Pensieve header: Perturbing the Heisenberg-algebra knot invariant. Continues "Solving to $k=3.nb" at pensieve://Projects/BabyDoPeGDO/.

$\mathbb{E}[\omega, Q, P\_\epsilon\text{Series}]$ represents ωe^{Q+P}, where ω is a scalar, Q is an ϵ-free quadratic, and $P=\sum_{k=0}^{\$k} P[\![k]\!]\,\epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: $\mathbb{E}[\_]//\mathbb{E}[\_]$ calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

Initialization and minor utilities

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\21-1350-KnotTheory\\Preps"];
Once[<< "../Common.m"];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[•]:=
```
$k=1;
```

In[•]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ ℰ;
CF[ℰ_] := Module[
   {vs = Cases[ℰ, (p | x | π | ξ)_, ∞] ⋃ {p | x | π | ξ}},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
  ];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

In[•]:=
```
ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
ϵSeries[0] := ϵSeries @@ Table[0, $k + 1];
ϵSeries /: S1_ϵSeries + S2_ϵSeries :=
  ϵSeries @@ Table[S1[[k]] + S2[[k]], {k, Min[Length@S1, Length@S2]}];
ϵSeries /: S1_ϵSeries * S2_ϵSeries := ϵSeries @@
   Table[Sum[S1[[j + 1]] * S2[[k - j + 1]], {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
ϵSeries /: c_ * S_ϵSeries := (c #) & /@ S;
ϵSeries /: ∂_vs___ S_ϵSeries := (s ↦ ∂_vs s) /@ S;
```

The Main Program

Variables and their duals:

```
{p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_i_)* := (u*)_i;
```

𝔼 operations:

```
𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
𝔼 /: 𝔼[ω1_, Q1_, P1_] 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
𝔼_{dr_}[ℰs___]_{$k_} := 𝔼_{dr} @@ 𝔼[ℰs]_{$k};
```

```
𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
  lvs = Flatten@Table[{x_{$@i}, p_{$@i}}, {i, is}];
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{lvs⋃lvs*}[lvs*.lvs, Times[
      𝔼[ℰ1s] /. Table[(v : x | p)_i → v_{$@i}, {i, is}],
      𝔼[ℰ2s] /. Table[(v : ξ | π)_i → v_{$@i}, {i, is}]
    ]])
 ]
```

$$[F:\mathcal{E}]_B := \mathfrak{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad \text{and} \quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathfrak{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

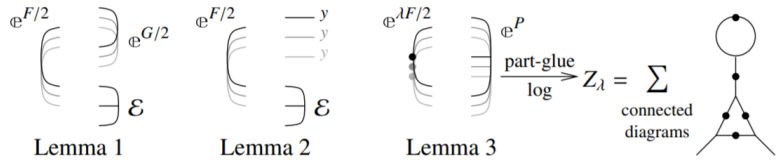
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\,\mathfrak{e}^{\sum_{i\in B} y_iz_i}\right\rangle_B = \mathfrak{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$.

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathfrak{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$




```
Zip_{vs_}[ℱ_, ℰ_] := ⟨ℱ, ℰ⟩ // Zip1_{vs} // Zip2_{vs} // Zip3_{vs}
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\, \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1} : \mathcal{E} \right\rangle_B$$

In[•]:=
```
Zip1{} = Identity;
Zip1vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{ℐ, F, G, u, v},
  ℐ = IdentityMatrix@Length@vs;
  F = Table[∂u,v ℱ, {u, vs*}, {v, vs*}];
  G = Table[∂u,v Q, {u, vs}, {v, vs}];
  CF /@ ⟨vs*.F.Inverse[ℐ - G.F].vs* / 2,
    𝔼[PowerExpand@Factor[ω Det[ℐ - G.F]^-1/2], Q - vs.G.vs / 2, P]⟩
 ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}\, \mathbb{e}^{\sum_{i\in B} y_i z_i} \right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \to z_B + F y_B} \right\rangle_B$.

In[•]:=
```
Zip2{} = Identity;
Zip2vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{F, Y, u, v},
  F = Table[∂u,v ℱ, {u, vs*}, {v, vs*}];
  Y = Table[∂v Q, {v, vs}];
  CF /@ ⟨ℱ, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
 ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

In[•]:=
```
Zip3vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{Z, u, v, m, j},
  Z[0] = P;
  For[m = 0, m < 2 $k, ++m,
   Z[m + 1] = CF[1/(2 (m + 1))
      Sum[∂u*,v* ℱ (∂u,v Z[m] + Sum[(∂u Z[j]) (∂v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
  ];
  𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k}] /. Table[v → 0, {v, vs}]]]
 ]
```

The Basic Tensors

```
In[•]:= η_i_ := 𝔼_{}→{i}[1, 0, ϵSeries[0]];
        m_i_,j_→k_ := 𝔼_{i,j}→{k}[1, -ξ_i π_j + (π_i + π_j) p_k + (ξ_i + ξ_j) x_k, ϵSeries[0]]
```

```
In[•]:= AllMonomials[{}, 0] = {1};
        AllMonomials[{}, d_Integer] /; d > 0 := {};
        AllMonomials[{v_, vs___}, d_Integer] :=
          Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
        AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
```

```
In[•]:= Basis[js_List, m_] := Flatten@Outer[Times,
           AllMonomials[Table[p_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
        Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 0, m}]
```

```
In[•]:= Basis[{i, j}, {2}]
```

```
Out[•]= {1, p_i x_i, p_i x_j, p_j x_i, p_j x_j, p_i^2 x_i^2, p_i^2 x_i x_j, p_i^2 x_j^2, p_i p_j x_i^2, p_i p_j x_i x_j, p_i p_j x_j^2, p_j^2 x_i^2, p_j^2 x_i x_j, p_j^2 x_j^2}
```

```
In[•]:= GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}];
        GenericCombination[bas_, c_k_] := bas.Table[c_k,j, {j, Length@bas}];
```

```
In[•]:= GenericCombination[Basis[{i, j}, {2}], c_1]
```

```
Out[•]= c_1,1 + p_i x_i c_1,2 + p_i x_j c_1,3 + p_j x_i c_1,4 + p_j x_j c_1,5 + p_i^2 x_i^2 c_1,6 + p_i^2 x_i x_j c_1,7 + p_i^2 x_j^2 c_1,8 +
        p_i p_j x_i^2 c_1,9 + p_i p_j x_i x_j c_1,10 + p_i p_j x_j^2 c_1,11 + p_j^2 x_i^2 c_1,12 + p_j^2 x_i x_j c_1,13 + p_j^2 x_j^2 c_1,14
```

```
In[•]:= R_i_,j_ := 𝔼_{}→{i,j}[T^(1/2), (T - 1) (p_i - p_j) x_j ,
           ϵSeries @@ Prepend[0]@Table[GenericCombination[Basis[{i, j}, {k + 1}], c_k], {k, $k}]];
        R̄_i_,j_ := 𝔼_{}→{i,j}[T^(-1/2), (T^-1 - 1) (p_i - p_j) x_j,
           ϵSeries @@ Prepend[0]@Table[GenericCombination[Basis[{i, j}, {k + 1}], d_k], {k, $k}]];
        C_i_ := 𝔼_{}→{i}[T^(1/2), 0, ϵSeries @@ Prepend[0]@
             Table[GenericCombination[Basis[{i}, {k + 1}], e_k], {k, $k}]];
        C̄_i_ := 𝔼_{}→{i}[T^(-1/2), 0, ϵSeries @@ Prepend[0]@
             Table[GenericCombination[Basis[{i}, {k + 1}], f_k], {k, $k}]];
```

In[•]:=
```
{R_{1,2}, R̄_{1,2}, C_1, C̄_1}
```

Out[•]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[\sqrt{T}\,,\ (-1+T)\,(p_1-p_2)\,x_2,$$
$$\in\text{Series}\Big[0,\ c_{1,1}+p_1\,x_1\,c_{1,2}+p_1\,x_2\,c_{1,3}+p_2\,x_1\,c_{1,4}+p_2\,x_2\,c_{1,5}+p_1^2\,x_1^2\,c_{1,6}+p_1^2\,x_1\,x_2\,c_{1,7}+p_1^2\,x_2^2\,c_{1,8}+$$
$$p_1\,p_2\,x_1^2\,c_{1,9}+p_1\,p_2\,x_1\,x_2\,c_{1,10}+p_1\,p_2\,x_2^2\,c_{1,11}+p_2^2\,x_1^2\,c_{1,12}+p_2^2\,x_1\,x_2\,c_{1,13}+p_2^2\,x_2^2\,c_{1,14}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[\frac{1}{\sqrt{T}},\ \Big(-1+\frac{1}{T}\Big)\,(p_1-p_2)\,x_2,\ \in\text{Series}\Big[0,$$
$$d_{1,1}+p_1\,x_1\,d_{1,2}+p_1\,x_2\,d_{1,3}+p_2\,x_1\,d_{1,4}+p_2\,x_2\,d_{1,5}+p_1^2\,x_1^2\,d_{1,6}+p_1^2\,x_1\,x_2\,d_{1,7}+p_1^2\,x_2^2\,d_{1,8}+$$
$$p_1\,p_2\,x_1^2\,d_{1,9}+p_1\,p_2\,x_1\,x_2\,d_{1,10}+p_1\,p_2\,x_2^2\,d_{1,11}+p_2^2\,x_1^2\,d_{1,12}+p_2^2\,x_1\,x_2\,d_{1,13}+p_2^2\,x_2^2\,d_{1,14}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T}\,,\ 0,\ \in\text{Series}\Big[0,\ e_{1,1}+p_1\,x_1\,e_{1,2}+p_1^2\,x_1^2\,e_{1,3}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}}\,,\ 0,\ \in\text{Series}\Big[0,\ f_{1,1}+p_1\,x_1\,f_{1,2}+p_1^2\,x_1^2\,f_{1,3}\Big]\Big]\Big\}$$

In[•]:=
```
RMoves := {
  (R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}),
  (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ (η_1 η_2),
  (C_1 C̄_2 // m_{1,2→1}) ≡ η_1,
  (R_{1,4} R̄_{5,2} C̄_3 // m_{2,4→2} // m_{1,3→1} // m_{1,5→1}) ≡ C̄_1 η_2,
  (C_3 R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1}),
  (C̄_2 R_{1,3} // m_{1,2→1} // m_{1,3→1}) ≡ η_1,  (C̄_2 R̄_{3,1} // m_{1,2→1} // m_{1,3→1}) ≡ η_1,
  (C_2 R̄_{1,3} // m_{1,2→1} // m_{1,3→1}) ≡ η_1, (C_2 R_{3,1} // m_{1,2→1} // m_{1,3→1}) ≡ η_1
 }
```

Solving for *R*, *C*, $k = 1

In[•]:=
```
$k = 1;
{R_{1,2}, C_1}
unknowns = Cases[{R_{1,2}, R̄_{1,2}, C_1, C̄_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

Out[•]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[\sqrt{T}\,,\ (-1+T)\,(p_1-p_2)\,x_2,$$
$$\in\text{Series}\Big[0,\ c_{1,1}+p_1\,x_1\,c_{1,2}+p_1\,x_2\,c_{1,3}+p_2\,x_1\,c_{1,4}+p_2\,x_2\,c_{1,5}+p_1^2\,x_1^2\,c_{1,6}+p_1^2\,x_1\,x_2\,c_{1,7}+p_1^2\,x_2^2\,c_{1,8}+$$
$$p_1\,p_2\,x_1^2\,c_{1,9}+p_1\,p_2\,x_1\,x_2\,c_{1,10}+p_1\,p_2\,x_2^2\,c_{1,11}+p_2^2\,x_1^2\,c_{1,12}+p_2^2\,x_1\,x_2\,c_{1,13}+p_2^2\,x_2^2\,c_{1,14}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T}\,,\ 0,\ \in\text{Series}\Big[0,\ e_{1,1}+p_1\,x_1\,e_{1,2}+p_1^2\,x_1^2\,e_{1,3}\Big]\Big]\Big\}$$

Out[•]=
$$\{c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}, c_{1,8}, c_{1,9}, c_{1,10}, c_{1,11}, c_{1,12}, c_{1,13}, c_{1,14}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, e_{1,1}, e_{1,2}, e_{1,3}, f_{1,1}, f_{1,2}, f_{1,3}\}$$

```
In[•]:= Short[errors = CCF /@ Cases[RMoves, a_ == b_ :> a - b], 25]
```

Out[•]//Short=

$$\Big\{T\,p_1\,x_3\,c_{1,2} - T^2\,p_1\,x_3\,c_{1,2} + p_1\,x_3\,c_{1,3} - T\,p_1\,x_3\,c_{1,3} - p_2\,x_1\,c_{1,4} + T\,p_2\,x_1\,c_{1,4} + p_3\,x_1\,c_{1,4} - T\,p_3\,x_1\,c_{1,4} +$$
$$p_1\,x_2\,c_{1,4} - T\,p_1\,x_2\,c_{1,4} - p_2\,x_2\,c_{1,4} + 2\,T\,p_2\,x_2\,c_{1,4} - T^2\,p_2\,x_2\,c_{1,4} - T\,p_3\,x_2\,c_{1,4} + T^2\,p_3\,x_2\,c_{1,4} +$$
$$T\,p_2\,x_3\,c_{1,4} - T^2\,p_2\,x_3\,c_{1,4} + p_1\,x_3\,c_{1,5} - T\,p_1\,x_3\,c_{1,5} - 2\,p_1^2\,x_1\,x_2\,c_{1,6} + 2\,T\,p_1^2\,x_1\,x_2\,c_{1,6} +$$
$$2\,T\,p_1\,p_2\,x_2^2\,c_{1,6} - 2\,T^2\,p_1\,p_2\,x_2^2\,c_{1,6} + 2\,T\,p_1^2\,x_1\,x_3\,c_{1,6} - 2\,T^2\,p_1^2\,x_1\,x_3\,c_{1,6} + T^2\,p_1^2\,x_3^2\,c_{1,6} -$$
$$2\,T^3\,p_1^2\,x_3^2\,c_{1,6} + T^4\,p_1^2\,x_3^2\,c_{1,6} + 2\,T\,p_1\,p_2\,x_2\,x_3\,c_{1,7} - 2\,T^2\,p_1\,p_2\,x_2\,x_3\,c_{1,7} + T\,p_1^2\,x_3^2\,c_{1,7} - 2\,T^2\,p_1^2\,x_3^2\,c_{1,7} +$$
$$T^3\,p_1^2\,x_3^2\,c_{1,7} + 2\,p_1^2\,x_2\,x_3\,c_{1,8} - 2\,T\,p_1^2\,x_2\,x_3\,c_{1,8} + p_1^2\,x_3^2\,c_{1,8} - 4\,T\,p_1^2\,x_3^2\,c_{1,8} + 3\,T^2\,p_1^2\,x_3^2\,c_{1,8} +$$
$$2\,T\,p_1\,p_2\,x_3^2\,c_{1,8} - 2\,T^2\,p_1\,p_2\,x_3^2\,c_{1,8} - p_1\,p_2\,x_1^2\,c_{1,9} + T\,p_1\,p_2\,x_1^2\,c_{1,9} + p_1\,p_3\,x_1^2\,c_{1,9} - T\,p_1\,p_3\,x_1^2\,c_{1,9} -$$
$$2\,p_1\,p_2\,x_1\,x_2\,c_{1,9} + \langle\langle 83\rangle\rangle + 2\,T\,p_1\,p_3\,x_2^2\,c_{1,12} - 2\,T^2\,p_1\,p_3\,x_2^2\,c_{1,12} - 2\,T\,p_2\,p_3\,x_2^2\,c_{1,12} +$$
$$6\,T^2\,p_2\,p_3\,x_2^2\,c_{1,12} - 6\,T^3\,p_2\,p_3\,x_2^2\,c_{1,12} + 2\,T^4\,p_2\,p_3\,x_2^2\,c_{1,12} - T^2\,p_3^2\,x_2^2\,c_{1,12} + 2\,T^3\,p_3^2\,x_2^2\,c_{1,12} -$$
$$T^4\,p_3^2\,x_2^2\,c_{1,12} + 2\,T\,p_2^2\,x_1\,x_3\,c_{1,12} - 2\,T^2\,p_2^2\,x_1\,x_3\,c_{1,12} + T^2\,p_2^2\,x_3^2\,c_{1,12} - 2\,T^3\,p_2^2\,x_3^2\,c_{1,12} + T^4\,p_2^2\,x_3^2\,c_{1,12} +$$
$$T\,p_2^2\,x_1\,x_3\,c_{1,13} - T^2\,p_2^2\,x_1\,x_3\,c_{1,13} - 2\,T\,p_2\,p_3\,x_1\,x_3\,c_{1,13} + 2\,T^2\,p_2\,p_3\,x_1\,x_3\,c_{1,13} + T\,p_3^2\,x_1\,x_3\,c_{1,13} -$$
$$T^2\,p_3^2\,x_1\,x_3\,c_{1,13} + p_1^2\,x_2\,x_3\,c_{1,13} - 2\,T\,p_1^2\,x_2\,x_3\,c_{1,13} + T^2\,p_1^2\,x_2\,x_3\,c_{1,13} - p_2^2\,x_2\,x_3\,c_{1,13} + 4\,T\,p_2^2\,x_2\,x_3\,c_{1,13} -$$
$$4\,T^2\,p_2^2\,x_2\,x_3\,c_{1,13} + T^3\,p_2^2\,x_2\,x_3\,c_{1,13} + 2\,T\,p_1\,p_3\,x_2\,x_3\,c_{1,13} - 2\,T^2\,p_1\,p_3\,x_2\,x_3\,c_{1,13} - 2\,T\,p_2\,p_3\,x_2\,x_3\,c_{1,13} +$$
$$4\,T^2\,p_2\,p_3\,x_2\,x_3\,c_{1,13} - 2\,T^3\,p_2\,p_3\,x_2\,x_3\,c_{1,13} - T^2\,p_3^2\,x_2\,x_3\,c_{1,13} + T^3\,p_3^2\,x_2\,x_3\,c_{1,13} + T\,p_2^2\,x_3^2\,c_{1,13} -$$
$$2\,T^2\,p_2^2\,x_3^2\,c_{1,13} + T^3\,p_2^2\,x_3^2\,c_{1,13} + 2\,p_2^2\,x_2\,x_3\,c_{1,14} - 2\,T\,p_2^2\,x_2\,x_3\,c_{1,14} + p_1^2\,x_3^2\,c_{1,14} - 2\,T\,p_1^2\,x_3^2\,c_{1,14} +$$
$$T^2\,p_1^2\,x_3^2\,c_{1,14} + 2\,T\,p_1\,p_3\,x_3^2\,c_{1,14} - 2\,T^2\,p_1\,p_3\,x_3^2\,c_{1,14} - 2\,T\,p_2\,p_3\,x_3^2\,c_{1,14} + 2\,T^2\,p_2\,p_3\,x_3^2\,c_{1,14},$$
$$\frac{\langle\langle 1\rangle\rangle}{T^2},\ e_{\langle\langle 1\rangle\rangle} + \langle\langle 5\rangle\rangle,\ \langle\langle 4\rangle\rangle,\ \frac{\langle\langle 1\rangle\rangle}{T^2},\ \frac{T^2\,c_{\langle\langle 1\rangle\rangle} + \langle\langle 36\rangle\rangle + \langle\langle 1\rangle\rangle}{T^2}\Big\}$$

```
In[•]:= eqns = Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, p1, p2, p3}][[ ;; , 2]] & /@ errors)]
```

