7}-\frac{1}{2}\times(-1+3T)\ c_{1,10}\right)\Big]\Big],\ \mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ \left(-1+\frac{1}{T}\right) x_2\ (y_1-y_2),$
$\epsilon\text{Series}\Big[0,\ -c_{1,1}-x_1 y_1 c_{1,2}-\frac{x_2 y_1 c_{1,3}}{T^2}+x_2 y_2\left(\frac{c_{1,2}}{T}+\frac{c_{1,3}}{T^2}\right)-\frac{(1-T)\ x_2^2 y_1^2 c_{1,10}}{2T^3}-$
$\frac{x_1 x_2 y_1 y_2 c_{1,10}}{T^2}+x_2^2 y_1 y_2\left(\frac{c_{1,7}}{T^2}-\frac{(-1-T)\ c_{1,10}}{2T^3}\right)+x_1 x_2 y_1^2\left(-\frac{c_{1,7}}{T}-\frac{(-1+T)\ c_{1,10}}{T^2}\right)\Big]\Big],$
$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\ 0,\ \epsilon\text{Series}\Big[0,\ \frac{c_{1,3}}{2T}-\frac{x_1 y_1 c_{1,10}}{T}\Big]\Big],\ \mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\ 0,\ \epsilon\text{Series}\Big[0,\ -\frac{c_{1,3}}{2T}+\frac{x_1 y_1 c_{1,10}}{T}\Big]\Big]\Big\}$

In[∘]:= $c_{1,1}=c_{1,2}=c_{1,3}=c_{1,7}=0;\ c_{1,10}=1;$
$\{R_{1,2}, \overline{R}_{1,2}, CC_1, \overline{CC}_1\}$

Out[∘]= $\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ (-1+T)\ x_2\ (y_1-y_2),$
$\epsilon\text{Series}\Big[0,\ \frac{1}{2}\times(-1+T)\ x_2^2 y_1^2+x_1 x_2 y_1 y_2+\frac{1}{2}\times(1-3T)\ x_2^2 y_1 y_2\Big]\Big],$
$\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ \left(-1+\frac{1}{T}\right) x_2\ (y_1-y_2),$
$\epsilon\text{Series}\Big[0,\ -\frac{(-1+T)\ x_1 x_2 y_1^2}{T^2}-\frac{(1-T)\ x_2^2 y_1^2}{2T^3}-\frac{x_1 x_2 y_1 y_2}{T^2}-\frac{(-1-T)\ x_2^2 y_1 y_2}{2T^3}\Big]\Big],$
$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\ 0,\ \epsilon\text{Series}\Big[0,\ -\frac{x_1 y_1}{T}\Big]\Big],\ \mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\ 0,\ \epsilon\text{Series}\Big[0,\ \frac{x_1 y_1}{T}\Big]\Big]\Big\}$

In[∘]:= $\{(R_{1,2} R_{4,3} R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}) \equiv (R_{2,3} R_{4,5} R_{1,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}),$
$(R_{1,2}\overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}) \equiv \mathbb{E}_{\{\}\to\{1,2\}}[1, 0, \epsilon\text{Series}[0]],$
$(CC_1\overline{CC}_2\ //\ m_{1,2\to1}) \equiv \mathbb{E}_{\{\}\to\{1\}}[1, 0, \epsilon\text{Series}[0]],$
$(CC_3 R_{1,2}\ //\ m_{2,3\to2}\ //\ m_{2,1\to1}) \equiv (\overline{CC}_3 R_{1,2}\ //\ m_{1,3\to1}\ //\ m_{1,2\to1})\}$

Out[∘]= {True, True, True, True}

Solving for R, CC, \$k = 2

In[∘]:=
```
$k = 2;
Short[#, 10] & [
 {(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3),
  (R1,2 R̄3,4 // m1,3→1 // m2,4→2) ≡ 𝔼{}→{1,2}[1, 0, ϵSeries[0]],
  (CC1 C̄C̄2 // m1,2→1 ) ≡ 𝔼{}→{1}[1, 0, ϵSeries[0]],
  (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)}]
```

Out[∘]//Short=

$$\{(-1+T)\,x_1 x_2 x_3 y_1^2 y_3 - 3T x_1 x_2 x_3 y_1 y_2 y_3 + (-2T+4T^2)\,x_1 x_3^2 y_1 y_2 y_3 + \ll 112\gg + x_3^3 y_2^3 (T^3 c_{2,18} + T^3 c_{2,27} - 3T^4 c_{2,27} + 3T^5 c_{2,27} - T^6 c_{2,27} + T^2 c_{2,28} - 3T^3 c_{2,28} + 3T^4 c_{2,28} - T^5 c_{2,28} + T c_{2,29} - 3T^2 c_{2,29} + 3T^3 c_{2,29} - T^4 c_{2,29} + c_{2,30} - 3T c_{2,30} + 3T^2 c_{2,30} - T^3 c_{2,30}) + x_3^3 y_1^2 y_3 (T^2 - 4T^3 + 3T^4 + T c_{2,22} - 2T^2 c_{2,22} + 2T^3 c_{2,22} + 2T c_{2,26} - 4T^2 c_{2,26} + 2T^3 c_{2,26} + 3T c_{2,30} - 6T^2 c_{2,30} + 3T^3 c_{2,30}) == 3 c_{2,1} + 2 x_1 y_1 c_{2,2} + \ll 143\gg + x_3^3 y_2^2 y_3 (T^3 c_{2,22} + 3T c_{2,30} - 6T^2 c_{2,30} + 3T^3 c_{2,30}),\ \ll 2\gg,\ \ll 1\gg == \ll 1\gg\}$$

In[∘]:=
```
unknowns = Cases[{R1,2, R̄1,2, CC1, C̄C̄1}, (c | d | e | f)$k,_, ∞] // Union
```

Out[∘]=

$$\{c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}, c_{2,5}, c_{2,6}, c_{2,7}, c_{2,8}, c_{2,9}, c_{2,10}, c_{2,11}, c_{2,12}, c_{2,13}, c_{2,14}, c_{2,15}, c_{2,16}, c_{2,17}, c_{2,18}, c_{2,19}, c_{2,20}, c_{2,21}, c_{2,22}, c_{2,23}, c_{2,24}, c_{2,25}, c_{2,26}, c_{2,27}, c_{2,28}, c_{2,29}, c_{2,30}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{2,11}, d_{2,12}, d_{2,13}, d_{2,14}, d_{2,15}, d_{2,16}, d_{2,17}, d_{2,18}, d_{2,19}, d_{2,20}, d_{2,21}, d_{2,22}, d_{2,23}, d_{2,24}, d_{2,25}, d_{2,26}, d_{2,27}, d_{2,28}, d_{2,29}, d_{2,30}, e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}\}$$

In[∘]:=
```
Short[errors = CF@{(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3)〚3, -1〛 -
     (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)〚3, -1〛,
    (R1,2 R̄3,4 // m1,3→1 // m2,4→2)〚3, -1〛,
    (CC1 C̄C̄2 // m1,2→1 )〚3, -1〛,
    (CC3 R1,2 // m2,3→2 // m2,1→1)〚3, -1〛 - (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)〚3, -1〛},
 10]
```

Out[∘]//Short=

$$\Big\{x_2 y_1 (c_{2,4} - T c_{2,4}) + x_1 y_3 (c_{2,4} - T c_{2,4}) + x_1 y_2 (-c_{2,4} + T c_{2,4}) + \ll 131\gg + x_3^3 y_1^2 y_3 (T^2 - 4T^3 + 3T^4 - 2T^2 c_{2,22} + 2T^3 c_{2,22} + 2T c_{2,26} - 4T^2 c_{2,26} + 2T^3 c_{2,26} + 3T c_{2,30} - 6T^2 c_{2,30} + 3T^3 c_{2,30}) + x_3^3 y_2 y_3^2 (-3T^2 c_{2,30} + 3T^3 c_{2,30}),\ c_{2,1} + d_{\ll 1\gg} + \ll 28\gg + \frac{\ll 1\gg}{2 \ll 1\gg},\ \ll 1\gg,$$

$$\frac{1 - 4T + 3T^2 - T^3 c_{2,3} + T^4 c_{2,4} + \ll 13\gg + 2T^4 e_{2,3} - 6T e_{2,4} + 18T^2 e_{2,4} - 18T^3 e_{2,4} + 6T^4 e_{2,4} - T^4 f_{2,1}}{T^4} + \frac{x_1 \ll 1\gg (-2 + 2T + \ll 46\gg + 18T^3 e_{2\,\ll 1\gg\,\ll 1\gg} - T^5 f_{2,2})}{T^4} + \frac{x_1^2 y_1^2 (\ll 1\gg)}{2T^4} + \frac{x_1^3 y_1^3 (-1 + T + T^3 - T^4 + \ll 45\gg + 2 c_{2,30} - 2T^3 c_{2,30} + 2 e_{2,4} - 2T^6 f_{2,4})}{2T^3}\Big\}$$

In[∘]:=
```
Short[#, 10] &[eqns =
  Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, y1, y2, y3}][[;; , 2]] & /@ errors)]]
```

Out[∘]//Short=

$$\Big\{0 == c_{2,4} - T\, c_{2,4},\ 0 == -c_{2,4} + T\, c_{2,4},\ 0 == T\, c_{2,4} - T^2 c_{2,4},\ \ll 168 \gg,\ 0 == e_{2,4} + f_{2,4},\ 0 == \frac{1}{2} - \frac{1}{2 T^3} + \frac{1}{2 T^2} - \frac{T}{2} + c_{2,15} - T^3 c_{2,15} + \frac{c_{2,16}}{T} - T^2 c_{2,16} + \frac{c_{2,17}}{T^2} - T\, c_{2,17} - c_{2,18} + \frac{c_{2,18}}{T^3} + c_{2,19} - T^3 c_{2,19} + \frac{c_{2,20}}{T} - T^2 c_{2,20} + \frac{c_{2,21}}{T^2} - T\, c_{2,21} - c_{2,22} + \frac{c_{2,22}}{T^3} + c_{2,23} - T^3 c_{2,23} + \frac{c_{2,24}}{T} - T^2 c_{2,24} + \frac{c_{2,25}}{T^2} - T\, c_{2,25} - c_{2,26} + \frac{c_{2,26}}{T^3} + c_{2,27} - T^3 c_{2,27} + \frac{c_{2,28}}{T} - T^2 c_{2,28} + \frac{c_{2,29}}{T^2} - T\, c_{2,29} - c_{2,30} + \frac{c_{2,30}}{T^3} + \frac{e_{2,4}}{T^3} - T^3 f_{2,4}\Big\}$$

In[∘]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=

$$\Big\{\Big\{c_{2,4} \to 0,\ c_{2,5} \to -T\, c_{2,2} - c_{2,3},\ c_{2,6} \to 0,\ c_{2,8} \to -\frac{1}{2} \times (1 - T)\, c_{2,10},\ c_{2,9} \to 0,\ c_{2,11} \to -\frac{1}{2} - T\, c_{2,7} - \frac{1}{2} \times (-1 + 3 T)\, c_{2,10},\ c_{2,12} \to 0,\ c_{2,13} \to 0,\ c_{2,14} \to 0,\ c_{2,15} \to 0,\ c_{2,17} \to -((-1 + T)\, c_{2,16}),\ c_{2,18} \to -\frac{-1 + 4 T - 3 T^2}{6 T},\ c_{2,19} \to 0,\ c_{2,20} \to -\frac{1}{2 T},\ c_{2,21} \to -\frac{1 - 3 T}{2 T},\ c_{2,22} \to -\frac{1 - 11 T + 16 T^2}{6 T} - \left(T - T^2\right) c_{2,16},\ c_{2,23} \to 0,\ c_{2,24} \to 0,\ c_{2,25} \to -\frac{1}{2},\ c_{2,26} \to \frac{1}{6} \times (-1 + 7 T) - T^2 c_{2,16},\ c_{2,27} \to 0,\ c_{2,28} \to 0,\ c_{2,29} \to 0,\ c_{2,30} \to 0,\ d_{2,1} \to -c_{2,1},\ d_{2,2} \to -c_{2,2},\ d_{2,3} \to -\frac{c_{2,3}}{T^2},\ d_{2,4} \to 0,\ d_{2,5} \to \frac{c_{2,2}}{T} + \frac{c_{2,3}}{T^2},\ d_{2,6} \to 0,\ d_{2,7} \to -\frac{1 - T}{T^3} - \frac{c_{2,7}}{T} - \frac{(-1 + T)\, c_{2,10}}{T^2},\ d_{2,8} \to -\frac{-1 + T}{2 T^4} - \frac{(1 - T)\, c_{2,10}}{2 T^3},\ d_{2,9} \to 0,\ d_{2,10} \to \frac{1}{T^3} - \frac{c_{2,10}}{T^2},\ d_{2,11} \to -\frac{1}{2 T^4} + \frac{c_{2,7}}{T^2} - \frac{(-1 - T)\, c_{2,10}}{2 T^3},\ d_{2,12} \to 0,\ d_{2,13} \to 0,\ d_{2,14} \to 0,\ d_{2,15} \to 0,\ d_{2,16} \to -\frac{-1 + T}{2 T^3} - \frac{c_{2,16}}{T},\ d_{2,17} \to -\frac{3 - 4 T + T^2}{2 T^4} - \frac{(-1 + T)\, c_{2,16}}{T^2},\ d_{2,18} \to -\frac{-3 + 4 T - T^2}{6 T^5},\ d_{2,19} \to 0,\ d_{2,20} \to -\frac{1}{2 T^3},\ d_{2,21} \to \frac{2}{T^4},\ d_{2,22} \to -\frac{4 + T + T^2}{6 T^5} - \frac{(1 - T)\, c_{2,16}}{T^3},\ d_{2,23} \to 0,\ d_{2,24} \to 0,\ d_{2,25} \to -\frac{1}{2 T^4},\ d_{2,26} \to -\frac{-1 + T}{6 T^5} + \frac{c_{2,16}}{T^3},\ d_{2,27} \to 0,\ d_{2,28} \to 0,\ d_{2,29} \to 0,\ d_{2,30} \to 0,\ e_{2,1} \to \frac{c_{2,3}}{2 T},\ e_{2,2} \to -\frac{c_{2,10}}{T},\ e_{2,3} \to 0,\ e_{2,4} \to 0,\ f_{2,1} \to -\frac{c_{2,3}}{2 T},\ f_{2,2} \to -\frac{1}{T^2} + \frac{c_{2,10}}{T},\ f_{2,3} \to 0,\ f_{2,4} \to 0\Big\}\Big\}$$