Out[•]=

$$\Big\{0 == c_{1,4} - T\,c_{1,4},\ 0 == -c_{1,4} + T\,c_{1,4},\ 0 == T\,c_{1,4} - T^2\,c_{1,4},\ 0 == -c_{1,4} + 2\,T\,c_{1,4} - T^2\,c_{1,4},$$
$$0 == -T\,c_{1,4} + T^2\,c_{1,4},\ 0 == T\,c_{1,2} - T^2\,c_{1,2} + c_{1,3} - T\,c_{1,3} + c_{1,5} - T\,c_{1,5},$$
$$0 == -2\,c_{1,6} + 2\,T\,c_{1,6},\ 0 == 2\,T\,c_{1,6} - 2\,T^2\,c_{1,6},\ 0 == c_{1,9} - T\,c_{1,9},$$
$$0 == -c_{1,9} + T\,c_{1,9},\ 0 == 2\,T\,c_{1,9} - 2\,T^2\,c_{1,9},\ 0 == -2\,c_{1,9} + 4\,T\,c_{1,9} - 2\,T^2\,c_{1,9},$$
$$0 == -2\,T\,c_{1,9} + 2\,T^2\,c_{1,9},\ 0 == 2\,T\,c_{1,6} - 2\,T^2\,c_{1,6} - c_{1,9} + 4\,T\,c_{1,9} - 4\,T^2\,c_{1,9} + T^3\,c_{1,9},$$
$$0 == 2\,T\,c_{1,8} - 2\,T^2\,c_{1,8} + T^2\,c_{1,9} - 2\,T^3\,c_{1,9} + T^4\,c_{1,9} + T\,c_{1,10} - 2\,T^2\,c_{1,10} + T^3\,c_{1,10},$$
$$0 == 2\,T\,c_{1,7} - 2\,T^2\,c_{1,7} - c_{1,10} + 4\,T\,c_{1,10} - 3\,T^2\,c_{1,10} + 2\,c_{1,11} - 2\,T\,c_{1,11},$$
$$0 == T^2\,c_{1,9} - T^3\,c_{1,9} + 2\,T\,c_{1,12} - 2\,T^2\,c_{1,12},\ 0 == c_{1,12} - T^2\,c_{1,12},\ 0 == -c_{1,12} + 2\,T\,c_{1,12} - T^2\,c_{1,12},$$
$$0 == c_{1,9} - 2\,T\,c_{1,9} + T^2\,c_{1,9} + c_{1,12} - 2\,T\,c_{1,12} + T^2\,c_{1,12},\ 0 == -2\,T\,c_{1,12} + 2\,T^2\,c_{1,12},$$
$$0 == -4\,T\,c_{1,12} + 8\,T^2\,c_{1,12} - 4\,T^3\,c_{1,12},\ 0 == -2\,c_{1,12} + 6\,T\,c_{1,12} - 6\,T^2\,c_{1,12} + 2\,T^3\,c_{1,12},$$
$$0 == -2\,T^2\,c_{1,12} + 2\,T^3\,c_{1,12},\ 0 == -T^2\,c_{1,12} + 2\,T^3\,c_{1,12} - T^4\,c_{1,12},$$
$$0 == -c_{1,12} + 4\,T\,c_{1,12} - 6\,T^2\,c_{1,12} + 4\,T^3\,c_{1,12} - T^4\,c_{1,12},\ 0 == -2\,T\,c_{1,12} + 6\,T^2\,c_{1,12} - 6\,T^3\,c_{1,12} + 2\,T^4\,c_{1,12},$$
$$0 == 2\,T\,c_{1,13} - 2\,T^2\,c_{1,13},\ 0 == T\,c_{1,13} - T^2\,c_{1,13},\ 0 == 2\,T\,c_{1,12} - 2\,T^2\,c_{1,12} + T\,c_{1,13} - T^2\,c_{1,13},$$
$$0 == 2\,c_{1,8} - 2\,T\,c_{1,8} + c_{1,10} - 2\,T\,c_{1,10} + T^2\,c_{1,10} + c_{1,13} - 2\,T\,c_{1,13} + T^2\,c_{1,13},\ 0 == -2\,T\,c_{1,13} + 2\,T^2\,c_{1,13},$$
$$0 == -2\,T\,c_{1,13} + 4\,T^2\,c_{1,13} - 2\,T^3\,c_{1,13},\ 0 == T^2\,c_{1,12} - 2\,T^3\,c_{1,12} + T^4\,c_{1,12} + T\,c_{1,13} - 2\,T^2\,c_{1,13} + T^3\,c_{1,13},$$
$$0 == -T^2\,c_{1,13} + T^3\,c_{1,13},\ 0 == -c_{1,13} + 4\,T\,c_{1,13} - 4\,T^2\,c_{1,13} + T^3\,c_{1,13} + 2\,c_{1,14} - 2\,T\,c_{1,14},$$
$$0 == 2\,T\,c_{1,14} - 2\,T^2\,c_{1,14},\ 0 == T^2\,c_{1,6} - 2\,T^3\,c_{1,6} + T^4\,c_{1,6} + T\,c_{1,7} - 2\,T^2\,c_{1,7} + T^3\,c_{1,7} +$$
$$c_{1,8} - 4\,T\,c_{1,8} + 3\,T^2\,c_{1,8} + c_{1,11} - 2\,T\,c_{1,11} + T^2\,c_{1,11} + c_{1,14} - 2\,T\,c_{1,14} + T^2\,c_{1,14},$$
$$0 == -2\,T\,c_{1,14} + 2\,T^2\,c_{1,14},\ 0 == c_{1,1} + d_{1,1},\ 0 == c_{1,1} - c_{1,4} + \frac{c_{1,4}}{T} + 2\,c_{1,12} + \frac{2\,c_{1,12}}{T^2} - \frac{4\,c_{1,12}}{T} + d_{1,1},$$

$$0 == c_{1,2} + c_{1,4} - \frac{c_{1,4}}{T} - 2\,c_{1,9} + \frac{2\,c_{1,9}}{T} - 4\,c_{1,12} - \frac{4\,c_{1,12}}{T^2} + \frac{8\,c_{1,12}}{T} + d_{1,2},$$

$$0 == \frac{c_{1,4}}{T} + \frac{4\,c_{1,12}}{T^2} - \frac{4\,c_{1,12}}{T} + d_{1,4},\ 0 == c_{1,2} + d_{1,2} + d_{1,4} - T\,d_{1,4},\ 0 == c_{1,4} + T\,d_{1,4},$$

$$0 == c_{1,2} - \frac{c_{1,2}}{T} + \frac{c_{1,3}}{T} + d_{1,3} + d_{1,5} - T\,d_{1,5},\ 0 == c_{1,4} - \frac{c_{1,4}}{T} + \frac{c_{1,5}}{T} + T\,d_{1,5},$$

$$0 == c_{1,4} - \frac{c_{1,4}}{T} + \frac{c_{1,5}}{T} - 4\,c_{1,12} - \frac{4\,c_{1,12}}{T^2} + \frac{8\,c_{1,12}}{T} + \frac{2\,c_{1,13}}{T^2} - \frac{2\,c_{1,13}}{T} + T\,d_{1,5},$$

$$0 == c_{1,6} + c_{1,9} - \frac{c_{1,9}}{T} + c_{1,12} + \frac{c_{1,12}}{T^2} - \frac{2\,c_{1,12}}{T} + d_{1,6},\ 0 == \frac{c_{1,9}}{T} - \frac{2\,c_{1,12}}{T^2} + \frac{2\,c_{1,12}}{T} + d_{1,9},$$

$$0 == 2\,c_{1,9} - \frac{2\,c_{1,9}}{T} + \frac{c_{1,10}}{T} + 4\,c_{1,12} + \frac{4\,c_{1,12}}{T^2} - \frac{8\,c_{1,12}}{T} - \frac{2\,c_{1,13}}{T^2} + \frac{2\,c_{1,13}}{T} + T\,d_{1,10},$$

$$0 == -2\,c_{1,9} + \frac{c_{1,9}}{T} + T\,c_{1,9} + c_{1,10} - \frac{c_{1,10}}{T} + \frac{c_{1,11}}{T} - 6\,c_{1,12} - \frac{2\,c_{1,12}}{T^2} + \frac{6\,c_{1,12}}{T} + 2\,T\,c_{1,12} + 2\,c_{1,13} + \frac{2\,c_{1,13}}{T^2} - \frac{4\,c_{1,13}}{T} - \frac{2\,c_{1,14}}{T^2} + \frac{2\,c_{1,14}}{T} + T^2\,d_{1,11},$$

$$0 == \frac{c_{1,12}}{T^2} + d_{1,12},\ 0 == c_{1,9} + T\,d_{1,9} + 2\,T\,d_{1,12} - 2\,T^2\,d_{1,12},\ 0 == c_{1,12} + T^2\,d_{1,12},$$

$$0 == c_{1,6} + d_{1,6} + d_{1,9} - T\,d_{1,9} + d_{1,12} - 2\,T\,d_{1,12} + T^2\,d_{1,12},\ 0 == -\frac{2\,c_{1,12}}{T^2} + \frac{2\,c_{1,12}}{T} + \frac{c_{1,13}}{T^2} + T\,d_{1,13},$$

$$0 == 2\,c_{1,9} - \frac{2\,c_{1,9}}{T} + \frac{c_{1,10}}{T} + T\,d_{1,10} + 2\,T\,d_{1,13} - 2\,T^2\,d_{1,13},\ 0 == 2\,c_{1,12} - \frac{2\,c_{1,12}}{T} + \frac{c_{1,13}}{T} + T^2\,d_{1,13},$$

$$0 == 2\,c_{1,6} - \frac{2\,c_{1,6}}{T} + \frac{c_{1,7}}{T} + d_{1,7} + d_{1,10} - T\,d_{1,10} + d_{1,13} - 2\,T\,d_{1,13} + T^2\,d_{1,13},$$

$$0 == c_{1,9} + \frac{c_{1,9}}{T^2} - \frac{2\,c_{1,9}}{T} - \frac{c_{1,10}}{T^2} + \frac{c_{1,10}}{T} + \frac{c_{1,11}}{T^2} + T\,d_{1,11} + 2\,T\,d_{1,14} - 2\,T^2\,d_{1,14},$$

$$0 == c_{1,12} + \frac{c_{1,12}}{T^2} - \frac{2\,c_{1,12}}{T} - \frac{c_{1,13}}{T^2} + \frac{c_{1,13}}{T} + \frac{c_{1,14}}{T^2} + T^2\,d_{1,14},$$

$$0 == c_{1,6} + \frac{c_{1,6}}{T^2} - \frac{2\,c_{1,6}}{T} - \frac{c_{1,7}}{T^2} + \frac{c_{1,7}}{T} + \frac{c_{1,8}}{T^2} + d_{1,8} + d_{1,11} - T\,d_{1,11} + d_{1,14} - 2\,T\,d_{1,14} + T^2\,d_{1,14},$$

$$0 == d_{1,1} - d_{1,4} + 2\,d_{1,12} + e_{1,1},\ 0 == \frac{d_{1,2}}{T} + d_{1,3} + \frac{d_{1,4}}{T} + d_{1,5} - \frac{2\,d_{1,9}}{T} - d_{1,10} - \frac{4\,d_{1,12}}{T} - 2\,d_{1,13} + \frac{e_{1,2}}{T},$$

$$0 == c_{1,6} + \frac{c_{1,7}}{T} + \frac{c_{1,8}}{T^2} + c_{1,9} + \frac{c_{1,10}}{T} + \frac{c_{1,11}}{T^2} + c_{1,12} + \frac{c_{1,13}}{T} + \frac{c_{1,14}}{T^2} + \frac{e_{1,3}}{T^2},$$

$$0 == \frac{d_{1,6}}{T^2} + \frac{d_{1,7}}{T} + d_{1,8} + \frac{d_{1,9}}{T^2} + \frac{d_{1,10}}{T} + d_{1,11} + \frac{d_{1,12}}{T^2} + \frac{d_{1,13}}{T} + d_{1,14} + \frac{e_{1,3}}{T^2},$$

$$0 == c_{1,1} - \frac{c_{1,3}}{T} + \frac{2\,c_{1,8}}{T^2} + e_{1,1} + e_{1,2} - \frac{e_{1,2}}{T} + 2\,e_{1,3} + \frac{2\,e_{1,3}}{T^2} - \frac{4\,e_{1,3}}{T},$$

$$0 == c_{1,2} + \frac{c_{1,3}}{T} + c_{1,4} + \frac{c_{1,5}}{T} - \frac{2\,c_{1,7}}{T} - \frac{4\,c_{1,8}}{T^2} - \frac{c_{1,10}}{T} - \frac{2\,c_{1,11}}{T^2} + \frac{e_{1,2}}{T} - \frac{4\,e_{1,3}}{T^2} + \frac{4\,e_{1,3}}{T},$$

$$0 == -\frac{c_{1,3}}{T} + c_{1,4} + \frac{2\,c_{1,8}}{T^2} - 2\,c_{1,12} + e_{1,1} + e_{1,2} - \frac{e_{1,2}}{T} + 2\,e_{1,3} + \frac{2\,e_{1,3}}{T^2} - \frac{4\,e_{1,3}}{T} - f_{1,1},$$

$$0 == c_{1,1} - c_{1,4} + 2\,c_{1,12} + f_{1,1},\ 0 == e_{1,1} + f_{1,1},\ 0 == e_{1,2} + f_{1,2},$$

$$0 == c_{1,2} - T\,c_{1,2} - c_{1,3} + \frac{c_{1,3}}{T} + c_{1,4} - T\,c_{1,4} - c_{1,5} + \frac{c_{1,5}}{T} - \frac{2\,c_{1,7}}{T} - \frac{4\,c_{1,8}}{T^2} +$$
$$2\,T\,c_{1,9} + c_{1,10} - \frac{c_{1,10}}{T} - \frac{2\,c_{1,11}}{T^2} + 4\,T\,c_{1,12} + 2\,c_{1,13} + \frac{e_{1,2}}{T} - \frac{4\,e_{1,3}}{T^2} + \frac{4\,e_{1,3}}{T} - T\,f_{1,2},$$
$$0 == T\,c_{1,2} + c_{1,3} + T\,c_{1,4} + c_{1,5} - 2\,T\,c_{1,9} - c_{1,10} - 4\,T\,c_{1,12} - 2\,c_{1,13} + T\,f_{1,2},$$
$$0 == -c_{1,2} + T\,c_{1,2} + c_{1,3} - 2\,c_{1,4} + \frac{c_{1,4}}{T} + T\,c_{1,4} + c_{1,5} - \frac{c_{1,5}}{T} + 4\,c_{1,9} - \frac{2\,c_{1,9}}{T} - 2\,T\,c_{1,9} - c_{1,10} + \frac{c_{1,10}}{T} +$$
$$12\,c_{1,12} + \frac{4\,c_{1,12}}{T^2} - \frac{12\,c_{1,12}}{T} - 4\,T\,c_{1,12} - 2\,c_{1,13} - \frac{2\,c_{1,13}}{T^2} + \frac{4\,c_{1,13}}{T} + T\,d_{1,3} - f_{1,2} + T\,f_{1,2},$$
$$0 == e_{1,3} + f_{1,3},\ 0 == -2\,c_{1,6} + 2\,T\,c_{1,6} + c_{1,7} - 4\,c_{1,9} + \frac{2\,c_{1,9}}{T} + 2\,T\,c_{1,9} + c_{1,10} - \frac{c_{1,10}}{T} -$$
$$6\,c_{1,12} - \frac{2\,c_{1,12}}{T^2} + \frac{6\,c_{1,12}}{T} + 2\,T\,c_{1,12} + c_{1,13} + \frac{c_{1,13}}{T^2} - \frac{2\,c_{1,13}}{T} + T\,d_{1,7} - 2\,f_{1,3} + 2\,T\,f_{1,3},$$
$$0 == d_{1,2} + T\,d_{1,3} + d_{1,4} + T\,d_{1,5} - 2\,T\,d_{1,7} - 4\,T^2\,d_{1,8} - T\,d_{1,10} - 2\,T^2\,d_{1,11} + T\,f_{1,2} + 4\,T\,f_{1,3} - 4\,T^2\,f_{1,3},$$
$$0 == c_{1,6} - T^2\,c_{1,6} + \frac{c_{1,7}}{T} - T\,c_{1,7} - c_{1,8} + \frac{c_{1,8}}{T^2} + c_{1,9} - T^2\,c_{1,9} + \frac{c_{1,10}}{T} - T\,c_{1,10} -$$
$$c_{1,11} + \frac{c_{1,11}}{T^2} + c_{1,12} - T^2\,c_{1,12} + \frac{c_{1,13}}{T} - T\,c_{1,13} - c_{1,14} + \frac{c_{1,14}}{T^2} + \frac{e_{1,3}}{T^2} - T^2\,f_{1,3},$$
$$0 == T^2\,c_{1,6} + T\,c_{1,7} + c_{1,8} + T^2\,c_{1,9} + T\,c_{1,10} + c_{1,11} + T^2\,c_{1,12} + T\,c_{1,13} + c_{1,14} + T^2\,f_{1,3},$$
$$0 == d_{1,6} + T\,d_{1,7} + T^2\,d_{1,8} + d_{1,9} + T\,d_{1,10} + T^2\,d_{1,11} + d_{1,12} + T\,d_{1,13} + T^2\,d_{1,14} + T^2\,f_{1,3},$$
$$0 == c_{1,6} - 2\,T\,c_{1,6} + T^2\,c_{1,6} - c_{1,7} + T\,c_{1,7} + c_{1,8} + 3\,c_{1,9} - \frac{c_{1,9}}{T} - 3\,T\,c_{1,9} + T^2\,c_{1,9} - 2\,c_{1,10} +$$
$$\frac{c_{1,10}}{T} + T\,c_{1,10} + c_{1,11} - \frac{c_{1,11}}{T} + 6\,c_{1,12} + \frac{c_{1,12}}{T^2} - \frac{4\,c_{1,12}}{T} - 4\,T\,c_{1,12} + T^2\,c_{1,12} - 3\,c_{1,13} -$$
$$\frac{c_{1,13}}{T^2} + \frac{3\,c_{1,13}}{T} + T\,c_{1,13} + c_{1,14} + \frac{c_{1,14}}{T^2} - \frac{2\,c_{1,14}}{T} + T^2\,d_{1,8} + f_{1,3} - 2\,T\,f_{1,3} + T^2\,f_{1,3},$$
$$0 == d_{1,1} - T\,d_{1,3} + 2\,T^2\,d_{1,8} + f_{1,1} + f_{1,2} - T\,f_{1,2} + 2\,f_{1,3} - 4\,T\,f_{1,3} + 2\,T^2\,f_{1,3}\Big\}$$

In[•]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=
$$\Big\{\Big\{c_{1,1} \to -\frac{c_{1,2}}{2} - \frac{c_{1,5}}{2\,T},\ c_{1,3} \to -T\,c_{1,2} - c_{1,5},\ c_{1,4} \to 0,\ c_{1,6} \to 0,\ c_{1,8} \to -\frac{1}{2} \times (1 - T)\,c_{1,10},\ c_{1,9} \to 0,$$
$$c_{1,11} \to -T\,c_{1,7} - \frac{1}{2} \times (-1 + 3\,T)\,c_{1,10},\ c_{1,12} \to 0,\ c_{1,13} \to 0,\ c_{1,14} \to 0,\ d_{1,1} \to \frac{c_{1,2}}{2} + \frac{c_{1,5}}{2\,T},$$
$$d_{1,2} \to -c_{1,2},\ d_{1,3} \to \frac{c_{1,2}}{T} + \frac{c_{1,5}}{T^2},\ d_{1,4} \to 0,\ d_{1,5} \to -\frac{c_{1,5}}{T^2},\ d_{1,6} \to 0,\ d_{1,7} \to -\frac{c_{1,7}}{T} - \frac{(-1 + T)\,c_{1,10}}{T^2},$$
$$d_{1,8} \to -\frac{(1 - T)\,c_{1,10}}{2\,T^3},\ d_{1,9} \to 0,\ d_{1,10} \to -\frac{c_{1,10}}{T^2},\ d_{1,11} \to \frac{c_{1,7}}{T^2} - \frac{(-1 - T)\,c_{1,10}}{2\,T^3},\ d_{1,12} \to 0,\ d_{1,13} \to 0,$$
$$d_{1,14} \to 0,\ e_{1,1} \to -\frac{c_{1,2}}{2} - \frac{c_{1,5}}{2\,T},\ e_{1,2} \to -\frac{c_{1,10}}{T},\ e_{1,3} \to 0,\ f_{1,1} \to \frac{c_{1,2}}{2} + \frac{c_{1,5}}{2\,T},\ f_{1,2} \to \frac{c_{1,10}}{T},\ f_{1,3} \to 0\Big\}\Big\}$$

In[•]:=
```
sol /. (a_ → b_) :> (a = b)
```

Out[•]=
$$\left\{-\frac{c_{1,2}}{2}-\frac{c_{1,5}}{2\,T},\ -T\,c_{1,2}-c_{1,5},\ 0,\ 0,\ -\frac{1}{2}\times(1-T)\,c_{1,10},\ 0,\ -T\,c_{1,7}-\frac{1}{2}\times(-1+3\,T)\,c_{1,10},\ 0,\ 0,\right.$$
$$0,\ \frac{c_{1,2}}{2}+\frac{c_{1,5}}{2\,T},\ -c_{1,2},\ \frac{c_{1,2}}{T}+\frac{c_{1,5}}{T^2},\ 0,\ -\frac{c_{1,5}}{T^2},\ 0,\ -\frac{c_{1,7}}{T}-\frac{(-1+T)\,c_{1,10}}{T^2},\ -\frac{(1-T)\,c_{1,10}}{2\,T^3},$$
$$\left.0,\ -\frac{c_{1,10}}{T^2},\ \frac{c_{1,7}}{T^2}-\frac{(-1-T)\,c_{1,10}}{2\,T^3},\ 0,\ 0,\ 0,\ -\frac{c_{1,2}}{2}-\frac{c_{1,5}}{2\,T},\ -\frac{c_{1,10}}{T},\ 0,\ \frac{c_{1,2}}{2}+\frac{c_{1,5}}{2\,T},\ \frac{c_{1,10}}{T},\ 0\right\}$$

In[•]:=
```
{R1,2, R̄1,2, C1, C̄1}
```

Out[•]=
$$\left\{\mathbb{E}_{\{\}\to\{1,2\}}\left[\sqrt{T},\ (-1+T)\,(p_1-p_2)\,x_2,\right.\right.$$
$$\in\text{Series}\left[0,\ -\frac{c_{1,2}}{2}+p_1\,x_1\,c_{1,2}+p_1\,x_2\,(-T\,c_{1,2}-c_{1,5})-\frac{c_{1,5}}{2\,T}+p_2\,x_2\,c_{1,5}+p_1^2\,x_1\,x_2\,c_{1,7}+\right.$$
$$\left.\left.p_1\,p_2\,x_1\,x_2\,c_{1,10}-\frac{1}{2}\times(1-T)\,p_1^2\,x_2^2\,c_{1,10}+p_1\,p_2\,x_2^2\left(-T\,c_{1,7}-\frac{1}{2}\times(-1+3\,T)\,c_{1,10}\right)\right]\right],$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\left[\frac{1}{\sqrt{T}},\ \left(-1+\frac{1}{T}\right)(p_1-p_2)\,x_2,\in\text{Series}\left[0,\right.\right.$$
$$\frac{c_{1,2}}{2}-p_1\,x_1\,c_{1,2}+\frac{c_{1,5}}{2\,T}-\frac{p_2\,x_2\,c_{1,5}}{T^2}+p_1\,x_2\left(\frac{c_{1,2}}{T}+\frac{c_{1,5}}{T^2}\right)-\frac{p_1\,p_2\,x_1\,x_2\,c_{1,10}}{T^2}-$$
$$\left.\left.\frac{(1-T)\,p_1^2\,x_2^2\,c_{1,10}}{2\,T^3}+p_1\,p_2\,x_2^2\left(\frac{c_{1,7}}{T^2}-\frac{(-1-T)\,c_{1,10}}{2\,T^3}\right)+p_1^2\,x_1\,x_2\left(-\frac{c_{1,7}}{T}-\frac{(-1+T)\,c_{1,10}}{T^2}\right)\right]\right],$$
$$\mathbb{E}_{\{\}\to\{1\}}\left[\sqrt{T},\ 0,\ \in\text{Series}\left[0,\ -\frac{c_{1,2}}{2}-\frac{c_{1,5}}{2\,T}-\frac{p_1\,x_1\,c_{1,10}}{T}\right]\right],$$
$$\left.\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{\sqrt{T}},\ 0,\ \in\text{Series}\left[0,\ \frac{c_{1,2}}{2}+\frac{c_{1,5}}{2\,T}+\frac{p_1\,x_1\,c_{1,10}}{T}\right]\right]\right\}$$

In[•]:=
```
Cases[{R1,2, R̄1,2, C1, C̄1}, (c | d | e | f)$k,_, ∞] // Union
```

Out[•]=
$$\{c_{1,2},\ c_{1,5},\ c_{1,7},\ c_{1,10}\}$$

In[•]:=
```
{c1,2 = 0, c1,5 = 0, c1,7 = 0, c1,10 = 1};
{R1,2, R̄1,2, C1, C̄1}
```

Out[•]=
$$\left\{\mathbb{E}_{\{\}\to\{1,2\}}\left[\sqrt{T},\ (-1+T)\,(p_1-p_2)\,x_2,\right.\right.$$
$$\in\text{Series}\left[0,\ p_1\,p_2\,x_1\,x_2+\frac{1}{2}\times(-1+T)\,p_1^2\,x_2^2+\frac{1}{2}\times(1-3\,T)\,p_1\,p_2\,x_2^2\right]\Big],$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\left[\frac{1}{\sqrt{T}},\ \left(-1+\frac{1}{T}\right)(p_1-p_2)\,x_2,\right.$$
$$\left.\in\text{Series}\left[0,\ -\frac{(-1+T)\,p_1^2\,x_1\,x_2}{T^2}-\frac{p_1\,p_2\,x_1\,x_2}{T^2}-\frac{(1-T)\,p_1^2\,x_2^2}{2\,T^3}-\frac{(-1-T)\,p_1\,p_2\,x_2^2}{2\,T^3}\right]\right],$$
$$\left.\mathbb{E}_{\{\}\to\{1\}}\left[\sqrt{T},\ 0,\ \in\text{Series}\left[0,\ -\frac{p_1\,x_1}{T}\right]\right],\ \mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{\sqrt{T}},\ 0,\ \in\text{Series}\left[0,\ \frac{p_1\,x_1}{T}\right]\right]\right\}$$

In[•]:=
```
RMoves
```

Out[•]=
```
{True, True, True, True, True, True, True, True, True}
```

Some Knot Theory at $k=1

In[•]:=
```
NewBit[K_] := Module[{Alex = Alexander[K][T]},
  T^3 Alex^2/(T - 1) ZF[K][[3, 2]] // Factor]
```

In[•]:=
```
NewBit /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=
$$\left\{2 - T + T^2,\ (1 + T) \times \left(1 - 3\,T + T^2\right),\ \frac{4 - 3\,T + 5\,T^2 - 3\,T^3 + 3\,T^4 - T^5 + T^6}{T^2},\ 9 - 11\,T + 7\,T^2 - T^3\right\}$$

In[•]:=
```
(*Two knots with equal Alexander, new bit does not agree*)
Alexander[Knot[6, 1]] == Alexander[Knot[9, 46]]
Timing[NewBit[Knot[6, 1]] == NewBit[Knot[9, 46]]]
```

Out[•]=
```
True
```

Out[•]=
$$\left\{33.,\ 5 - 11\,T - T^2 + 3\,T^3 == 7 - 21\,T + 9\,T^2 + T^3\right\}$$

In[•]:=
```
equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
Length@Union@Echo[ZF /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

» Out[•]=
$$\left\{\mathbb{E}_{\{\}\to\{1\}}\left[\frac{T^4}{1 - 4\,T + 9\,T^2 - 15\,T^3 + 17\,T^4 - 15\,T^5 + 9\,T^6 - 4\,T^7 + T^8},\ 0,\ \epsilon\text{Series}\left[0,\ \left(-3 + 20\,T - 69\,T^2 + 161\,T^3 - 272\,T^4 + 328\,T^5 - 225\,T^6 - 92\,T^7 + 548\,T^8 - 952\,T^9 + 1113\,T^{10} - 980\,T^{11} + 668\,T^{12} - 349\,T^{13} + 135\,T^{14} - 36\,T^{15} + 5\,T^{16}\right) \Big/ \left(T - 8\,T^2 + 34\,T^3 - 102\,T^4 + 235\,T^5 - 436\,T^6 + 669\,T^7 - 860\,T^8 + 935\,T^9 - 860\,T^{10} + 669\,T^{11} - 436\,T^{12} + 235\,T^{13} - 102\,T^{14} + 34\,T^{15} - 8\,T^{16} + T^{17}\right)\right]\right],$$
$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{T^4}{1 - 4\,T + 9\,T^2 - 15\,T^3 + 17\,T^4 - 15\,T^5 + 9\,T^6 - 4\,T^7 + T^8},\ 0,\ \epsilon\text{Series}\left[0,\ \left(-3 + 20\,T - 69\,T^2 + 161\,T^3 - 272\,T^4 + 328\,T^5 - 225\,T^6 - 92\,T^7 + 548\,T^8 - 952\,T^9 + 1113\,T^{10} - 980\,T^{11} + 668\,T^{12} - 349\,T^{13} + 135\,T^{14} - 36\,T^{15} + 5\,T^{16}\right) \Big/ \left(T - 8\,T^2 + 34\,T^3 - 102\,T^4 + 235\,T^5 - 436\,T^6 + 669\,T^7 - 860\,T^8 + 935\,T^9 - 860\,T^{10} + 669\,T^{11} - 436\,T^{12} + 235\,T^{13} - 102\,T^{14} + 34\,T^{15} - 8\,T^{16} + T^{17}\right)\right]\right]\right\}$$