In[∘]:= **sol /. (a_ → b_) ⧴ (a = b)**

Out[∘]=
$$\left\{0,\ -T\,c_{2,2}-c_{2,3},\ 0,\ -\frac{1}{2}\times(1-T)\,c_{2,10},\ 0,\ -\frac{1}{2}-T\,c_{2,7}-\frac{1}{2}\times(-1+3T)\,c_{2,10},\ 0,\ 0,\ 0,\ 0,\ -((-1+T)\,c_{2,16}),\ -\frac{-1+4T-3T^2}{6T},\ 0,\ -\frac{1}{2T},\ -\frac{1-3T}{2T},\ -\frac{1-11T+16T^2}{6T}-\left(T-T^2\right)c_{2,16},\ 0,\ 0,\ -\frac{1}{2},\ \frac{1}{6}\times(-1+7T)-T^2\,c_{2,16},\ 0,\ 0,\ 0,\ 0,\ -c_{2,1},\ -c_{2,2},\ -\frac{c_{2,3}}{T^2},\ 0,\ \frac{c_{2,2}}{T}+\frac{c_{2,3}}{T^2},\ 0,\ -\frac{1-T}{T^3}-\frac{c_{2,7}}{T}-\frac{(-1+T)\,c_{2,10}}{T^2},\ -\frac{-1+T}{2T^4}-\frac{(1-T)\,c_{2,10}}{2T^3},\ 0,\ \frac{1}{T^3}-\frac{c_{2,10}}{T^2},\ -\frac{1}{2T^4}+\frac{c_{2,7}}{T^2}-\frac{(-1-T)\,c_{2,10}}{2T^3},\ 0,\ 0,\ 0,\ 0,\ -\frac{-1+T}{2T^3}-\frac{c_{2,16}}{T},\ -\frac{3-4T+T^2}{2T^4}-\frac{(-1+T)\,c_{2,16}}{T^2},\ -\frac{-3+4T-T^2}{6T^5},\ 0,\ -\frac{1}{2T^3},\ \frac{2}{T^4},\ -\frac{4+T+T^2}{6T^5}-\frac{(1-T)\,c_{2,16}}{T^3},\ 0,\ 0,\ -\frac{1}{2T^4},\ -\frac{-1+T}{6T^5}+\frac{c_{2,16}}{T^3},\ 0,\ 0,\ 0,\ 0,\ \frac{c_{2,3}}{2T},\ -\frac{c_{2,10}}{T},\ 0,\ 0,\ -\frac{c_{2,3}}{2T},\ -\frac{1}{T^2}+\frac{c_{2,10}}{T},\ 0,\ 0\right\}$$

In[∘]:= $c_{2,1}=c_{2,2}=c_{2,3}=c_{2,7}=c_{2,10}=c_{2,16}=0;$
$\{R_{1,2},\ \overline{R}_{1,2},\ CC_1,\ \overline{CC}_1\}$

Out[∘]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ (-1+T)\,x_2\,(y_1-y_2),\ \epsilon\text{Series}\Big[0,\ \frac{1}{2}\times(-1+T)\,x_2^2 y_1^2+x_1 x_2 y_1 y_2+\frac{1}{2}\times(1-3T)\,x_2^2 y_1 y_2,\ -\frac{\left(-1+4T-3T^2\right)x_2^3 y_1^3}{6T}-\frac{1}{2}x_2^2 y_1 y_2-\frac{x_1^2 x_2 y_1^2 y_2}{2T}-\frac{(1-3T)\,x_1 x_2^2 y_1^2 y_2}{2T}-\frac{\left(1-11T+16T^2\right)x_2^3 y_1^2 y_2}{6T}-\frac{1}{2}x_1 x_2^2 y_1 y_2^2+\frac{1}{6}\times(-1+7T)\,x_2^3 y_1 y_2^2\Big]\Big],\ \mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ \left(-1+\frac{1}{T}\right)x_2\,(y_1-y_2),$$
$$\epsilon\text{Series}\Big[0,\ -\frac{(-1+T)\,x_1 x_2 y_1^2}{T^2}-\frac{(1-T)\,x_2^2 y_1^2}{2T^3}-\frac{x_1 x_2 y_1 y_2}{T^2}-\frac{(-1-T)\,x_2^2 y_1 y_2}{2T^3},\ -\frac{(1-T)\,x_1 x_2 y_1^2}{T^3}-\frac{(-1+T)\,x_2^2 y_1^2}{2T^4}-\frac{(-1+T)\,x_1^2 x_2 y_1^3}{2T^3}-\frac{\left(3-4T+T^2\right)x_1 x_2^2 y_1^3}{2T^4}-\frac{\left(-3+4T-T^2\right)x_2^3 y_1^3}{6T^5}+\frac{x_1 x_2 y_1 y_2}{T^3}-\frac{x_2^2 y_1 y_2}{2T^4}-\frac{x_1^2 x_2 y_1^2 y_2}{2T^3}+\frac{2x_1 x_2^2 y_1^2 y_2}{T^4}-\frac{\left(4+T+T^2\right)x_2^3 y_1^2 y_2}{6T^5}-\frac{x_1 x_2^2 y_1 y_2^2}{2T^4}-\frac{(-1+T)\,x_2^3 y_1 y_2^2}{6T^5}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\ 0,\ \epsilon\text{Series}\Big[0,\ -\frac{x_1 y_1}{T},\ 0\Big]\Big],\ \mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\ 0,\ \epsilon\text{Series}\Big[0,\ \frac{x_1 y_1}{T},\ -\frac{x_1 y_1}{T^2}\Big]\Big]\Big\}$$