Out[•]=
```
1
```

In[•]:=
```
equiv =
  {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
Length@Union[ZF /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

Out[•]=
```
1
```

Solving for R, C, \$k = 2

In[•]:=
```
$k = 2;
{R_{1,2}, C_1}
Short[RMoves, 20]
```

Out[•]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[\sqrt{T}\,,\ (-1+T)\ (p_1-p_2)\ x_2,$$
$$\in\text{Series}\Big[0,\ p_1\,p_2\,x_1\,x_2+\frac{1}{2}\times(-1+T)\ p_1^2\,x_2^2+\frac{1}{2}\times(1-3\,T)\ p_1\,p_2\,x_2^2,$$
$$c_{2,1}+p_1\,x_1\,c_{2,2}+p_1\,x_2\,c_{2,3}+p_2\,x_1\,c_{2,4}+p_2\,x_2\,c_{2,5}+p_1^2\,x_1^2\,c_{2,6}+p_1^2\,x_1\,x_2\,c_{2,7}+$$
$$p_1^2\,x_2^2\,c_{2,8}+p_1\,p_2\,x_1^2\,c_{2,9}+p_1\,p_2\,x_1\,x_2\,c_{2,10}+p_1\,p_2\,x_2^2\,c_{2,11}+p_2^2\,x_1^2\,c_{2,12}+p_2^2\,x_1\,x_2\,c_{2,13}+$$
$$p_2^2\,x_2^2\,c_{2,14}+p_1^3\,x_1^3\,c_{2,15}+p_1^3\,x_1^2\,x_2\,c_{2,16}+p_1^3\,x_1\,x_2^2\,c_{2,17}+p_1^3\,x_2^3\,c_{2,18}+p_1^2\,p_2\,x_1^3\,c_{2,19}+$$
$$p_1^2\,p_2\,x_1^2\,x_2\,c_{2,20}+p_1^2\,p_2\,x_1\,x_2^2\,c_{2,21}+p_1^2\,p_2\,x_2^3\,c_{2,22}+p_1\,p_2^2\,x_1^3\,c_{2,23}+p_1\,p_2^2\,x_1^2\,x_2\,c_{2,24}+$$
$$p_1\,p_2^2\,x_1\,x_2^2\,c_{2,25}+p_1\,p_2^2\,x_2^3\,c_{2,26}+p_2^3\,x_1^3\,c_{2,27}+p_2^3\,x_1^2\,x_2\,c_{2,28}+p_2^3\,x_1\,x_2^2\,c_{2,29}+p_2^3\,x_2^3\,c_{2,30}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T}\,,\ 0,\ \in\text{Series}\Big[0,\ -\frac{p_1\,x_1}{T},\ e_{2,1}+p_1\,x_1\,e_{2,2}+p_1^2\,x_1^2\,e_{2,3}+p_1^3\,x_1^3\,e_{2,4}\Big]\Big]\Big\}$$

Out[•]//Short=
$$\Big\{(-1+T)\ p_1^2\,p_3\,x_1\,x_2\,x_3-3\,T\,p_1\,p_2\,p_3\,x_1\,x_2\,x_3+\left(-2\,T+4\,T^2\right)\ p_1\,p_2\,p_3\,x_1\,x_3^2+3\,c_{2,1}+$$
$$2\,p_1\,x_1\,c_{2,2}+p_2\,x_1\,c_{2,4}+p_3\,x_1\,c_{2,4}+T\,p_3\,x_2\,c_{2,4}+p_1\,x_2\,(c_{2,2}-T\,c_{2,2}+c_{2,3}+c_{2,4}-T\,c_{2,4})+$$
$$2\,T\,p_3\,x_3\,c_{2,5}+\text{<<98>>}+\frac{1}{2}\,p_1^2\,p_3\,x_2\,x_3^2\,\left(T-T^3+2\,T\,c_{2,21}-4\,T^2\,c_{2,21}+2\,T^3\,c_{2,21}+\right.$$
$$\left.4\,T\,c_{2,25}-8\,T^2\,c_{2,25}+4\,T^3\,c_{2,25}+6\,T\,c_{2,29}-12\,T^2\,c_{2,29}+6\,T^3\,c_{2,29}\right)+2\,T^3\,p_3^3\,x_3^3\,c_{2,30}+$$
$$p_2^3\,x_2^3\,\left(T^3\,c_{2,15}+c_{2,30}\right)+p_2^3\,x_2^2\,x_3\,\left(T^3\,c_{2,16}+T\,c_{2,29}-T^2\,c_{2,29}+3\,c_{2,30}-3\,T\,c_{2,30}\right)+$$
$$p_2^3\,x_2\,x_3^2\,\left(T^3\,c_{2,17}+T^2\,c_{2,28}-2\,T^3\,c_{2,28}+T^4\,c_{2,28}+2\,T\,c_{2,29}-4\,T^2\,c_{2,29}+2\,T^3\,c_{2,29}+\right.$$
$$\left.3\,c_{2,30}-6\,T\,c_{2,30}+3\,T^2\,c_{2,30}\right)+p_1\,p_3^2\,x_3^3\,\left(T^2\,c_{2,26}+3\,T^2\,c_{2,30}-3\,T^3\,c_{2,30}\right)+$$
$$\frac{1}{2}\,p_1^3\,x_3^3\,\left(-1+3\,T-5\,T^2+5\,T^3-2\,T^4+2\,T^3\,c_{2,15}-6\,T^4\,c_{2,15}+6\,T^5\,c_{2,15}-2\,T^6\,c_{2,15}+\right.$$
$$2\,T^2\,c_{2,16}-6\,T^3\,c_{2,16}+6\,T^4\,c_{2,16}-2\,T^5\,c_{2,16}+2\,T\,c_{2,17}-6\,T^2\,c_{2,17}+6\,T^3\,c_{2,17}-2\,T^4\,c_{2,17}+$$
$$4\,c_{2,18}-12\,T\,c_{2,18}+12\,T^2\,c_{2,18}-2\,T^3\,c_{2,18}+2\,c_{2,22}-6\,T\,c_{2,22}+6\,T^2\,c_{2,22}-2\,T^3\,c_{2,22}+$$
$$2\,c_{2,26}-6\,T\,c_{2,26}+6\,T^2\,c_{2,26}-2\,T^3\,c_{2,26}+2\,c_{2,30}-6\,T\,c_{2,30}+6\,T^2\,c_{2,30}-2\,T^3\,c_{2,30}\Big)+$$
$$p_2^3\,x_3^3\,\left(T^3\,c_{2,18}+T^3\,c_{2,27}-3\,T^4\,c_{2,27}+3\,T^5\,c_{2,27}-T^6\,c_{2,27}+T^2\,c_{2,28}-3\,T^3\,c_{2,28}+3\,T^4\,c_{2,28}-\right.$$
$$\left.T^5\,c_{2,28}+T\,c_{2,29}-3\,T^2\,c_{2,29}+3\,T^3\,c_{2,29}-T^4\,c_{2,29}+c_{2,30}-3\,T\,c_{2,30}+3\,T^2\,c_{2,30}-T^3\,c_{2,30}\right)+$$
$$p_1^2\,p_3\,x_3^3\,\left(T^2-4\,T^3+3\,T^4+T\,c_{2,22}-2\,T^2\,c_{2,22}+2\,T^3\,c_{2,22}+2\,T\,c_{2,26}-4\,T^2\,c_{2,26}+\right.$$
$$\left.2\,T^3\,c_{2,26}+3\,T\,c_{2,30}-6\,T^2\,c_{2,30}+3\,T^3\,c_{2,30}\right)==$$
$$3\,c_{2,1}+2\,p_1\,x_1\,c_{2,2}+\text{<<143>>}+p_2^2\,p_3\,x_3^3\,\left(T^3\,c_{2,22}+3\,T\,c_{2,30}-6\,T^2\,c_{2,30}+3\,T^3\,c_{2,30}\right),$$
$$\text{<<7>>},\ \frac{p_1^3\ \text{<<1>>}\ (-1+\text{<<18>>})}{2\,T^3}+\frac{\text{<<1>>}}{2\ \text{<<1>>}}+\frac{\text{<<1>>}}{\text{<<1>>}}+\frac{\text{<<1>>}}{T^4}==0\Big\}$$

```
In[•]:= unknowns = Cases[{R_{1,2}, R̄_{1,2}, C_1, C̄_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

```
Out[•]= {c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}, c_{2,5}, c_{2,6}, c_{2,7}, c_{2,8}, c_{2,9}, c_{2,10}, c_{2,11}, c_{2,12}, c_{2,13}, c_{2,14},
  c_{2,15}, c_{2,16}, c_{2,17}, c_{2,18}, c_{2,19}, c_{2,20}, c_{2,21}, c_{2,22}, c_{2,23}, c_{2,24}, c_{2,25}, c_{2,26}, c_{2,27},
  c_{2,28}, c_{2,29}, c_{2,30}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{2,11},
  d_{2,12}, d_{2,13}, d_{2,14}, d_{2,15}, d_{2,16}, d_{2,17}, d_{2,18}, d_{2,19}, d_{2,20}, d_{2,21}, d_{2,22}, d_{2,23}, d_{2,24},
  d_{2,25}, d_{2,26}, d_{2,27}, d_{2,28}, d_{2,29}, d_{2,30}, e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}}
```

```
In[•]:= Short[errors = CCF /@ Cases[RMoves, a_ == b_ :> a - b], 25]
```

```
Out[•]//Short= {1/2 ×
    (2 p1 p2 x2 x3 - 2 T p1 p2 x2 x3 - 2 p1^2 p2 x1 x2 x3 + 2 T p1^2 p2 x1 x2 x3 + 2 p1 p2^2 x1 x2 x3 - 2 T p1 p2^2 x1 x2 x3 -
      2 p1^2 p3 x1 x2 x3 + 2 T p1^2 p3 x1 x2 x3 + 2 p1^3 x2^2 x3 - 4 T p1^3 x2^2 x3 + 2 T^2 p1^3 x2^2 x3 - 6 p1^2 p2 x2^2 x3 +
      16 T p1^2 p2 x2^2 x3 - 10 T^2 p1^2 p2 x2^2 x3 + p1 p2^2 x2^2 x3 - 6 T p1 p2^2 x2^2 x3 + 5 T^2 p1 p2^2 x2^2 x3 + 2 T p1 p2 p3 x2^2 x3 -
      2 T^2 p1 p2 p3 x2^2 x3 + p1^2 x3^2 - 2 T p1^2 x3^2 + T^2 p1^2 x3^2 + p1^2 p2 x1 x3^2 - T^2 p1^2 p2 x1 x3^2 - p1^2 p3 x1 x3^2 +
      4 T p1^2 p3 x1 x3^2 - 3 T^2 p1^2 p3 x1 x3^2 - 2 T p1 p2 p3 x1 x3^2 + 2 T^2 p1 p2 p3 x1 x3^2 + p1^3 x2 x3^2 -
      3 T p1^3 x2 x3^2 + 3 T^2 p1^3 x2 x3^2 - T^3 p1^3 x2 x3^2 - 4 p1^2 p2 x2 x3^2 + 14 T p1^2 p2 x2 x3^2 - 16 T^2 p1^2 p2 x2 x3^2 +
      6 T^3 p1^2 p2 x2 x3^2 - 3 T p1 p2^2 x2 x3^2 + 4 T^2 p1 p2^2 x2 x3^2 - T^3 p1 p2^2 x2 x3^2 + <<999>> + 12 T p2^3 x2 x3^2 c_{2,29} -
      20 T^2 p2^3 x2 x3^2 c_{2,29} + 12 T^3 p2^3 x2 x3^2 c_{2,29} - 2 T^4 p2^3 x2 x3^2 c_{2,29} + 6 T p1^2 p3 x2 x3^2 c_{2,29} -
      12 T^2 p1^2 p3 x2 x3^2 c_{2,29} + 6 T^3 p1^2 p3 x2 x3^2 c_{2,29} - 6 T p2^2 p3 x2 x3^2 c_{2,29} + 18 T^2 p2^2 p3 x2 x3^2 c_{2,29} -
      18 T^3 p2^2 p3 x2 x3^2 c_{2,29} + 6 T^4 p2^2 p3 x2 x3^2 c_{2,29} + 6 T^2 p1 p3^2 x2 x3^2 c_{2,29} - 6 T^3 p1 p3^2 x2 x3^2 c_{2,29} -
      6 T^2 p2 p3^2 x2 x3^2 c_{2,29} + 12 T^3 p2 p3^2 x2 x3^2 c_{2,29} - 6 T^4 p2 p3^2 x2 x3^2 c_{2,29} - 2 T^3 p3^3 x2 x3^2 c_{2,29} +
      2 T^4 p3^3 x2 x3^2 c_{2,29} + 2 T p2^3 x3^3 c_{2,29} - 6 T^2 p2^3 x3^3 c_{2,29} + 6 T^3 p2^3 x3^3 c_{2,29} - 2 T^4 p2^3 x3^3 c_{2,29} +
      6 p2^3 x2^2 x3 c_{2,30} - 6 T p2^3 x2^2 x3 c_{2,30} + 6 p2^3 x2 x3^2 c_{2,30} - 12 T p2^3 x2 x3^2 c_{2,30} + 6 T^2 p2^3 x2 x3^2 c_{2,30} +
      2 p1^3 x3^3 c_{2,30} - 6 T p1^3 x3^3 c_{2,30} + 6 T^2 p1^3 x3^3 c_{2,30} - 2 T^3 p1^3 x3^3 c_{2,30} + 6 T p1^2 p3 x3^3 c_{2,30} -
      12 T^2 p1^2 p3 x3^3 c_{2,30} + 6 T^3 p1^2 p3 x3^3 c_{2,30} - 6 T p2^2 p3 x3^3 c_{2,30} + 12 T^2 p2^2 p3 x3^3 c_{2,30} -
      6 T^3 p2^2 p3 x3^3 c_{2,30} + 6 T^2 p1 p3^3 x3^3 c_{2,30} - 6 T^3 p1 p3^3 x3^3 c_{2,30} - 6 T^2 p2 p3^3 x3^3 c_{2,30} + 6 T^3 p2 p3^3 x3^3 c_{2,30}),
  <<1>>/(2 T^4), <<1>>/<<1>>, <<4>>, <<1>>/(2 <<1>>), (<<106>> + 2 T <<1>> <<1>> e_{2,4})/(2 T^4)}
```

```
In[•]:= Short[#, 10] &[eqns =
  Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, p1, p2, p3}][[ ;; , 2]] & /@ errors)]]
```

```
Out[•]//Short= {0 == c_{2,4} - T c_{2,4}, 0 == -c_{2,4} + T c_{2,4}, <<215>>,
  0 == 1 + 2/T + d_{2,2} + T d_{2,3} + d_{2,4} + T d_{2,5} - 2 T d_{2,7} - 4 T^2 d_{2,8} - T d_{2,10} - 2 T^2 d_{2,11} + 6 T^2 d_{2,17} +
    18 T^3 d_{2,18} + 2 T^2 d_{2,21} + 6 T^3 d_{2,22} + T f_{2,2} + 4 T f_{2,3} - 4 T^2 f_{2,3} + 18 T f_{2,4} - 36 T^2 f_{2,4} + 18 T^3 f_{2,4}}
```

In[•]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

$$\left\{\left\{c_{2,1}\to -\frac{c_{2,2}}{2}-\frac{c_{2,5}}{2\,T},\ c_{2,3}\to -T\,c_{2,2}-c_{2,5},\ c_{2,4}\to 0,\ c_{2,6}\to 0,\ c_{2,8}\to -\frac{1}{2}\times(1-T)\,c_{2,10},\right.\right.$$
$$c_{2,9}\to 0,\ c_{2,11}\to -\frac{1}{2}-T\,c_{2,7}-\frac{1}{2}\times(-1+3\,T)\,c_{2,10},\ c_{2,12}\to 0,\ c_{2,13}\to 0,\ c_{2,14}\to 0,$$
$$c_{2,15}\to 0,\ c_{2,17}\to -((-1+T)\,c_{2,16}),\ c_{2,18}\to -\frac{-1+4\,T-3\,T^2}{6\,T},\ c_{2,19}\to 0,\ c_{2,20}\to -\frac{1}{2\,T},$$
$$c_{2,21}\to -\frac{1-3\,T}{2\,T},\ c_{2,22}\to -\frac{1-11\,T+16\,T^2}{6\,T}-\left(T-T^2\right)c_{2,16},\ c_{2,23}\to 0,\ c_{2,24}\to 0,$$
$$c_{2,25}\to -\frac{1}{2},\ c_{2,26}\to \frac{1}{6}\times(-1+7\,T)-T^2\,c_{2,16},\ c_{2,27}\to 0,\ c_{2,28}\to 0,\ c_{2,29}\to 0,\ c_{2,30}\to 0,$$
$$d_{2,1}\to \frac{c_{2,2}}{2}+\frac{c_{2,5}}{2\,T},\ d_{2,2}\to -c_{2,2},\ d_{2,3}\to \frac{c_{2,2}}{T}+\frac{c_{2,5}}{T^2},\ d_{2,4}\to 0,\ d_{2,5}\to -\frac{c_{2,5}}{T^2},\ d_{2,6}\to 0,$$
$$d_{2,7}\to -\frac{1-T}{T^3}-\frac{c_{2,7}}{T}-\frac{(-1+T)\,c_{2,10}}{T^2},\ d_{2,8}\to -\frac{-1+T}{2\,T^4}-\frac{(1-T)\,c_{2,10}}{2\,T^3},\ d_{2,9}\to 0,\ d_{2,10}\to \frac{1}{T^3}-\frac{c_{2,10}}{T^2},$$
$$d_{2,11}\to -\frac{1}{2\,T^4}+\frac{c_{2,7}}{T^2}-\frac{(-1-T)\,c_{2,10}}{2\,T^3},\ d_{2,12}\to 0,\ d_{2,13}\to 0,\ d_{2,14}\to 0,\ d_{2,15}\to 0,$$
$$d_{2,16}\to -\frac{-1+T}{2\,T^3}-\frac{c_{2,16}}{T},\ d_{2,17}\to -\frac{3-4\,T+T^2}{2\,T^4}-\frac{(-1+T)\,c_{2,16}}{T^2},\ d_{2,18}\to -\frac{-3+4\,T-T^2}{6\,T^5},\ d_{2,19}\to 0,$$
$$d_{2,20}\to -\frac{1}{2\,T^3},\ d_{2,21}\to \frac{2}{T^4},\ d_{2,22}\to -\frac{4+T+T^2}{6\,T^5}-\frac{(1-T)\,c_{2,16}}{T^3},\ d_{2,23}\to 0,\ d_{2,24}\to 0,\ d_{2,25}\to -\frac{1}{2\,T^4},$$
$$d_{2,26}\to -\frac{-1+T}{6\,T^5}+\frac{c_{2,16}}{T^3},\ d_{2,27}\to 0,\ d_{2,28}\to 0,\ d_{2,29}\to 0,\ d_{2,30}\to 0,\ e_{2,1}\to -\frac{c_{2,2}}{2}-\frac{c_{2,5}}{2\,T},$$
$$\left.\left.e_{2,2}\to -\frac{c_{2,10}}{T},\ e_{2,3}\to 0,\ e_{2,4}\to 0,\ f_{2,1}\to \frac{c_{2,2}}{2}+\frac{c_{2,5}}{2\,T},\ f_{2,2}\to -\frac{1}{T^2}+\frac{c_{2,10}}{T},\ f_{2,3}\to 0,\ f_{2,4}\to 0\right\}\right\}$$

In[•]:=
```
sol /. (a_ → b_) :> (a = b)
```

Out[•]=
$$\left\{-\frac{c_{2,2}}{2}-\frac{c_{2,5}}{2\,T},\ -T\,c_{2,2}-c_{2,5},\ 0,\ 0,\ -\frac{1}{2}\times(1-T)\,c_{2,10},\ 0,\ -\frac{1}{2}-T\,c_{2,7}-\frac{1}{2}\times(-1+3\,T)\,c_{2,10},\ 0,\ 0,\ 0,\right.$$
$$0,\ -((-1+T)\,c_{2,16}),\ -\frac{-1+4\,T-3\,T^2}{6\,T},\ 0,\ -\frac{1}{2\,T},\ -\frac{1-3\,T}{2\,T},\ -\frac{1-11\,T+16\,T^2}{6\,T}-\left(T-T^2\right)c_{2,16},$$
$$0,\ 0,\ -\frac{1}{2},\ \frac{1}{6}\times(-1+7\,T)-T^2\,c_{2,16},\ 0,\ 0,\ 0,\ 0,\ \frac{c_{2,2}}{2}+\frac{c_{2,5}}{2\,T},\ -c_{2,2},\ \frac{c_{2,2}}{T}+\frac{c_{2,5}}{T^2},\ 0,$$
$$-\frac{c_{2,5}}{T^2},\ 0,\ -\frac{1-T}{T^3}-\frac{c_{2,7}}{T}-\frac{(-1+T)\,c_{2,10}}{T^2},\ -\frac{-1+T}{2\,T^4}-\frac{(1-T)\,c_{2,10}}{2\,T^3},\ 0,\ \frac{1}{T^3}-\frac{c_{2,10}}{T^2},$$
$$-\frac{1}{2\,T^4}+\frac{c_{2,7}}{T^2}-\frac{(-1-T)\,c_{2,10}}{2\,T^3},\ 0,\ 0,\ 0,\ 0,\ -\frac{-1+T}{2\,T^3}-\frac{c_{2,16}}{T},\ -\frac{3-4\,T+T^2}{2\,T^4}-\frac{(-1+T)\,c_{2,16}}{T^2},$$
$$-\frac{-3+4\,T-T^2}{6\,T^5},\ 0,\ -\frac{1}{2\,T^3},\ \frac{2}{T^4},\ -\frac{4+T+T^2}{6\,T^5}-\frac{(1-T)\,c_{2,16}}{T^3},\ 0,\ 0,\ -\frac{1}{2\,T^4},\ -\frac{-1+T}{6\,T^5}+\frac{c_{2,16}}{T^3},$$
$$\left.0,\ 0,\ 0,\ 0,\ -\frac{c_{2,2}}{2}-\frac{c_{2,5}}{2\,T},\ -\frac{c_{2,10}}{T},\ 0,\ 0,\ \frac{c_{2,2}}{2}+\frac{c_{2,5}}{2\,T},\ -\frac{1}{T^2}+\frac{c_{2,10}}{T},\ 0,\ 0\right\}$$

In[•]:=
```
Cases[{R_{1,2}, \[RoundImplies]R_{1,2}, C_1, \[RoundImplies]C_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

Out[•]=
$$\{c_{2,2},\ c_{2,5},\ c_{2,7},\ c_{2,10},\ c_{2,16}\}$$

In[•]:=
```
(*{c2,2=0,c2,5=0,c2,7=0,c2,10=1,c2,16=0};*)
{R1,2, R̄1,2, C1, C̄1}
```

Out[•]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[\sqrt{T},\ (-1+T)\ (p_1-p_2)\ x_2,$$
$$\in\text{Series}\Big[0,\ p_1\,p_2\,x_1\,x_2+\frac{1}{2}\times(-1+T)\ p_1^2\,x_2^2+\frac{1}{2}\times(1-3\,T)\ p_1\,p_2\,x_2^2,$$
$$-\frac{p_1^2\,p_2\,x_1^2\,x_2}{2\,T}-\frac{(1-3\,T)\ p_1^2\,p_2\,x_1\,x_2^2}{2\,T}-\frac{1}{2}\,p_1\,p_2^2\,x_1\,x_2^2-\frac{\left(-1+4\,T-3\,T^2\right)\ p_1^3\,x_2^3}{6\,T}-\frac{c_{2,2}}{2}+p_1\,x_1\,c_{2,2}+$$
$$p_1\,x_2\ (-T\,c_{2,2}-c_{2,5})-\frac{c_{2,5}}{2\,T}+p_2\,x_2\,c_{2,5}+p_1^2\,x_1\,x_2\,c_{2,7}+p_1\,p_2\,x_1\,x_2\,c_{2,10}-\frac{1}{2}\times(1-T)\ p_1^2\,x_2^2\,c_{2,10}+$$
$$p_1\,p_2\,x_2^2\left(-\frac{1}{2}-T\,c_{2,7}-\frac{1}{2}\times(-1+3\,T)\ c_{2,10}\right)+p_1^3\,x_1^2\,x_2\,c_{2,16}-(-1+T)\ p_1^3\,x_1\,x_2^2\,c_{2,16}+$$
$$p_1\,p_2^2\,x_2^3\left(\frac{1}{6}\times(-1+7\,T)-T^2\,c_{2,16}\right)+p_1^2\,p_2\,x_2^3\left(-\frac{1-11\,T+16\,T^2}{6\,T}-\left(T-T^2\right)\ c_{2,16}\right)\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[\frac{1}{\sqrt{T}},\ \left(-1+\frac{1}{T}\right)\ (p_1-p_2)\ x_2,\ \in\text{Series}\Big[0,$$
$$-\frac{(-1+T)\ p_1^2\,x_1\,x_2}{T^2}-\frac{p_1\,p_2\,x_1\,x_2}{T^2}-\frac{(1-T)\ p_1^2\,x_2^2}{2\,T^3}-\frac{(-1-T)\ p_1\,p_2\,x_2^2}{2\,T^3},$$
$$-\frac{p_1^2\,p_2\,x_1^2\,x_2}{2\,T^3}+\frac{2\,p_1^2\,p_2\,x_1\,x_2^2}{T^4}-\frac{p_1\,p_2^2\,x_1\,x_2^2}{2\,T^4}-\frac{\left(-3+4\,T-T^2\right)\ p_1^3\,x_2^3}{6\,T^5}+\frac{c_{2,2}}{2}-p_1\,x_1\,c_{2,2}+\frac{c_{2,5}}{2\,T}-\frac{p_2\,x_2\,c_{2,5}}{T^2}+$$
$$p_1\,x_2\left(\frac{c_{2,2}}{T}+\frac{c_{2,5}}{T^2}\right)+p_1\,p_2\,x_2^2\left(-\frac{1}{2\,T^4}+\frac{c_{2,7}}{T^2}-\frac{(-1-T)\ c_{2,10}}{2\,T^3}\right)+p_1^2\,x_2^2\left(-\frac{-1+T}{2\,T^4}-\frac{(1-T)\ c_{2,10}}{2\,T^3}\right)+$$
$$p_1\,p_2\,x_1\,x_2\left(\frac{1}{T^3}-\frac{c_{2,10}}{T^2}\right)+p_1^2\,x_1\,x_2\left(-\frac{1-T}{T^3}-\frac{c_{2,7}}{T}-\frac{(-1+T)\ c_{2,10}}{T^2}\right)+$$
$$p_1\,p_2^2\,x_2^3\left(-\frac{-1+T}{6\,T^5}+\frac{c_{2,16}}{T^3}\right)+p_1^2\,p_2\,x_2^3\left(-\frac{4+T+T^2}{6\,T^5}-\frac{(1-T)\ c_{2,16}}{T^3}\right)+$$
$$p_1^3\,x_1\,x_2^2\left(-\frac{3-4\,T+T^2}{2\,T^4}-\frac{(-1+T)\ c_{2,16}}{T^2}\right)+p_1^3\,x_1^2\,x_2\left(-\frac{-1+T}{2\,T^3}-\frac{c_{2,16}}{T}\right)\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\ 0,\ \in\text{Series}\Big[0,\ -\frac{p_1\,x_1}{T},\ -\frac{c_{2,2}}{2}-\frac{c_{2,5}}{2\,T}-\frac{p_1\,x_1\,c_{2,10}}{T}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\ 0,\ \in\text{Series}\Big[0,\ \frac{p_1\,x_1}{T},\ \frac{c_{2,2}}{2}+\frac{c_{2,5}}{2\,T}+p_1\,x_1\left(-\frac{1}{T^2}+\frac{c_{2,10}}{T}\right)\Big]\Big]\Big\}$$

In[•]:=
```
RMoves
```

Out[•]=
```
{True, True, True, True, True, True, True, True, True}
```

Some Knot Theory at $k=2

According to 12XingStats.nb at pensieve://Projects/SL2Invariant/k=2/ the following pair have equal ρ_1 but different ρ_2:

```
In[•]:= equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
        Length@Union[ZF /@ equiv]
```

KnotTheory: Loading precomputed data in PD4Knots`.

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