In[∘]:=
$$\{(R_{1,2}\,R_{4,3}\,R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3})\equiv(R_{2,3}\,R_{4,5}\,R_{1,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}),$$
$$(R_{1,2}\,\overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2})\equiv\mathbb{E}_{\{\}\to\{1,2\}}[1,\ 0,\ \epsilon\text{Series}[0]],$$
$$(CC_1\,\overline{CC}_2\ //\ m_{1,2\to1})\equiv\mathbb{E}_{\{\}\to\{1\}}[1,\ 0,\ \epsilon\text{Series}[0]],$$
$$(CC_3\,R_{1,2}\ //\ m_{2,3\to2}\ //\ m_{2,1\to1})\equiv(\overline{CC}_3\,R_{1,2}\ //\ m_{1,3\to1}\ //\ m_{1,2\to1})\}$$

Out[∘]= {True, True, True, True}

Solving for *R*, CC, $k = 3

In[◦]:=
```
$k = 3;
Short[#, 10] & [
 {(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3),
  (R1,2 R̄3,4 // m1,3→1 // m2,4→2) ≡ 𝔼{}→{1,2}[1, 0, ϵSeries[0]],
  (CC1 C̄C̄2 // m1,2→1 ) ≡ 𝔼{}→{1}[1, 0, ϵSeries[0]],
  (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)}]
```

Out[◦]= $Aborted

In[◦]:=
```
unknowns = Cases[{R1,2, R̄1,2, CC1, C̄C̄1}, (c | d | e | f)$k,_, ∞] // Union
```

Out[◦]= $\{c_{3,1}, c_{3,2}, c_{3,3}, c_{3,4}, c_{3,5}, c_{3,6}, c_{3,7}, c_{3,8}, c_{3,9}, c_{3,10}, c_{3,11}, c_{3,12}, c_{3,13}, c_{3,14}, c_{3,15}, c_{3,16}, c_{3,17}, c_{3,18}, c_{3,19}, c_{3,20}, c_{3,21}, c_{3,22}, c_{3,23}, c_{3,24}, c_{3,25}, c_{3,26}, c_{3,27}, c_{3,28}, c_{3,29}, c_{3,30}, c_{3,31}, c_{3,32}, c_{3,33}, c_{3,34}, c_{3,35}, c_{3,36}, c_{3,37}, c_{3,38}, c_{3,39}, c_{3,40}, c_{3,41}, c_{3,42}, c_{3,43}, c_{3,44}, c_{3,45}, c_{3,46}, c_{3,47}, c_{3,48}, c_{3,49}, c_{3,50}, c_{3,51}, c_{3,52}, c_{3,53}, c_{3,54}, c_{3,55}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, d_{3,5}, d_{3,6}, d_{3,7}, d_{3,8}, d_{3,9}, d_{3,10}, d_{3,11}, d_{3,12}, d_{3,13}, d_{3,14}, d_{3,15}, d_{3,16}, d_{3,17}, d_{3,18}, d_{3,19}, d_{3,20}, d_{3,21}, d_{3,22}, d_{3,23}, d_{3,24}, d_{3,25}, d_{3,26}, d_{3,27}, d_{3,28}, d_{3,29}, d_{3,30}, d_{3,31}, d_{3,32}, d_{3,33}, d_{3,34}, d_{3,35}, d_{3,36}, d_{3,37}, d_{3,38}, d_{3,39}, d_{3,40}, d_{3,41}, d_{3,42}, d_{3,43}, d_{3,44}, d_{3,45}, d_{3,46}, d_{3,47}, d_{3,48}, d_{3,49}, d_{3,50}, d_{3,51}, d_{3,52}, d_{3,53}, d_{3,54}, d_{3,55}, e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, f_{3,5}\}$

In[◦]:=
```
Short[errors = CF@{(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3)⟦3, -1⟧ -
     (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)⟦3, -1⟧,
    (R1,2 R̄3,4 // m1,3→1 // m2,4→2)⟦3, -1⟧,
    (CC1 C̄C̄2 // m1,2→1 )⟦3, -1⟧,
    (CC3 R1,2 // m2,3→2 // m2,1→1)⟦3, -1⟧ - (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)⟦3, -1⟧},
 10]
```

Out[◦]//Short=
$$\Big\{x_2 y_1 (c_{3,4} - T c_{3,4}) + x_1 y_3 (c_{3,4} - T c_{3,4}) + x_1 y_2 (-c_{3,4} + T c_{3,4}) + x_3 y_2 (T c_{3,4} - T^2 c_{\ll 1\gg}) + \ll 348\gg + \ll 1\gg + x_3^4 y_2 y_3^3 (-4 T^3 c_{3,55} + 4 T^4 c_{3,55}) + \frac{1}{24} x_3^4 y_1^2 y_3^2 (7 T^2 - 97 T^3 + 329 T^4 - 239 T^5 - 48 T^3 c_{3,45} + 48 T^4 c_{3,45} + 72 T^2 c_{3,50} - 144 T^3 c_{3,50} + 72 T^4 c_{3,50} + 144 T^2 c_{3,55} - 288 T^3 c_{3,55} + 144 T^4 c_{3,55}) + \frac{1}{12 T} x_3^4 y_1^4 (5 - 19 T + 13 T^2 + 38 T^3 - 89 T^4 + 77 T^5 - 25 T^6 + 12 T^5 c_{3,31} - 48 T^6 c_{3,31} + \ll 46\gg + 12 T c_{3,50} - 48 T^2 c_{3,50} + 72 T^3 c_{3,50} - 48 T^4 c_{3,50} + 12 T^5 c_{3,50} + 12 T c_{3,55} - 48 T^2 c_{3,55} + 72 T^3 c_{3,55} - 48 T^4 c_{3,55} + 12 T^5 c_{3,55}), \ll 2\gg, \ll 1\gg\Big\}$$