```
Out[•]= 2
```

```
In[•]:= equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
        res21 = ZF /@ equiv
```

$$\text{Out}[\bullet]= \left\{\mathbb{E}_{\{\}\to\{1\}}\left[\frac{T^4}{1-4T+9T^2-15T^3+17T^4-15T^5+9T^6-4T^7+T^8},\ 0,\right.\right.$$
$$\in\text{Series}\left[0,\ \left(-3+20T-69T^2+161T^3-272T^4+328T^5-225T^6-92T^7+548T^8-952T^9+1113T^{10}-\right.\right.$$
$$\left.980T^{11}+668T^{12}-349T^{13}+135T^{14}-36T^{15}+5T^{16}\right)\Big/\left(T-8T^2+34T^3-102T^4+235T^5-436T^6+\right.$$
$$\left.669T^7-860T^8+935T^9-860T^{10}+669T^{11}-436T^{12}+235T^{13}-102T^{14}+34T^{15}-8T^{16}+T^{17}\right),$$
$$\left(3-40T+264T^2-1128T^3+3437T^4-7552T^5+10297T^6+2304T^7-67324T^8+259472T^9-\right.$$
$$699066T^{10}+1539252T^{11}-2919131T^{12}+4882760T^{13}-7290870T^{14}+9779044T^{15}-$$
$$11816854T^{16}+12877354T^{17}-12651386T^{18}+11191592T^{19}-8896165T^{20}+6336738T^{21}-$$
$$4030390T^{22}+2278962T^{23}-1139320T^{24}+500046T^{25}-190857T^{26}+62504T^{27}-17215T^{28}+$$
$$3862T^{29}-668T^{30}+80T^{31}-5T^{32}-8T^2c_{2,2}+120T^3c_{2,2}-924T^4c_{2,2}+4862T^5c_{2,2}-$$
$$19548T^6c_{2,2}+63624T^7c_{2,2}-173490T^8c_{2,2}+405054T^9c_{2,2}-821164T^{10}c_{2,2}+$$
$$1457946T^{11}c_{2,2}-2275932T^{12}c_{2,2}+3120910T^{13}c_{2,2}-3730618T^{14}c_{2,2}+3810432T^{15}c_{2,2}-$$
$$3161604T^{16}c_{2,2}+1801302T^{17}c_{2,2}-1801302T^{19}c_{2,2}+3161604T^{20}c_{2,2}-3810432T^{21}c_{2,2}+$$
$$3730618T^{22}c_{2,2}-3120910T^{23}c_{2,2}+2275932T^{24}c_{2,2}-1457946T^{25}c_{2,2}+821164T^{26}c_{2,2}-$$
$$405054T^{27}c_{2,2}+173490T^{28}c_{2,2}-63624T^{29}c_{2,2}+19548T^{30}c_{2,2}-4862T^{31}c_{2,2}+$$
$$924T^{32}c_{2,2}-120T^{33}c_{2,2}+8T^{34}c_{2,2}-8Tc_{2,5}+120T^2c_{2,5}-924T^3c_{2,5}+4862T^4c_{2,5}-$$
$$19548T^5c_{2,5}+63624T^6c_{2,5}-173490T^7c_{2,5}+405054T^8c_{2,5}-821164T^9c_{2,5}+$$
$$1457946T^{10}c_{2,5}-2275932T^{11}c_{2,5}+3120910T^{12}c_{2,5}-3730618T^{13}c_{2,5}+3810432T^{14}c_{2,5}-$$
$$3161604T^{15}c_{2,5}+1801302T^{16}c_{2,5}-1801302T^{18}c_{2,5}+3161604T^{19}c_{2,5}-3810432T^{20}c_{2,5}+$$
$$3730618T^{21}c_{2,5}-3120910T^{22}c_{2,5}+2275932T^{23}c_{2,5}-1457946T^{24}c_{2,5}+821164T^{25}c_{2,5}-$$
$$405054T^{26}c_{2,5}+173490T^{27}c_{2,5}-63624T^{28}c_{2,5}+19548T^{29}c_{2,5}-4862T^{30}c_{2,5}+$$
$$924T^{31}c_{2,5}-120T^{32}c_{2,5}+8T^{33}c_{2,5}-6Tc_{2,10}+88T^2c_{2,10}-662T^3c_{2,10}+3398T^4c_{2,10}-$$
$$13302T^5c_{2,10}+42048T^6c_{2,10}-110922T^7c_{2,10}+248966T^8c_{2,10}-480208T^9c_{2,10}+$$
$$796974T^{10}c_{2,10}-1126576T^{11}c_{2,10}+1312494T^{12}c_{2,10}-1137250T^{13}c_{2,10}+400328T^{14}c_{2,10}+$$
$$969964T^{15}c_{2,10}-2827546T^{16}c_{2,10}+4806348T^{17}c_{2,10}-6430150T^{18}c_{2,10}+$$
$$7293172T^{19}c_{2,10}-7220536T^{20}c_{2,10}+6323986T^{21}c_{2,10}-4929326T^{22}c_{2,10}+$$
$$3425288T^{23}c_{2,10}-2118918T^{24}c_{2,10}+1162120T^{25}c_{2,10}-561142T^{26}c_{2,10}+236058T^{27}c_{2,10}-$$
$$\left.85200T^{28}c_{2,10}+25794T^{29}c_{2,10}-6326T^{30}c_{2,10}+1186T^{31}c_{2,10}-152T^{32}c_{2,10}+10T^{33}c_{2,10}\right)\Big/$$
$$\left(2T^2-32T^3+264T^4-1496T^5+6516T^6-23136T^7+69396T^8-180024T^9+410582T^{10}-\right.$$
$$833112T^{11}+1517288T^{12}-2496728T^{13}+3730618T^{14}-5080576T^{15}+6323208T^{16}-$$
$$7205208T^{17}+7524878T^{18}-7205208T^{19}+6323208T^{20}-5080576T^{21}+$$
$$3730618T^{22}-2496728T^{23}+1517288T^{24}-833112T^{25}+410582T^{26}-180024T^{27}+$$

$$69\,396\,T^{28} - 23\,136\,T^{29} + 6516\,T^{30} - 1496\,T^{31} + 264\,T^{32} - 32\,T^{33} + 2\,T^{34}\big)\big]\big],$$

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{T^4}{1 - 4\,T + 9\,T^2 - 15\,T^3 + 17\,T^4 - 15\,T^5 + 9\,T^6 - 4\,T^7 + T^8},$$

$$0,$$

$$\epsilon\text{Series}\big[0,$$

$$\big(-3 + 20\,T - 69\,T^2 + 161\,T^3 - 272\,T^4 + 328\,T^5 - 225\,T^6 - 92\,T^7 + 548\,T^8 -$$
$$952\,T^9 + 1113\,T^{10} - 980\,T^{11} + 668\,T^{12} - 349\,T^{13} + 135\,T^{14} - 36\,T^{15} + 5\,T^{16}\big) \Big/$$
$$\big(T - 8\,T^2 + 34\,T^3 - 102\,T^4 + 235\,T^5 - 436\,T^6 + 669\,T^7 - 860\,T^8 + 935\,T^9 - 860\,T^{10} +$$
$$669\,T^{11} - 436\,T^{12} + 235\,T^{13} - 102\,T^{14} + 34\,T^{15} - 8\,T^{16} + T^{17}\big),$$

$$\big(3 - 40\,T + 264\,T^2 - 1120\,T^3 + 3333\,T^4 - 6896\,T^5 + 7641\,T^6 + 9944\,T^7 - 83\,404\,T^8 + 283\,088\,T^9 -$$
$$716\,082\,T^{10} + 1\,514\,140\,T^{11} - 2\,796\,883\,T^{12} + 4\,607\,952\,T^{13} - 6\,839\,214\,T^{14} + 9\,183\,044\,T^{15} -$$
$$11\,164\,950\,T^{16} + 12\,281\,354\,T^{17} - 12\,199\,730\,T^{18} + 10\,916\,784\,T^{19} - 8\,773\,917\,T^{20} + 6\,311\,626\,T^{21} -$$
$$4\,047\,406\,T^{22} + 2\,302\,578\,T^{23} - 1\,155\,400\,T^{24} + 507\,686\,T^{25} - 193\,513\,T^{26} + 63\,160\,T^{27} - 17\,319\,T^{28} +$$
$$3870\,T^{29} - 668\,T^{30} + 80\,T^{31} - 5\,T^{32} - 8\,T^2\,c_{2,2} + 120\,T^3\,c_{2,2} - 924\,T^4\,c_{2,2} + 4862\,T^5\,c_{2,2} -$$
$$19\,548\,T^6\,c_{2,2} + 63\,624\,T^7\,c_{2,2} - 173\,490\,T^8\,c_{2,2} + 405\,054\,T^9\,c_{2,2} - 821\,164\,T^{10}\,c_{2,2} +$$
$$1\,457\,946\,T^{11}\,c_{2,2} - 2\,275\,932\,T^{12}\,c_{2,2} + 3\,120\,910\,T^{13}\,c_{2,2} - 3\,730\,618\,T^{14}\,c_{2,2} + 3\,810\,432\,T^{15}\,c_{2,2} -$$
$$3\,161\,604\,T^{16}\,c_{2,2} + 1\,801\,302\,T^{17}\,c_{2,2} - 1\,801\,302\,T^{19}\,c_{2,2} + 3\,161\,604\,T^{20}\,c_{2,2} - 3\,810\,432\,T^{21}\,c_{2,2} +$$
$$3\,730\,618\,T^{22}\,c_{2,2} - 3\,120\,910\,T^{23}\,c_{2,2} + 2\,275\,932\,T^{24}\,c_{2,2} - 1\,457\,946\,T^{25}\,c_{2,2} + 821\,164\,T^{26}\,c_{2,2} -$$
$$405\,054\,T^{27}\,c_{2,2} + 173\,490\,T^{28}\,c_{2,2} - 63\,624\,T^{29}\,c_{2,2} + 19\,548\,T^{30}\,c_{2,2} - 4862\,T^{31}\,c_{2,2} +$$
$$924\,T^{32}\,c_{2,2} - 120\,T^{33}\,c_{2,2} + 8\,T^{34}\,c_{2,2} - 8\,T\,c_{2,5} + 120\,T^2\,c_{2,5} - 924\,T^3\,c_{2,5} + 4862\,T^4\,c_{2,5} -$$
$$19\,548\,T^5\,c_{2,5} + 63\,624\,T^6\,c_{2,5} - 173\,490\,T^7\,c_{2,5} + 405\,054\,T^8\,c_{2,5} - 821\,164\,T^9\,c_{2,5} +$$
$$1\,457\,946\,T^{10}\,c_{2,5} - 2\,275\,932\,T^{11}\,c_{2,5} + 3\,120\,910\,T^{12}\,c_{2,5} - 3\,730\,618\,T^{13}\,c_{2,5} + 3\,810\,432\,T^{14}\,c_{2,5} -$$
$$3\,161\,604\,T^{15}\,c_{2,5} + 1\,801\,302\,T^{16}\,c_{2,5} - 1\,801\,302\,T^{18}\,c_{2,5} + 3\,161\,604\,T^{19}\,c_{2,5} - 3\,810\,432\,T^{20}\,c_{2,5} +$$
$$3\,730\,618\,T^{21}\,c_{2,5} - 3\,120\,910\,T^{22}\,c_{2,5} + 2\,275\,932\,T^{23}\,c_{2,5} - 1\,457\,946\,T^{24}\,c_{2,5} + 821\,164\,T^{25}\,c_{2,5} -$$
$$405\,054\,T^{26}\,c_{2,5} + 173\,490\,T^{27}\,c_{2,5} - 63\,624\,T^{28}\,c_{2,5} + 19\,548\,T^{29}\,c_{2,5} - 4862\,T^{30}\,c_{2,5} +$$
$$924\,T^{31}\,c_{2,5} - 120\,T^{32}\,c_{2,5} + 8\,T^{33}\,c_{2,5} - 6\,T\,c_{2,10} + 88\,T^2\,c_{2,10} - 662\,T^3\,c_{2,10} + 3398\,T^4\,c_{2,10} -$$
$$13\,302\,T^5\,c_{2,10} + 42\,048\,T^6\,c_{2,10} - 110\,922\,T^7\,c_{2,10} + 248\,966\,T^8\,c_{2,10} - 480\,208\,T^9\,c_{2,10} +$$
$$796\,974\,T^{10}\,c_{2,10} - 1\,126\,576\,T^{11}\,c_{2,10} + 1\,312\,494\,T^{12}\,c_{2,10} - 1\,137\,250\,T^{13}\,c_{2,10} + 400\,328\,T^{14}\,c_{2,10} +$$
$$969\,964\,T^{15}\,c_{2,10} - 2\,827\,546\,T^{16}\,c_{2,10} + 4\,806\,348\,T^{17}\,c_{2,10} - 6\,430\,150\,T^{18}\,c_{2,10} +$$
$$7\,293\,172\,T^{19}\,c_{2,10} - 7\,220\,536\,T^{20}\,c_{2,10} + 6\,323\,986\,T^{21}\,c_{2,10} - 4\,929\,326\,T^{22}\,c_{2,10} +$$
$$3\,425\,288\,T^{23}\,c_{2,10} - 2\,118\,918\,T^{24}\,c_{2,10} + 1\,162\,120\,T^{25}\,c_{2,10} - 561\,142\,T^{26}\,c_{2,10} + 236\,058\,T^{27}\,c_{2,10} -$$
$$85\,200\,T^{28}\,c_{2,10} + 25\,794\,T^{29}\,c_{2,10} - 6326\,T^{30}\,c_{2,10} + 1186\,T^{31}\,c_{2,10} - 152\,T^{32}\,c_{2,10} + 10\,T^{33}\,c_{2,10}\big) \Big/$$
$$\big(2\,T^2 - 32\,T^3 + 264\,T^4 - 1496\,T^5 + 6516\,T^6 - 23\,136\,T^7 + 69\,396\,T^8 - 180\,024\,T^9 + 410\,582\,T^{10} -$$
$$833\,112\,T^{11} + 1\,517\,288\,T^{12} - 2\,496\,728\,T^{13} + 3\,730\,618\,T^{14} - 5\,080\,576\,T^{15} + 6\,323\,208\,T^{16} -$$
$$7\,205\,208\,T^{17} + 7\,524\,878\,T^{18} - 7\,205\,208\,T^{19} + 6\,323\,208\,T^{20} - 5\,080\,576\,T^{21} +$$
$$3\,730\,618\,T^{22} - 2\,496\,728\,T^{23} + 1\,517\,288\,T^{24} - 833\,112\,T^{25} + 410\,582\,T^{26} - 180\,024\,T^{27} +$$
$$69\,396\,T^{28} - 23\,136\,T^{29} + 6516\,T^{30} - 1496\,T^{31} + 264\,T^{32} - 32\,T^{33} + 2\,T^{34}\big)\big]\big]\Big\}$$

According to 12XingStats.nb at pensieve://Projects/SL2Invariant/k=2/ the following triple have equal ρ_1 but different ρ_2:

```
In[•]:= equiv =
          {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
        Length@Union[res22 = ZF /@ equiv]

        KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

Out[•]= $Aborted
```

Solving for $R, C, \$k = 3$

```
In[•]:= $k = 3;
        Short[RMoves, 20]
```

Out[•]//Short=

$$\Big\{(-2+2\,T)\,p_1^2\,p_3\,x_1\,x_2\,x_3+(-1-6\,T)\,p_1\,p_2\,p_3\,x_1\,x_2\,x_3+\frac{(1-T)\,p_1^3\,p_3\,x_1^2\,x_2\,x_3}{2\,T}+4\,p_1^2\,p_2\,p_3\,x_1^2\,x_2\,x_3+$$

$$\frac{\left(1-4\,T+3\,T^2\right)\,p_1^3\,p_3\,x_1\,x_2^2\,x_3}{2\,T}+\text{<<263>>}+p_1\,p_3^3\,x_3^4\,\left(T^3\,c_{3,50}+4\,T^3\,c_{3,55}-4\,T^4\,c_{3,55}\right)+$$

$$p_2^4\,x_3^4\,\left(T^4\,c_{3,35}+T^4\,c_{3,51}-4\,T^5\,c_{3,51}+6\,T^6\,c_{3,51}-4\,T^7\,c_{3,51}+T^8\,c_{3,51}+T^3\,c_{3,52}-4\,T^4\,c_{3,52}+6\,T^5\,c_{3,52}-\right.$$
$$\left.4\,T^6\,c_{3,52}+T^7\,c_{3,52}+T^2\,c_{3,53}-4\,T^3\,c_{3,53}+6\,T^4\,c_{3,53}-4\,T^5\,c_{3,53}+T^6\,c_{3,53}+T\,c_{3,54}-4\,T^2\,c_{3,54}+\right.$$
$$\left.6\,T^3\,c_{3,54}-4\,T^4\,c_{3,54}+T^5\,c_{3,54}+c_{3,55}-4\,T\,c_{3,55}+6\,T^2\,c_{3,55}-4\,T^3\,c_{3,55}+T^4\,c_{3,55}\right)+$$

$$\frac{1}{24}\,p_1^2\,p_3^2\,x_3^4\,\left(7\,T^2-97\,T^3+329\,T^4-239\,T^5+24\,T^2\,c_{3,45}-48\,T^3\,c_{3,45}+48\,T^4\,c_{3,45}+\right.$$
$$\left.72\,T^2\,c_{3,50}-144\,T^3\,c_{3,50}+72\,T^4\,c_{3,50}+144\,T^2\,c_{3,55}-288\,T^3\,c_{3,55}+144\,T^4\,c_{3,55}\right)+$$

$$\frac{1}{12\,T}\,p_1^4\,x_3^4\,\left(5-19\,T+13\,T^2+38\,T^3-89\,T^4+77\,T^5-25\,T^6+12\,T^5\,c_{3,31}-48\,T^6\,c_{3,31}+72\,T^7\,c_{3,31}-\right.$$
$$48\,T^8\,c_{3,31}+12\,T^9\,c_{3,31}+12\,T^4\,c_{3,32}-48\,T^5\,c_{3,32}+72\,T^6\,c_{3,32}-48\,T^7\,c_{3,32}+12\,T^8\,c_{3,32}+$$
$$12\,T^3\,c_{3,33}-48\,T^4\,c_{3,33}+72\,T^5\,c_{3,33}-48\,T^6\,c_{3,33}+12\,T^7\,c_{3,33}+12\,T^2\,c_{3,34}-48\,T^3\,c_{3,34}+$$
$$72\,T^4\,c_{3,34}-48\,T^5\,c_{3,34}+12\,T^6\,c_{3,34}+24\,T\,c_{3,35}-96\,T^2\,c_{3,35}+144\,T^3\,c_{3,35}-96\,T^4\,c_{3,35}+$$
$$36\,T^5\,c_{3,35}+12\,T\,c_{3,40}-48\,T^2\,c_{3,40}+72\,T^3\,c_{3,40}-48\,T^4\,c_{3,40}+12\,T^5\,c_{3,40}+12\,T\,c_{3,45}-$$
$$48\,T^2\,c_{3,45}+72\,T^3\,c_{3,45}-48\,T^4\,c_{3,45}+12\,T^5\,c_{3,45}+12\,T\,c_{3,50}-48\,T^2\,c_{3,50}+72\,T^3\,c_{3,50}-$$
$$\left.48\,T^4\,c_{3,50}+12\,T^5\,c_{3,50}+12\,T\,c_{3,55}-48\,T^2\,c_{3,55}+72\,T^3\,c_{3,55}-48\,T^4\,c_{3,55}+12\,T^5\,c_{3,55}\right)==$$

$$3\,c_{3,1}+2\,p_1\,x_1\,c_{3,2}+\text{<<368>>}+p_2^2\,p_3^2\,x_3^4\,\left(T^4\,c_{3,45}+6\,T^2\,c_{3,55}-12\,T^3\,c_{3,55}+6\,T^4\,c_{3,55}\right),$$

$$\text{<<7>>},$$

$$\frac{p_1^4\,\text{<<1>>}\,(\text{<<1>>})}{12\,T^5}+\text{<<3>>}+\frac{\text{<<1>>}}{\text{<<1>>}}==$$

$$0\Big\}$$

In[•]:=
```
unknowns = Cases[{R_{1,2}, R̄_{1,2}, C_1, C̄_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

Out[•]=
$$\{c_{3,1}, c_{3,2}, c_{3,3}, c_{3,4}, c_{3,5}, c_{3,6}, c_{3,7}, c_{3,8}, c_{3,9}, c_{3,10}, c_{3,11}, c_{3,12}, c_{3,13}, c_{3,14}, c_{3,15}, c_{3,16},$$
$$c_{3,17}, c_{3,18}, c_{3,19}, c_{3,20}, c_{3,21}, c_{3,22}, c_{3,23}, c_{3,24}, c_{3,25}, c_{3,26}, c_{3,27}, c_{3,28}, c_{3,29}, c_{3,30}, c_{3,31},$$
$$c_{3,32}, c_{3,33}, c_{3,34}, c_{3,35}, c_{3,36}, c_{3,37}, c_{3,38}, c_{3,39}, c_{3,40}, c_{3,41}, c_{3,42}, c_{3,43}, c_{3,44}, c_{3,45},$$
$$c_{3,46}, c_{3,47}, c_{3,48}, c_{3,49}, c_{3,50}, c_{3,51}, c_{3,52}, c_{3,53}, c_{3,54}, c_{3,55}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, d_{3,5},$$
$$d_{3,6}, d_{3,7}, d_{3,8}, d_{3,9}, d_{3,10}, d_{3,11}, d_{3,12}, d_{3,13}, d_{3,14}, d_{3,15}, d_{3,16}, d_{3,17}, d_{3,18}, d_{3,19}, d_{3,20},$$
$$d_{3,21}, d_{3,22}, d_{3,23}, d_{3,24}, d_{3,25}, d_{3,26}, d_{3,27}, d_{3,28}, d_{3,29}, d_{3,30}, d_{3,31}, d_{3,32}, d_{3,33}, d_{3,34}, d_{3,35},$$
$$d_{3,36}, d_{3,37}, d_{3,38}, d_{3,39}, d_{3,40}, d_{3,41}, d_{3,42}, d_{3,43}, d_{3,44}, d_{3,45}, d_{3,46}, d_{3,47}, d_{3,48}, d_{3,49},$$
$$d_{3,50}, d_{3,51}, d_{3,52}, d_{3,53}, d_{3,54}, d_{3,55}, e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, f_{3,5}\}$$