In[∘]:=
```
Short[#, 10] &[eqns =
  Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, y1, y2, y3}][[;; , 2]] & /@ errors)]]
```

Out[∘]//Short=

$$\Big\{0 == c_{3,4} - T\,c_{3,4},\ \ll 418 \gg,$$

$$0 == \frac{3}{4} + \frac{5}{12\,T^5} - \frac{3}{4\,T^4} - \frac{1}{6\,T^3} - \frac{5}{12\,T} + \frac{T}{6} + c_{3,31} - T^4 c_{3,31} + \frac{c_{3,32}}{T} - T^3 c_{3,32} + \frac{c_{3,33}}{T^2} - T^2 c_{3,33} + \frac{c_{3,34}}{T^3} - T\,c_{3,34} - c_{3,35} + \frac{c_{3,35}}{T^4} + c_{3,36} - T^4 c_{3,36} + \frac{c_{3,37}}{T} - T^3 c_{3,37} + \frac{c_{3,38}}{T^2} - T^2 c_{3,38} + \frac{c_{3,39}}{T^3} - T\,c_{3,39} - c_{3,40} + \frac{c_{3,40}}{T^4} + c_{3,41} - T^4 c_{3,41} + \frac{c_3 \ll 1 \gg \ll 2 \gg}{T} - T^3 c_{3,42} + \frac{c_{3,43}}{T^2} - T^2 c_{3,43} + \frac{c_{3,44}}{T^3} - T\,c_{3,44} - c_{3,45} + \frac{c_{3,45}}{T^4} + c_{3,46} - T^4 c_{3,46} + \frac{c_{3,47}}{T} - T^3 c_{3,47} + \frac{c_{3,48}}{T^2} - T^2 c_{3,48} + \frac{c_{3,49}}{T^3} - T\,c_{3,49} - c_{3,50} + \frac{c_{3,50}}{T^4} + c_{3,51} - T^4 c_{3,51} + \frac{c_{3,52}}{T} - T^3 c_{3,52} + \frac{c_{3,53}}{T^2} - T^2 c_{3,53} + \frac{c_{3,54}}{T^3} - T\,c_{3,54} - c_{3,55} + \frac{c_{3,55}}{T^4} + \frac{e_{3,5}}{T^4} - T^4 f_{3,5}\Big\}$$

In[∘]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=

$$\Big\{\Big\{c_{3,4} \to 0,\ c_{3,5} \to -T\,c_{3,2} - c_{3,3},\ c_{3,6} \to 0,\ c_{3,8} \to -\frac{1}{2} \times (1 - T)\,c_{3,10},\ c_{3,9} \to 0,$$

$$c_{3,11} \to -T\,c_{3,7} - \frac{1}{2} \times (-1 + 3\,T)\,c_{3,10},\ c_{3,12} \to 0,\ c_{3,13} \to 0,\ c_{3,14} \to 0,\ c_{3,15} \to 0,$$

$$c_{3,17} \to -((-1 + T)\,c_{3,16}),\ c_{3,18} \to -\frac{1 - T}{6\,T},\ c_{3,19} \to 0,\ c_{3,20} \to 0,\ c_{3,21} \to \frac{1}{2\,T},$$

$$c_{3,22} \to -\frac{-2 + 5\,T}{2\,T} - \left(T - T^2\right) c_{3,16},\ c_{3,23} \to 0,\ c_{3,24} \to 0,\ c_{3,25} \to 0,\ c_{3,26} \to \frac{5}{6} - T^2 c_{3,16},$$

$$c_{3,27} \to 0,\ c_{3,28} \to 0,\ c_{3,29} \to 0,\ c_{3,30} \to 0,\ c_{3,31} \to 0,\ c_{3,33} \to -\frac{3}{2} \times (-1 + T)\,c_{3,32},$$

$$c_{3,34} \to -\left(\left(-1 + 2\,T - T^2\right) c_{3,32}\right),\ c_{3,35} \to -\frac{1 - 12\,T + 27\,T^2 - 16\,T^3}{24\,T^2},$$

$$c_{3,36} \to 0,\ c_{3,37} \to \frac{1}{6\,T^2},\ c_{3,38} \to -\frac{-1 + 3\,T}{4\,T^2},\ c_{3,39} \to -\frac{-1 + 11\,T - 16\,T^2}{6\,T^2},$$

$$c_{3,40} \to -\frac{-1 + 31\,T - 131\,T^2 + 125\,T^3}{24\,T^2} - \left(T - 2\,T^2 + T^3\right) c_{3,32},\ c_{3,41} \to 0,\ c_{3,42} \to 0,\ c_{3,43} \to \frac{1}{T},$$

$$c_{3,44} \to -\frac{-5 + 23\,T}{6\,T},\ c_{3,45} \to -\frac{-5 + 69\,T - 142\,T^2}{24\,T} + \frac{3}{2} \times (-1 + T)\,T^2 c_{3,32},\ c_{3,46} \to 0,\ c_{3,47} \to 0,$$

$$c_{3,48} \to 0,\ c_{3,49} \to \frac{1}{6},\ c_{3,50} \to \frac{1}{24} \times (1 - 15\,T) - T^3 c_{3,32},\ c_{3,51} \to 0,\ c_{3,52} \to 0,\ c_{3,53} \to 0,$$

$$c_{3,54} \to 0,\ c_{3,55} \to 0,\ d_{3,1} \to -c_{3,1},\ d_{3,2} \to -c_{3,2},\ d_{3,3} \to -\frac{c_{3,3}}{T^2},\ d_{3,4} \to 0,\ d_{3,5} \to \frac{c_{3,2}}{T} + \frac{c_{3,3}}{T^2},$$

$$d_{3,6} \to 0,\ d_{3,7} \to -\frac{-1 + T}{T^4} - \frac{c_{3,7}}{T} - \frac{(-1 + T)\,c_{3,10}}{T^2},\ d_{3,8} \to -\frac{1 - T}{2\,T^5} - \frac{(1 - T)\,c_{3,10}}{2\,T^3},\ d_{3,9} \to 0,$$

$$d_{3,10} \to -\frac{1}{T^4} - \frac{c_{3,10}}{T^2},\ d_{3,11} \to \frac{1}{2\,T^5} + \frac{c_{3,7}}{T^2} - \frac{(-1 - T)\,c_{3,10}}{2\,T^3},\ d_{3,12} \to 0,\ d_{3,13} \to 0,\ d_{3,14} \to 0,\ d_{3,15} \to 0,$$

$d_{3,16} \to -\frac{1-T}{T^4} - \frac{c_{3,16}}{T}$, $d_{3,17} \to -\frac{-7+9T-2T^2}{2T^5} - \frac{(-1+T)\,c_{3,16}}{T^2}$, $d_{3,18} \to -\frac{7-9T+2T^2}{6T^6}$, $d_{3,19} \to 0$,
$d_{3,20} \to \frac{1}{T^4}$, $d_{3,21} \to -\frac{9-T}{2T^5}$, $d_{3,22} \to \frac{3}{2T^6} - \frac{(1-T)\,c_{3,16}}{T^3}$, $d_{3,23} \to 0$, $d_{3,24} \to 0$, $d_{3,25} \to \frac{1}{T^5}$,
$d_{3,26} \to -\frac{1}{3T^6} + \frac{c_{3,16}}{T^3}$, $d_{3,27} \to 0$, $d_{3,28} \to 0$, $d_{3,29} \to 0$, $d_{3,30} \to 0$, $d_{3,31} \to 0$, $d_{3,32} \to -\frac{-1+T}{6T^4} - \frac{c_{3,32}}{T}$,
$d_{3,33} \to -\frac{2-3T+T^2}{T^5} - \frac{3\times(-1+T)\,c_{3,32}}{2T^2}$, $d_{3,34} \to -\frac{-16+27T-12T^2+T^3}{6T^6} - \frac{\left(1-2T+T^2\right)c_{3,32}}{T^3}$,
$d_{3,35} \to -\frac{16-27T+12T^2-T^3}{24T^7}$, $d_{3,36} \to 0$, $d_{3,37} \to -\frac{1}{6T^4}$, $d_{3,38} \to -\frac{-3+T}{T^5}$,
$d_{3,39} \to \frac{3\times(-3+T)}{2T^6}$, $d_{3,40} \to -\frac{-27+5T-T^2-T^3}{24T^7} - \frac{\left(-1+2T-T^2\right)c_{3,32}}{T^4}$, $d_{3,41} \to 0$,
$d_{3,42} \to 0$, $d_{3,43} \to -\frac{1}{T^5}$, $d_{3,44} \to \frac{2}{T^6}$, $d_{3,45} \to -\frac{12-T-5T^2}{24T^7} + \frac{3\times(-1+T)\,c_{3,32}}{2T^4}$,
$d_{3,46} \to 0$, $d_{3,47} \to 0$, $d_{3,48} \to 0$, $d_{3,49} \to -\frac{1}{6T^6}$, $d_{3,50} \to -\frac{-1-T}{24T^7} + \frac{c_{3,32}}{T^4}$, $d_{3,51} \to 0$,
$d_{3,52} \to 0$, $d_{3,53} \to 0$, $d_{3,54} \to 0$, $d_{3,55} \to 0$, $e_{3,1} \to \frac{c_{3,3}}{2T}$, $e_{3,2} \to -\frac{c_{3,10}}{T}$, $e_{3,3} \to 0$,
$e_{3,4} \to 0$, $e_{3,5} \to 0$, $f_{3,1} \to -\frac{c_{3,3}}{2T}$, $f_{3,2} \to \frac{1}{T^3} + \frac{c_{3,10}}{T}$, $f_{3,3} \to 0$, $f_{3,4} \to 0$, $f_{3,5} \to 0\}\}$

```
In[]:= sol /. (a_ → b_) ⧴ (a = b)
```

Out[$\circ$]=

$$\Big\{0,\ -T\,c_{3,2}-c_{3,3},\ 0,\ -\frac{1}{2}\times(1-T)\,c_{3,10},\ 0,\ -T\,c_{3,7}-\frac{1}{2}\times(-1+3T)\,c_{3,10},\ 0,\ 0,\ 0,\ 0,$$
$$-((-1+T)\,c_{3,16}),\ -\frac{1-T}{6T},\ 0,\ 0,\ \frac{1}{2T},\ -\frac{-2+5T}{2T}-\left(T-T^2\right)c_{3,16},\ 0,\ 0,\ 0,\ \frac{5}{6}-T^2\,c_{3,16},\ 0,$$
$$0,\ 0,\ 0,\ 0,\ -\frac{3}{2}\times(-1+T)\,c_{3,32},\ -\left(\left(-1+2T-T^2\right)c_{3,32}\right),\ -\frac{1-12T+27T^2-16T^3}{24T^2},\ 0,\ \frac{1}{6T^2},$$
$$-\frac{-1+3T}{4T^2},\ -\frac{-1+11T-16T^2}{6T^2},\ -\frac{-1+31T-131T^2+125T^3}{24T^2}-\left(T-2T^2+T^3\right)c_{3,32},\ 0,\ 0,\ \frac{1}{T},$$
$$-\frac{-5+23T}{6T},\ -\frac{-5+69T-142T^2}{24T}+\frac{3}{2}\times(-1+T)\,T^2\,c_{3,32},\ 0,\ 0,\ 0,\ \frac{1}{6},\ \frac{1}{24}\times(1-15T)-T^3\,c_{3,32},$$
$$0,\ 0,\ 0,\ 0,\ 0,\ -c_{3,1},\ -c_{3,2},\ -\frac{c_{3,3}}{T^2},\ 0,\ \frac{c_{3,2}}{T}+\frac{c_{3,3}}{T^2},\ 0,\ -\frac{-1+T}{T^4}-\frac{c_{3,7}}{T}-\frac{(-1+T)\,c_{3,10}}{T^2},$$
$$-\frac{1-T}{2T^5}-\frac{(1-T)\,c_{3,10}}{2T^3},\ 0,\ -\frac{1}{T^4}-\frac{c_{3,10}}{T^2},\ \frac{1}{2T^5}+\frac{c_{3,7}}{T^2}-\frac{(-1-T)\,c_{3,10}}{2T^3},\ 0,\ 0,\ 0,\ 0,$$
$$-\frac{1-T}{T^4}-\frac{c_{3,16}}{T},\ -\frac{-7+9T-2T^2}{2T^5}-\frac{(-1+T)\,c_{3,16}}{T^2},\ -\frac{7-9T+2T^2}{6T^6},\ 0,\ \frac{1}{T^4},\ -\frac{9-T}{2T^5},$$
$$\frac{3}{2T^6}-\frac{(1-T)\,c_{3,16}}{T^3},\ 0,\ 0,\ \frac{1}{T^5},\ -\frac{1}{3T^6}+\frac{c_{3,16}}{T^3},\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{-1+T}{6T^4}-\frac{c_{3,32}}{T},$$
$$-\frac{2-3T+T^2}{T^5}-\frac{3\times(-1+T)\,c_{3,32}}{2T^2},\ -\frac{-16+27T-12T^2+T^3}{6T^6}-\frac{\left(1-2T+T^2\right)c_{3,32}}{T^3},$$
$$-\frac{16-27T+12T^2-T^3}{24T^7},\ 0,\ -\frac{1}{6T^4},\ -\frac{-3+T}{T^5},\ \frac{3\times(-3+T)}{2T^6},\ -\frac{-27+5T-T^2-T^3}{24T^7}-\frac{\left(-1+2T-T^2\right)c_{3,32}}{T^4},$$
$$0,\ 0,\ -\frac{1}{T^5},\ \frac{2}{T^6},\ -\frac{12-T-5T^2}{24T^7}+\frac{3\times(-1+T)\,c_{3,32}}{2T^4},\ 0,\ 0,\ 0,\ -\frac{1}{6T^6},$$
$$-\frac{-1-T}{24T^7}+\frac{c_{3,32}}{T^4},\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{c_{3,3}}{2T},\ -\frac{c_{3,10}}{T},\ 0,\ 0,\ 0,\ -\frac{c_{3,3}}{2T},\ \frac{1}{T^3}+\frac{c_{3,10}}{T},\ 0,\ 0,\ 0\Big\}$$