In[•]:=
```
Short[errors = CCF /@ Cases[RMoves, a_ == b_ :> a - b], 25]
```

Out[•]//Short=
$$\Big\{\frac{1}{24\,T}\big(48\,T\,p_1\,p_2\,x_2\,x_3 - 48\,T^2\,p_1\,p_2\,x_2\,x_3 - 24\,p_1^2\,p_2\,x_1\,x_2\,x_3 - 24\,T\,p_1^2\,p_2\,x_1\,x_2\,x_3 + 48\,T^2\,p_1^2\,p_2\,x_1\,x_2\,x_3 +$$
$$48\,T\,p_1\,p_2^2\,x_1\,x_2\,x_3 - 48\,T^2\,p_1\,p_2^2\,x_1\,x_2\,x_3 - 48\,T\,p_1^2\,p_3\,x_1\,x_2\,x_3 + 48\,T^2\,p_1^2\,p_3\,x_1\,x_2\,x_3 + 12\,p_1^3\,p_2\,x_1^2\,x_2\,x_3 -$$
$$12\,T\,p_1^3\,p_2\,x_1^2\,x_2\,x_3 - 48\,p_1^2\,p_2^2\,x_1^2\,x_2\,x_3 + 48\,T\,p_1^2\,p_2^2\,x_1^2\,x_2\,x_3 + 12\,p_1^3\,p_3\,x_1^2\,x_2\,x_3 - 12\,T\,p_1^3\,p_3\,x_1^2\,x_2\,x_3 +$$
$$12\,p_1^3\,x_2^2\,x_3 + 24\,T\,p_1^3\,x_2^2\,x_3 - 84\,T^2\,p_1^3\,x_2^2\,x_3 + \ll 3718 \gg + 96\,T^2\,p_1^3\,p_3\,x_3^4\,c_{3,55} - 288\,T^3\,p_1^3\,p_3\,x_3^4\,c_{3,55} +$$
$$288\,T^4\,p_1^3\,p_3\,x_3^4\,c_{3,55} - 96\,T^5\,p_1^3\,p_3\,x_3^4\,c_{3,55} - 96\,T^2\,p_2^3\,p_3\,x_3^4\,c_{3,55} + 288\,T^3\,p_2^3\,p_3\,x_3^4\,c_{3,55} -$$
$$288\,T^4\,p_2^3\,p_3\,x_3^4\,c_{3,55} + 96\,T^5\,p_2^3\,p_3\,x_3^4\,c_{3,55} + 144\,T^3\,p_1^2\,p_2^2\,x_3^4\,c_{3,55} - 288\,T^4\,p_1^2\,p_2^2\,x_3^4\,c_{3,55} +$$
$$144\,T^5\,p_1^2\,p_2^2\,x_3^4\,c_{3,55} - 144\,T^3\,p_2^2\,p_3^2\,x_3^4\,c_{3,55} + 288\,T^4\,p_2^2\,p_3^2\,x_3^4\,c_{3,55} - 144\,T^5\,p_2^2\,p_3^2\,x_3^4\,c_{3,55} +$$
$$96\,T^4\,p_1\,p_3^3\,x_3^4\,c_{3,55} - 96\,T^5\,p_1\,p_3^3\,x_3^4\,c_{3,55} - 96\,T^4\,p_2\,p_3^3\,x_3^4\,c_{3,55} + 96\,T^5\,p_2\,p_3^3\,x_3^4\,c_{3,55}\big),$$
$$\frac{1}{24\,T^6}\big(24\,T^3\,p_1\,p_2\,x_1\,x_2 - 48\,T^4\,p_1\,p_2\,x_1\,x_2 - 24\,T^3\,p_1^2\,p_2\,x_1^2\,x_2 + 48\,T^4\,p_1^2\,p_2\,x_1^2\,x_2 - 12\,T^2\,p_1\,p_2\,x_2^2 +$$
$$\ll 858 \gg + 144\,T^{10}\,p_1^2\,p_2^2\,x_2^4\,d_{3,55} + 96\,T^9\,p_1\,p_2^3\,x_2^4\,d_{3,55} - 96\,T^{10}\,p_1\,p_2^3\,x_2^4\,d_{3,55} + 24\,T^{10}\,p_2^4\,x_2^4\,d_{3,55}\big),$$
$$\frac{-p_1\,x_1 + 2\,T\,p_1\,x_1 + \ll 8 \gg + T^3\,p_1^3\,x_1^3\,f_{3,4} + T^3\,p_1^4\,x_1^4\,f_{3,5}}{T^3},$$
$$\frac{\ll 1 \gg}{24\,T^4},$$
$$\frac{\ll 1 \gg}{12\,T^6},$$
$$\frac{-12\,p_1\,x_1 + \ll 158 \gg + 12\,\ll 4 \gg}{12\,T^2},$$
$$\frac{108 - 144\,T + \ll 208 \gg + 12\,T^7\,p_1^4\,x_1^4\,f_{3,5}}{12\,T^3}, \frac{1}{12\,T^6}$$
$$\big(-12\,T^2\,p_1\,x_1 + 24\,T^3\,p_1\,x_1 + 18\,T\,p_1^2\,x_1^2 + \ll 153 \gg +$$
$$12\,T^5\,p_1\,x_1\,e_{3,2} + 12\,T^4\,p_1^2\,x_1^2\,e_{3,3} + 12\,T^3\,p_1^3\,x_1^3\,e_{3,4} + 12\,T^2\,p_1^4\,x_1^4\,e_{3,5}\big),$$
$$\frac{1}{12\,T^6}\big(12 - 144\,T + 336\,T^2 - 276\,T^3 + 72\,T^4 - 36\,p_1\,x_1 + 216\,T\,p_1\,x_1 + 60\,T^2\,p_1\,x_1 - 360\,T^3\,p_1\,x_1 +$$
$$72\,T^4\,p_1\,x_1 + 18\,p_1^2\,x_1^2 + 60\,T\,p_1^2\,x_1^2 - 588\,T^2\,p_1^2\,x_1^2 + \ll 186 \gg + 1728\,T^4\,e_{3,5} - 1152\,T^5\,e_{3,5} + 288\,T^6\,e_{3,5} -$$
$$1152\,T^2\,p_1\,x_1\,e_{3,5} + 3456\,T^3\,p_1\,x_1\,e_{3,5} - 3456\,T^4\,p_1\,x_1\,e_{3,5} + 1152\,T^5\,p_1\,x_1\,e_{3,5} + 864\,T^2\,p_1^2\,x_1^2\,e_{3,5} -$$
$$1728\,T^3\,p_1^2\,x_1^2\,e_{3,5} + 864\,T^4\,p_1^2\,x_1^2\,e_{3,5} - 192\,T^2\,p_1^3\,x_1^3\,e_{3,5} + 192\,T^3\,p_1^3\,x_1^3\,e_{3,5} + 12\,T^2\,p_1^4\,x_1^4\,e_{3,5}\big)\Big\}$$

In[•]:=
```
Short[#, 10] &[eqns =
  Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, p1, p2, p3}][[ ;; , 2]] & /@ errors)]]
```

Out[•]//Short=

$$\Big\{0 == c_{3,4} - T\,c_{3,4},\ 0 == -c_{3,4} + T\,c_{3,4},\ 0 == T\,c_{3,4} - T^2\,c_{3,4},$$

$$\ll 489 \gg,\ 0 == 1 + \frac{9}{T^3} - \frac{12}{T^2} + \frac{2}{T} + d_{3,1} - T\,d_{3,3} + 2\,T^2\,d_{3,8} - 6\,T^3\,d_{3,18} +$$

$$24\,T^4\,d_{3,35} + f_{3,1} + f_{3,2} - T\,f_{3,2} + 2\,f_{3,3} - 4\,T\,f_{3,3} + 2\,T^2\,f_{3,3} + 6\,f_{3,4} - 18\,T\,f_{3,4} +$$

$$18\,T^2\,f_{3,4} - 6\,T^3\,f_{3,4} + 24\,f_{3,5} - 96\,T\,f_{3,5} + 144\,T^2\,f_{3,5} - 96\,T^3\,f_{3,5} + 24\,T^4\,f_{3,5},$$

$$0 == -\frac{7}{2} - \frac{4}{T^3} + \frac{22}{T^2} - \frac{20}{T} + d_{3,6} + T\,d_{3,7} + T^2\,d_{3,8} + d_{3,9} + T\,d_{3,10} + T^2\,d_{3,11} + d_{3,12} + T\,d_{3,13} +$$

$$T^2\,d_{3,14} - 3\,T\,d_{3,16} - 6\,T^2\,d_{3,17} - 9\,T^3\,d_{3,18} - 2\,T\,d_{3,20} - 4\,T^2\,d_{3,21} - 6\,T^3\,d_{3,22} - T\,d_{3,24} - 2\,T^2\,d_{3,25} -$$

$$3\,T^3\,d_{3,26} + 12\,T^2\,d_{3,33} + 36\,T^3\,d_{3,34} + 72\,T^4\,d_{3,35} + 6\,T^2\,d_{3,38} + 18\,T^3\,d_{3,39} + 36\,T^4\,d_{3,40} + 2\,T^2\,d_{3,43} +$$

$$6\,T^3\,d_{3,44} + 12\,T^4\,d_{3,45} + T^2\,f_{3,3} + 9\,T^2\,f_{3,4} - 9\,T^3\,f_{3,4} + 72\,T^2\,f_{3,5} - 144\,T^3\,f_{3,5} + 72\,T^4\,f_{3,5}\Big\}$$

In[•]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

$$\Big\{\Big\{c_{3,1} \to -\frac{c_{3,2}}{2} - \frac{c_{3,5}}{2\,T},\ c_{3,3} \to -T\,c_{3,2} - c_{3,5},\ c_{3,4} \to 0,\ c_{3,6} \to 0,\ c_{3,8} \to -\frac{1}{2} \times (1 - T)\,c_{3,10},$$

$$c_{3,9} \to 0,\ c_{3,11} \to -1 - T\,c_{3,7} - \frac{1}{2} \times (-1 + 3\,T)\,c_{3,10},\ c_{3,12} \to 0,\ c_{3,13} \to 0,\ c_{3,14} \to 0,$$

$$c_{3,15} \to 0,\ c_{3,17} \to -((-1 + T)\,c_{3,16}),\ c_{3,18} \to -\frac{-1 + 7\,T - 6\,T^2}{6\,T},\ c_{3,19} \to 0,\ c_{3,20} \to -\frac{1}{T},$$

$$c_{3,21} \to -\frac{1 - 6\,T}{2\,T},\ c_{3,22} \to -\frac{-4 - 7\,T + 32\,T^2}{6\,T} - \left(T - T^2\right) c_{3,16},\ c_{3,23} \to 0,\ c_{3,24} \to 0,\ c_{3,25} \to -1,$$

$$c_{3,26} \to \frac{1}{6} \times (3 + 14\,T) - T^2\,c_{3,16},\ c_{3,27} \to 0,\ c_{3,28} \to 0,\ c_{3,29} \to 0,\ c_{3,30} \to 0,\ c_{3,31} \to 0,$$

$$c_{3,33} \to -\frac{3}{2} \times (-1 + T)\,c_{3,32},\ c_{3,34} \to -\left(\left(-1 + 2\,T - T^2\right) c_{3,32}\right),\ c_{3,35} \to -\frac{1 - 12\,T + 27\,T^2 - 16\,T^3}{24\,T^2},$$

$$c_{3,36} \to 0,\ c_{3,37} \to \frac{1}{6\,T^2},\ c_{3,38} \to -\frac{-1 + 3\,T}{4\,T^2},\ c_{3,39} \to -\frac{-1 + 11\,T - 16\,T^2}{6\,T^2},$$

$$c_{3,40} \to -\frac{-1 + 31\,T - 131\,T^2 + 125\,T^3}{24\,T^2} - \left(T - 2\,T^2 + T^3\right) c_{3,32},\ c_{3,41} \to 0,\ c_{3,42} \to 0,\ c_{3,43} \to \frac{1}{T},$$

$$c_{3,44} \to -\frac{-5 + 23\,T}{6\,T},\ c_{3,45} \to -\frac{-5 + 69\,T - 142\,T^2}{24\,T} + \frac{3}{2} \times (-1 + T)\,T^2\,c_{3,32},\ c_{3,46} \to 0,\ c_{3,47} \to 0,$$

$$c_{3,48} \to 0,\ c_{3,49} \to \frac{1}{6},\ c_{3,50} \to \frac{1}{24} \times (1 - 15\,T) - T^3\,c_{3,32},\ c_{3,51} \to 0,\ c_{3,52} \to 0,\ c_{3,53} \to 0,\ c_{3,54} \to 0,$$

$$c_{3,55} \to 0,\ d_{3,1} \to \frac{c_{3,2}}{2} + \frac{c_{3,5}}{2\,T},\ d_{3,2} \to -c_{3,2},\ d_{3,3} \to \frac{c_{3,2}}{T} + \frac{c_{3,5}}{T^2},\ d_{3,4} \to 0,\ d_{3,5} \to -\frac{c_{3,5}}{T^2},$$

$$d_{3,6} \to 0,\ d_{3,7} \to -\frac{-1 + 3\,T - 2\,T^2}{T^4} - \frac{c_{3,7}}{T} - \frac{(-1 + T)\,c_{3,10}}{T^2},\ d_{3,8} \to -\frac{1 - 3\,T + 2\,T^2}{2\,T^5} - \frac{(1 - T)\,c_{3,10}}{2\,T^3},$$

$$d_{3,9} \to 0,\ d_{3,10} \to -\frac{1 - 2\,T}{T^4} - \frac{c_{3,10}}{T^2},\ d_{3,11} \to -\frac{-1 + 2\,T}{2\,T^5} + \frac{c_{3,7}}{T^2} - \frac{(-1 - T)\,c_{3,10}}{2\,T^3},$$

$$d_{3,12}\to 0,\ d_{3,13}\to 0,\ d_{3,14}\to 0,\ d_{3,15}\to 0,\ d_{3,16}\to -\frac{1-2\,T+T^2}{T^4}-\frac{c_{3,16}}{T},$$

$$d_{3,17}\to -\frac{-7+15\,T-10\,T^2+2\,T^3}{2\,T^5}-\frac{(-1+T)\,c_{3,16}}{T^2},\ d_{3,18}\to -\frac{7-15\,T+10\,T^2-2\,T^3}{6\,T^6},\ d_{3,19}\to 0,$$

$$d_{3,20}\to -\frac{-1+T}{T^4},\ d_{3,21}\to \frac{9\times(-1+T)}{2\,T^5},\ d_{3,22}\to -\frac{-9+8\,T+2\,T^2+2\,T^3}{6\,T^6}-\frac{(1-T)\,c_{3,16}}{T^3},$$

$$d_{3,23}\to 0,\ d_{3,24}\to 0,\ d_{3,25}\to -\frac{-1+T}{T^5},\ d_{3,26}\to -\frac{1-T+T^2}{3\,T^6}+\frac{c_{3,16}}{T^3},\ d_{3,27}\to 0,$$

$$d_{3,28}\to 0,\ d_{3,29}\to 0,\ d_{3,30}\to 0,\ d_{3,31}\to 0,\ d_{3,32}\to -\frac{-1+T}{6\,T^4}-\frac{c_{3,32}}{T},$$

$$d_{3,33}\to -\frac{2-3\,T+T^2}{T^5}-\frac{3\times(-1+T)\,c_{3,32}}{2\,T^2},\ d_{3,34}\to -\frac{-16+27\,T-12\,T^2+T^3}{6\,T^6}-\frac{\left(1-2\,T+T^2\right)c_{3,32}}{T^3},$$

$$d_{3,35}\to -\frac{16-27\,T+12\,T^2-T^3}{24\,T^7},\ d_{3,36}\to 0,\ d_{3,37}\to -\frac{1}{6\,T^4},\ d_{3,38}\to -\frac{-3+T}{T^5},$$

$$d_{3,39}\to \frac{3\times(-3+T)}{2\,T^6},\ d_{3,40}\to -\frac{-27+5\,T-T^2-T^3}{24\,T^7}-\frac{\left(-1+2\,T-T^2\right)c_{3,32}}{T^4},\ d_{3,41}\to 0,$$

$$d_{3,42}\to 0,\ d_{3,43}\to -\frac{1}{T^5},\ d_{3,44}\to \frac{2}{T^6},\ d_{3,45}\to -\frac{12-T-5\,T^2}{24\,T^7}+\frac{3\times(-1+T)\,c_{3,32}}{2\,T^4},\ d_{3,46}\to 0,$$

$$d_{3,47}\to 0,\ d_{3,48}\to 0,\ d_{3,49}\to -\frac{1}{6\,T^6},\ d_{3,50}\to -\frac{-1-T}{24\,T^7}+\frac{c_{3,32}}{T^4},\ d_{3,51}\to 0,\ d_{3,52}\to 0,$$

$$d_{3,53}\to 0,\ d_{3,54}\to 0,\ d_{3,55}\to 0,\ e_{3,1}\to -\frac{c_{3,2}}{2}-\frac{c_{3,5}}{2\,T},\ e_{3,2}\to -\frac{c_{3,10}}{T},\ e_{3,3}\to 0,\ e_{3,4}\to 0,$$

$$e_{3,5}\to 0,\ f_{3,1}\to \frac{c_{3,2}}{2}+\frac{c_{3,5}}{2\,T},\ f_{3,2}\to -\frac{-1+2\,T}{T^3}+\frac{c_{3,10}}{T},\ f_{3,3}\to 0,\ f_{3,4}\to 0,\ f_{3,5}\to 0\Big\}\Big\}$$