In[∘]:= $c_{3,1} = c_{3,2} = c_{3,3} = c_{3,7} = c_{3,10} = c_{3,16} = c_{3,32} = 0;$
$\{R_{1,2}, \overline{R}_{1,2}, CC_1, \overline{CC}_1\}$

Out[∘]=

$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1, (-1+T)\, x_2 (y_1-y_2), \epsilon\text{Series}\Big[0, \frac{1}{2}\times(-1+T)\, x_2^2 y_1^2 + x_1 x_2 y_1 y_2 + \frac{1}{2}\times(1-3T)\, x_2^2 y_1 y_2,$$

$$-\frac{(-1+4T-3T^2)\, x_2^3 y_1^3}{6T} - \frac{1}{2} x_2^2 y_1 y_2 - \frac{x_1^2 x_2 y_1^2 y_2}{2T} - \frac{(1-3T)\, x_1 x_2^2 y_1^2 y_2}{2T} - \frac{(1-11T+16T^2)\, x_2^3 y_1^2 y_2}{6T} - \frac{1}{2} x_1 x_2^2 y_1 y_2^2 + \frac{1}{6}\times(-1+7T)\, x_2^3 y_1 y_2^2,$$

$$-\frac{(1-T)\, x_2^3 y_1^3}{6T} - \frac{(1-12T+27T^2-16T^3)\, x_2^4 y_1^4}{24T^2} + \frac{x_1 x_2^2 y_1^2 y_2}{2T} - \frac{(-2+5T)\, x_2^3 y_1^2 y_2}{2T} + \frac{x_1^3 x_2 y_1^3 y_2}{6T^2} - \frac{(-1+3T)\, x_1^2 x_2^2 y_1^3 y_2}{4T^2} - \frac{(-1+11T-16T^2)\, x_1 x_2^3 y_1^3 y_2}{6T^2} - \frac{(-1+31T-131T^2+125T^3)\, x_2^4 y_1^3 y_2}{24T^2} + \frac{5}{6} x_2^3 y_1 y_2^2 + \frac{x_1^2 x_2^2 y_1^2 y_2^2}{T} - \frac{(-5+23T)\, x_1 x_2^3 y_1^2 y_2^2}{6T} - \frac{(-5+69T-142T^2)\, x_2^4 y_1^2 y_2^2}{24T} + \frac{1}{6} x_1 x_2^3 y_1 y_2^3 + \frac{1}{24}\times(1-15T)\, x_2^4 y_1 y_2^3\Big]\Big], \mathbb{E}_{\{\}\to\{1,2\}}\Big[1, \Big(-1+\frac{1}{T}\Big) x_2 (y_1-y_2),$$

$$\epsilon\text{Series}\Big[0, -\frac{(-1+T)\, x_1 x_2 y_1^2}{T^2} - \frac{(1-T)\, x_2^2 y_1^2}{2T^3} - \frac{x_1 x_2 y_1 y_2}{T^2} - \frac{(-1-T)\, x_2^2 y_1 y_2}{2T^3},$$

$$-\frac{(1-T)\, x_1 x_2 y_1^2}{T^3} - \frac{(-1+T)\, x_2^2 y_1^2}{2T^4} - \frac{(-1+T)\, x_1^2 x_2 y_1^3}{2T^3} - \frac{(3-4T+T^2)\, x_1 x_2^2 y_1^3}{2T^4} - \frac{(-3+4T-T^2)\, x_2^3 y_1^3}{6T^5} + \frac{x_1 x_2 y_1 y_2}{T^3} - \frac{x_2^2 y_1 y_2}{2T^4} - \frac{x_1^2 x_2 y_1^2 y_2}{2T^3} + \frac{2 x_1 x_2^2 y_1^2 y_2}{T^4} - \frac{(4+T+T^2)\, x_2^3 y_1^2 y_2}{6T^5} - \frac{x_1 x_2^2 y_1 y_2^2}{2T^4} - \frac{(-1+T)\, x_2^3 y_1 y_2^2}{6T^5}, -\frac{(-1+T)\, x_1 x_2 y_1^2}{T^4} - \frac{(1-T)\, x_2^2 y_1^2}{2T^5} - \frac{(1-T)\, x_1^2 x_2 y_1^3}{T^4} -$$

$$\frac{(-7+9T-2T^2)\, x_1 x_2^2 y_1^3}{2T^5} - \frac{(7-9T+2T^2)\, x_2^3 y_1^3}{6T^6} - \frac{(-1+T)\, x_1^3 x_2 y_1^4}{6T^4} - \frac{(2-3T+T^2)\, x_1^2 x_2^2 y_1^4}{T^5} - \frac{(-16+27T-12T^2+T^3)\, x_1 x_2^3 y_1^4}{6T^6} - \frac{(16-27T+12T^2-T^3)\, x_2^4 y_1^4}{24T^7} - \frac{x_1 x_2 y_1 y_2}{T^4} + \frac{x_2^2 y_1 y_2}{2T^5} + \frac{x_1^2 x_2 y_1^2 y_2}{T^4} - \frac{(9-T)\, x_1 x_2^2 y_1^2 y_2}{2T^5} + \frac{3 x_2^3 y_1^2 y_2}{2T^6} - \frac{x_1^3 x_2 y_1^3 y_2}{6T^4} - \frac{(-3+T)\, x_1^2 x_2^2 y_1^3 y_2}{T^5} + \frac{3\times(-3+T)\, x_1 x_2^3 y_1^3 y_2}{2T^6} - \frac{(-27+5T-T^2-T^3)\, x_2^4 y_1^3 y_2}{24T^7} + \frac{x_1 x_2^2 y_1 y_2^2}{T^5} - \frac{x_2^3 y_1 y_2^2}{3T^6} - \frac{x_1^2 x_2^2 y_1^2 y_2^2}{T^5} + \frac{2 x_1 x_2^3 y_1^2 y_2^2}{T^6} - \frac{(12-T-5T^2)\, x_2^4 y_1^2 y_2^2}{24T^7} - \frac{x_1 x_2^3 y_1 y_2^3}{6T^6} - \frac{(-1-T)\, x_2^4 y_1 y_2^3}{24T^7}\Big]\Big],$$

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T}, 0, \epsilon\text{Series}\Big[0, -\frac{x_1 y_1}{T}, 0, 0\Big]\Big],$$

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}}, 0, \epsilon\text{Series}\Big[0, \frac{x_1 y_1}{T}, -\frac{x_1 y_1}{T^2}, \frac{x_1 y_1}{T^3}\Big]\Big]\Big\}$$

```
In[∘]:= {(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3),
  (R1,2 R̄3,4 // m1,3→1 // m2,4→2) ≡ 𝔼{}→{1,2}[1, 0, ϵSeries[0]],
  (CC1 C̄C̄2 // m1,2→1 ) ≡ 𝔼{}→{1}[1, 0, ϵSeries[0]],
  (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)}

Out[∘]= {True, True, True, True}
```

Some Knot Theory

```
In[∘]:= Define[Kinki = CC3 R1,2 // m2,3→2 // m2,1→i, K̄īn̄k̄i = CC3 R̄1,2 // m1,3→1 // m1,2→i]
```

```
In[∘]:= RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
    n = Length@pd; rots = Table[0, {2 n}];
    xs = Cases[pd, x_X ⧴ { Xp[x⟦4⟧, x⟦1⟧]  PositiveQ@x
                          Xm[x⟦2⟧, x⟦1⟧]  True      ];
    For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → (xs /. {
              Xp[k + 1, l_] | Xm[l_, k + 1] ⧴ {l, k + 1, 1 - l},
              Xp[l_, k + 1] | Xm[k + 1, l_] ⧴ (++rots⟦l⟧; {1 - l, k + 1, l})
            })],
      Cases[front, k | -k] /. {k, -k} ⧴ --rots⟦k + 1⟧;
     ]];
    RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

```
In[∘]:= rot[i_, 0] := 𝔼{}→{i}[1, 0, ϵSeries@0];
rot[i_, n_] := Module[{j},
    rot[i, n] = If[n > 0, rot[i, n - 1] CCj, rot[i, n + 1] C̄C̄j] // mi,j→i];
```

In[ ]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{}→{0}[1, 0, ϵSeries@0];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
     Xp, (R_{i,j} OverBar[Kink_k]) // m_{j,k→j},
     Xm, (OverBar[R]_{i,j} Kink_k) // m_{j,k→j}
    ];
   ξ1 = (rot[k, rots[[i]]] ξ1) // m_{k,i→i}; rots[[i]] = 0;
   ξ1 = (ξ1 rot[k, rots[[i + 1]]]) // m_{i,k→i}; rots[[i + 1]] = 0;
   ξ1 = (rot[k, rots[[j]]] ξ1) // m_{k,j→j}; rots[[j]] = 0;
   ξ1 = (ξ1 rot[k, rots[[j + 1]]]) // m_{j,k→j}; rots[[j + 1]] = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // m_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ξ = ξ // m_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ξ = ξ // m_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ξ = ξ // m_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ (*/. {x_0→x,y_0→y,a_0→a}*))
 ]
```

In[ ]:=
```
$k = 1
```

Out[ ]= 1

In[ ]:=
```
NewBit[K_] := Module[{Alex = Alexander[K][T]},
  T^3 Alex^2/(T - 1) Z[K][[3, 2]] // Factor]
```

In[ ]:=
```
NewBit /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]= $\left\{2 - T + T^2,\ (1 + T) \times \left(1 - 3\,T + T^2\right),\ \frac{4 - 3\,T + 5\,T^2 - 3\,T^3 + 3\,T^4 - T^5 + T^6}{T^2},\ 9 - 11\,T + 7\,T^2 - T^3\right\}$

In[∘]:=
```
(*Two knots with equal Alexander, new bit does not agree*)
Alexander[Knot[6, 1]] == Alexander[Knot[9, 46]]
Timing[NewBit[Knot[6, 1]] == NewBit[Knot[9, 46]]]
```

Out[∘]= True

Out[∘]= $\{40.5781, 5 - 11\,T - T^2 + 3\,T^3 == 7 - 21\,T + 9\,T^2 + T^3\}$

In[∘]:=
```
equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
Length@Union[Z /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[∘]= 1

In[∘]:=
```
equiv =
  {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
Length@Union[Z /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

Out[∘]= 1

In[∘]:=
```
$k = 2
```

Out[∘]= 2

In[∘]:=
```
equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
Length@Union[Z /@ equiv]
```

Out[∘]= 2

In[∘]:=
```
equiv =
  {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
Length@Union[Z /@ equiv]
```

Out[∘]= $Aborted