In[•]:=
```
sol /. (a_ → b_) :> (a = b)
```

Out[•]=
$$\Big\{-\frac{c_{3,2}}{2}-\frac{c_{3,5}}{2\,T},\ -T\,c_{3,2}-c_{3,5},\ 0,\ 0,\ -\frac{1}{2}\times(1-T)\,c_{3,10},\ 0,\ -1-T\,c_{3,7}-\frac{1}{2}\times(-1+3\,T)\,c_{3,10},$$
$$0,\ 0,\ 0,\ 0,\ -((-1+T)\,c_{3,16}),\ -\frac{-1+7\,T-6\,T^2}{6\,T},\ 0,\ -\frac{1}{T},\ -\frac{1-6\,T}{2\,T},$$
$$-\frac{-4-7\,T+32\,T^2}{6\,T}-\left(T-T^2\right)c_{3,16},\ 0,\ 0,\ -1,\ \frac{1}{6}\times(3+14\,T)-T^2\,c_{3,16},\ 0,\ 0,\ 0,\ 0,\ 0,$$
$$-\frac{3}{2}\times(-1+T)\,c_{3,32},\ -\left(\left(-1+2\,T-T^2\right)c_{3,32}\right),\ -\frac{1-12\,T+27\,T^2-16\,T^3}{24\,T^2},\ 0,\ \frac{1}{6\,T^2},\ -\frac{-1+3\,T}{4\,T^2},$$
$$-\frac{-1+11\,T-16\,T^2}{6\,T^2},\ -\frac{-1+31\,T-131\,T^2+125\,T^3}{24\,T^2}-\left(T-2\,T^2+T^3\right)c_{3,32},\ 0,\ 0,\ \frac{1}{T},\ -\frac{-5+23\,T}{6\,T},$$
$$-\frac{-5+69\,T-142\,T^2}{24\,T}+\frac{3}{2}\times(-1+T)\,T^2\,c_{3,32},\ 0,\ 0,\ 0,\ \frac{1}{6},\ \frac{1}{24}\times(1-15\,T)-T^3\,c_{3,32},\ 0,\ 0,\ 0,$$
$$0,\ 0,\ \frac{c_{3,2}}{2}+\frac{c_{3,5}}{2\,T},\ -c_{3,2},\ \frac{c_{3,2}}{T}+\frac{c_{3,5}}{T^2},\ 0,\ -\frac{c_{3,5}}{T^2},\ 0,\ -\frac{-1+3\,T-2\,T^2}{T^4}-\frac{c_{3,7}}{T}-\frac{(-1+T)\,c_{3,10}}{T^2},$$
$$-\frac{1-3\,T+2\,T^2}{2\,T^5}-\frac{(1-T)\,c_{3,10}}{2\,T^3},\ 0,\ -\frac{1-2\,T}{T^4}-\frac{c_{3,10}}{T^2},\ -\frac{-1+2\,T}{2\,T^5}+\frac{c_{3,7}}{T^2}-\frac{(-1-T)\,c_{3,10}}{2\,T^3},\ 0,\ 0,\ 0,\ 0,$$
$$-\frac{1-2\,T+T^2}{T^4}-\frac{c_{3,16}}{T},\ -\frac{-7+15\,T-10\,T^2+2\,T^3}{2\,T^5}-\frac{(-1+T)\,c_{3,16}}{T^2},\ -\frac{7-15\,T+10\,T^2-2\,T^3}{6\,T^6},\ 0,\ -\frac{-1+T}{T^4},$$
$$\frac{9\times(-1+T)}{2\,T^5},\ -\frac{-9+8\,T+2\,T^2+2\,T^3}{6\,T^6}-\frac{(1-T)\,c_{3,16}}{T^3},\ 0,\ 0,\ -\frac{-1+T}{T^5},\ -\frac{1-T+T^2}{3\,T^6}+\frac{c_{3,16}}{T^3},\ 0,\ 0,\ 0,\ 0,$$
$$0,\ -\frac{-1+T}{6\,T^4}-\frac{c_{3,32}}{T},\ -\frac{2-3\,T+T^2}{T^5}-\frac{3\times(-1+T)\,c_{3,32}}{2\,T^2},\ -\frac{-16+27\,T-12\,T^2+T^3}{6\,T^6}-\frac{\left(1-2\,T+T^2\right)c_{3,32}}{T^3},$$
$$-\frac{16-27\,T+12\,T^2-T^3}{24\,T^7},\ 0,\ -\frac{1}{6\,T^4},\ -\frac{-3+T}{T^5},\ \frac{3\times(-3+T)}{2\,T^6},\ -\frac{-27+5\,T-T^2-T^3}{24\,T^7}-\frac{\left(-1+2\,T-T^2\right)c_{3,32}}{T^4},$$
$$0,\ 0,\ -\frac{1}{T^5},\ \frac{2}{T^6},\ -\frac{12-T-5\,T^2}{24\,T^7}+\frac{3\times(-1+T)\,c_{3,32}}{2\,T^4},\ 0,\ 0,\ 0,\ -\frac{1}{6\,T^6},\ -\frac{-1-T}{24\,T^7}+\frac{c_{3,32}}{T^4},$$
$$0,\ 0,\ 0,\ 0,\ 0,\ -\frac{c_{3,2}}{2}-\frac{c_{3,5}}{2\,T},\ -\frac{c_{3,10}}{T},\ 0,\ 0,\ 0,\ \frac{c_{3,2}}{2}+\frac{c_{3,5}}{2\,T},\ -\frac{-1+2\,T}{T^3}+\frac{c_{3,10}}{T},\ 0,\ 0,\ 0\Big\}$$

In[•]:=
```
Cases[{R_{1,2}, R̄_{1,2}, C_1, C̄_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

Out[•]=
$$\{c_{3,2},\ c_{3,5},\ c_{3,7},\ c_{3,10},\ c_{3,16},\ c_{3,32}\}$$

In[•]:=
```
{c_{3,2} = 0, c_{3,5} = 0, c_{3,7} = 0, c_{3,10} = 0, c_{3,16} = 0, c_{3,32} = 1};
{R_{1,2}, R̄_{1,2}, C_1, C̄_1}
```

Out[•]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[\sqrt{T},\ (-1+T)\,(p_1-p_2)\,x_2,$$
$$\in\text{Series}\Big[0,\ p_1\,p_2\,x_1\,x_2+\frac{1}{2}\times(-1+T)\,p_1^2\,x_2^2+\frac{1}{2}\times(1-3\,T)\,p_1\,p_2\,x_2^2,$$
$$p_1\,p_2\,x_1\,x_2-\frac{p_1^2\,p_2\,x_1^2\,x_2}{2\,T}+\frac{1}{2}\times(-1+T)\,p_1^2\,x_2^2+\left(-\frac{1}{2}+\frac{1}{2}\times(1-3\,T)\right)p_1\,p_2\,x_2^2-\frac{(1-3\,T)\,p_1^2\,p_2\,x_1\,x_2^2}{2\,T}-$$

$$\frac{1}{2}\,p_1\,p_2^2\,x_1\,x_2^2-\frac{\left(-1+4\,T-3\,T^2\right)\,p_1^3\,x_2^3}{6\,T}-\frac{\left(1-11\,T+16\,T^2\right)\,p_1^2\,p_2\,x_2^3}{6\,T}+\frac{1}{6}\times(-1+7\,T)\,p_1\,p_2^2\,x_2^3,$$

$$-\frac{p_1^2\,p_2\,x_1^2\,x_2}{T}+p_1^4\,x_1^3\,x_2+\frac{p_1^3\,p_2\,x_1^3\,x_2}{6\,T^2}-p_1\,p_2\,x_2^2-\frac{(1-6\,T)\,p_1^2\,p_2\,x_1\,x_2^2}{2\,T}-p_1\,p_2^2\,x_1\,x_2^2-$$

$$\frac{3}{2}\times(-1+T)\,p_1^4\,x_1^2\,x_2^2-\frac{(-1+3\,T)\,p_1^3\,p_2\,x_1^2\,x_2^2}{4\,T^2}+\frac{p_1^2\,p_2^2\,x_1^2\,x_2^2}{T}-\frac{\left(-1+7\,T-6\,T^2\right)\,p_1^3\,x_2^3}{6\,T}-$$

$$\frac{\left(-4-7\,T+32\,T^2\right)\,p_1^2\,p_2\,x_2^3}{6\,T}+\frac{1}{6}\times(3+14\,T)\,p_1\,p_2^2\,x_2^3+\left(1-2\,T+T^2\right)\,p_1^4\,x_1\,x_2^3-$$

$$\frac{\left(-1+11\,T-16\,T^2\right)\,p_1^3\,p_2\,x_1\,x_2^3}{6\,T^2}-\frac{(-5+23\,T)\,p_1^2\,p_2^2\,x_1\,x_2^3}{6\,T}+\frac{1}{6}\,p_1\,p_2^3\,x_1\,x_2^3-$$

$$\frac{\left(1-12\,T+27\,T^2-16\,T^3\right)\,p_1^4\,x_2^4}{24\,T^2}+\left(-T+2\,T^2-T^3-\frac{-1+31\,T-131\,T^2+125\,T^3}{24\,T^2}\right)\,p_1^3\,p_2\,x_2^4+$$

$$\left(\frac{3}{2}\times(-1+T)\,T^2-\frac{-5+69\,T-142\,T^2}{24\,T}\right)\,p_1^2\,p_2^2\,x_2^4+\left(\frac{1}{24}\times(1-15\,T)-T^3\right)\,p_1\,p_2^3\,x_2^4\Big]\Big],$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[\frac{1}{\sqrt{T}},\left(-1+\frac{1}{T}\right)\,(p_1-p_2)\,x_2,\in\text{Series}\Big[0,$$

$$-\frac{(-1+T)\,p_1^2\,x_1\,x_2}{T^2}-\frac{p_1\,p_2\,x_1\,x_2}{T^2}-\frac{(1-T)\,p_1^2\,x_2^2}{2\,T^3}-\frac{(-1-T)\,p_1\,p_2\,x_2^2}{2\,T^3},$$

$$\left(-\frac{1-T}{T^3}-\frac{-1+T}{T^2}\right)\,p_1^2\,x_1\,x_2+\left(\frac{1}{T^3}-\frac{1}{T^2}\right)\,p_1\,p_2\,x_1\,x_2-\frac{(-1+T)\,p_1^3\,x_1^2\,x_2}{2\,T^3}-\frac{p_1^2\,p_2\,x_1^2\,x_2}{2\,T^3}+$$

$$\left(-\frac{-1+T}{2\,T^4}-\frac{1-T}{2\,T^3}\right)\,p_1^2\,x_2^2+\left(-\frac{1}{2\,T^4}-\frac{-1-T}{2\,T^3}\right)\,p_1\,p_2\,x_2^2-\frac{\left(3-4\,T+T^2\right)\,p_1^3\,x_1\,x_2^2}{2\,T^4}+$$

$$\frac{2\,p_1^2\,p_2\,x_1\,x_2^2}{T^4}-\frac{p_1\,p_2^2\,x_1\,x_2^2}{2\,T^4}-\frac{\left(-3+4\,T-T^2\right)\,p_1^3\,x_2^3}{6\,T^5}-\frac{\left(4+T+T^2\right)\,p_1^2\,p_2\,x_2^3}{6\,T^5}-\frac{(-1+T)\,p_1\,p_2^2\,x_2^3}{6\,T^5},$$

$$-\frac{\left(-1+3\,T-2\,T^2\right)\,p_1^2\,x_1\,x_2}{T^4}-\frac{(1-2\,T)\,p_1\,p_2\,x_1\,x_2}{T^4}-\frac{\left(1-2\,T+T^2\right)\,p_1^3\,x_1^2\,x_2}{T^4}-\frac{(-1+T)\,p_1^2\,p_2\,x_1^2\,x_2}{T^4}+$$

$$\left(-\frac{-1+T}{6\,T^4}-\frac{1}{T}\right)\,p_1^4\,x_1^3\,x_2-\frac{p_1^3\,p_2\,x_1^3\,x_2}{6\,T^4}-\frac{\left(1-3\,T+2\,T^2\right)\,p_1^2\,x_2^2}{2\,T^5}-\frac{(-1+2\,T)\,p_1\,p_2\,x_2^2}{2\,T^5}-$$

$$\frac{\left(-7+15\,T-10\,T^2+2\,T^3\right)\,p_1^3\,x_1\,x_2^2}{2\,T^5}+\frac{9\times(-1+T)\,p_1^2\,p_2\,x_1\,x_2^2}{2\,T^5}-\frac{(-1+T)\,p_1\,p_2^2\,x_1\,x_2^2}{T^5}+$$

$$\left(-\frac{3\times(-1+T)}{2\,T^2}-\frac{2-3\,T+T^2}{T^5}\right)\,p_1^4\,x_1^2\,x_2^2-\frac{(-3+T)\,p_1^3\,p_2\,x_1^2\,x_2^2}{T^5}-\frac{p_1^2\,p_2^2\,x_1^2\,x_2^2}{T^5}-$$

$$\frac{\left(7-15\,T+10\,T^2-2\,T^3\right)\,p_1^3\,x_2^3}{6\,T^6}-\frac{\left(-9+8\,T+2\,T^2+2\,T^3\right)\,p_1^2\,p_2\,x_2^3}{6\,T^6}-\frac{\left(1-T+T^2\right)\,p_1\,p_2^2\,x_2^3}{3\,T^6}+$$

$$\left(-\frac{1-2\,T+T^2}{T^3}-\frac{-16+27\,T-12\,T^2+T^3}{6\,T^6}\right)\,p_1^4\,x_1\,x_2^3+\frac{3\times(-3+T)\,p_1^3\,p_2\,x_1\,x_2^3}{2\,T^6}+\frac{2\,p_1^2\,p_2^2\,x_1\,x_2^3}{T^6}-$$

$$\frac{p_1\,p_2^3\,x_1\,x_2^3}{6\,T^6}-\frac{\left(16-27\,T+12\,T^2-T^3\right)\,p_1^4\,x_2^4}{24\,T^7}+\left(-\frac{-1+2\,T-T^2}{T^4}-\frac{-27+5\,T-T^2-T^3}{24\,T^7}\right)\,p_1^3\,p_2\,x_2^4+$$

$$\left(\frac{3\times(-1+T)}{2\,T^4}-\frac{12-T-5\,T^2}{24\,T^7}\right)\,p_1^2\,p_2^2\,x_2^4+\left(-\frac{-1-T}{24\,T^7}+\frac{1}{T^4}\right)\,p_1\,p_2^3\,x_2^4\Big]\Big],$$

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\,0,\,\in\text{Series}\Big[0,\,-\frac{p_1\,x_1}{T},\,-\frac{p_1\,x_1}{T},\,0\Big]\Big],$$

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\,0,\,\in\text{Series}\Big[0,\,\frac{p_1\,x_1}{T},\,\left(-\frac{1}{T^2}+\frac{1}{T}\right)\,p_1\,x_1,\,-\frac{(-1+2\,T)\,p_1\,x_1}{T^3}\Big]\Big]\Big\}$$

In[•]:= **RMoves**

Out[•]= {True, True, True, True, True, True, True, True, True}