Pensieve header: Perturbing the Heisenberg-algebra knot invariant. Continues “Solving to $k=3.nb” at pensieve://Projects/BabyDoPeGDO/.

$\mathbb{E}[\omega, Q,$ P_ϵSeries] represents $\omega\, e^{Q+P}$, where ω is a scalar, Q is an ϵ-free quadratic, and $P = \sum_{k=0}^{\$k} P[\![k]\!]\, \epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: 𝔼_[_] // 𝔼_[_] calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

Initialization and minor utilities

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\21-1350-KnotTheory\\Preps"];
Once[<< "../Common.m"];
PP_ = Identity;
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
$k=1;
```

In[∘]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ ℰ;
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (p | x | π | ξ)_, ∞] ⋃ {p | x | π | ξ}},
    Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] (Times @@ vs^ps)]
  ];
(*CF[ℰ_] := PP_CF@CCF[ℰ];*)
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

In[∘]:=
```
ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
ϵSeries[0] := ϵSeries @@ Table[0, $k + 1];
ϵSeries /: S1_ϵSeries + S2_ϵSeries :=
  ϵSeries @@ Table[S1[[k]] + S2[[k]], {k, Min[Length@S1, Length@S2]}];
ϵSeries /: S1_ϵSeries ∗ S2_ϵSeries := ϵSeries @@
   Table[Sum[S1[[j + 1]] ∗ S2[[k - j + 1]], {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
ϵSeries /: c_ ∗ S_ϵSeries := (c #) & /@ S;
ϵSeries /: ∂_vs___ S_ϵSeries := (s ↦ ∂_vs s) /@ S;
```

The Basic Tensors

```
In[]:= η_i_ := 𝔼_{}→{i}[1, 0, ϵSeries[0]];
m_i_,j_→k_ := 𝔼_{i,j}→{k}[1, -ξ_i π_j + (π_i + π_j) p_k + (ξ_i + ξ_j) x_k, ϵSeries[0]]
```

```
In[]:= AllMonomials[{}, 0] = {1};
AllMonomials[{}, d_Integer] /; d > 0 := {};
AllMonomials[{v_, vs___}, d_Integer] :=
  Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
```

```
In[]:= Basis[js_List, m_] := Flatten@Outer[Times,
    AllMonomials[Table[p_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 0, m}]
```

```
In[]:= Basis[{i, j}, {2}]
```

Out[]= $\{1, p_i x_i, p_i x_j, p_j x_i, p_j x_j, p_i^2 x_i^2, p_i^2 x_i x_j, p_i^2 x_j^2, p_i p_j x_i^2, p_i p_j x_i x_j, p_i p_j x_j^2, p_j^2 x_i^2, p_j^2 x_i x_j, p_j^2 x_j^2\}$

```
In[]:= GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}];
GenericCombination[bas_, c_k_] := bas.Table[c_k,j, {j, Length@bas}];
```

```
In[]:= GenericCombination[Basis[{i, j}, {2}], c_1]
```

Out[]= $c_{1,1} + p_i x_i c_{1,2} + p_i x_j c_{1,3} + p_j x_i c_{1,4} + p_j x_j c_{1,5} + p_i^2 x_i^2 c_{1,6} + p_i^2 x_i x_j c_{1,7} + p_i^2 x_j^2 c_{1,8} + p_i p_j x_i^2 c_{1,9} + p_i p_j x_i x_j c_{1,10} + p_i p_j x_j^2 c_{1,11} + p_j^2 x_i^2 c_{1,12} + p_j^2 x_i x_j c_{1,13} + p_j^2 x_j^2 c_{1,14}$

```
In[]:= R_i_,j_ := 𝔼_{}→{i,j}[1, (T - 1) (p_i - p_j) x_j ,
   ϵSeries @@ Prepend[0]@Table[GenericCombination[Basis[{i, j}, {k + 1}], c_k], {k, $k}]];
R̄_i_,j_ := 𝔼_{}→{i,j}[1, (T^-1 - 1) (p_i - p_j) x_j,
   ϵSeries @@ Prepend[0]@Table[GenericCombination[Basis[{i, j}, {k + 1}], d_k], {k, $k}]];
C_i_ := 𝔼_{}→{i}[√T, 0, ϵSeries @@ Prepend[0]@
    Table[GenericCombination[Basis[{i}, {k + 1}], e_k], {k, $k}]];
C̄_i_ := 𝔼_{}→{i}[1/√T, 0, ϵSeries @@ Prepend[0]@
    Table[GenericCombination[Basis[{i}, {k + 1}], f_k], {k, $k}]];
```

In[•]:= $\{R_{1,2}, \overline{R}_{1,2}, C_1, \overline{C}_1\}$

Out[•]=
$$\{\mathbb{E}_{\{\}\to\{1,2\}}[1, (-1+T)(p_1-p_2)x_2, \in\text{Series}[0, c_{1,1}+p_1x_1c_{1,2}+p_1x_2c_{1,3}+p_2x_1c_{1,4}+p_2x_2c_{1,5}+p_1^2x_1^2c_{1,6}+p_1^2x_1x_2c_{1,7}+p_1^2x_2^2c_{1,8}+p_1p_2x_1^2c_{1,9}+p_1p_2x_1x_2c_{1,10}+p_1p_2x_2^2c_{1,11}+p_2^2x_1^2c_{1,12}+p_2^2x_1x_2c_{1,13}+p_2^2x_2^2c_{1,14}]],$$
$$\mathbb{E}_{\{\}\to\{1,2\}}[1, \left(-1+\frac{1}{T}\right)(p_1-p_2)x_2, \in\text{Series}[0, d_{1,1}+p_1x_1d_{1,2}+p_1x_2d_{1,3}+p_2x_1d_{1,4}+p_2x_2d_{1,5}+p_1^2x_1^2d_{1,6}+p_1^2x_1x_2d_{1,7}+p_1^2x_2^2d_{1,8}+p_1p_2x_1^2d_{1,9}+p_1p_2x_1x_2d_{1,10}+p_1p_2x_2^2d_{1,11}+p_2^2x_1^2d_{1,12}+p_2^2x_1x_2d_{1,13}+p_2^2x_2^2d_{1,14}]],$$
$$\mathbb{E}_{\{\}\to\{1\}}[\sqrt{T}, 0, \in\text{Series}[0, e_{1,1}+p_1x_1e_{1,2}+p_1^2x_1^2e_{1,3}]],$$
$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{\sqrt{T}}, 0, \in\text{Series}[0, f_{1,1}+p_1x_1f_{1,2}+p_1^2x_1^2f_{1,3}]\right]\}$$

The Main Program

Variables and their duals:

In[•]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_i_)* := (u*)_i;
```

𝔼 operations:

In[•]:=
```
𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
𝔼 /: 𝔼[ω1_, Q1_, P1_] 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
𝔼_{dr_}[ℰs___]_{$k_} := 𝔼_{dr} @@ 𝔼[ℰs]_{$k};
```

In[•]:=
```
𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
  lvs = Flatten@Table[{x_{$@i}, p_{$@i}}, {i, is}];
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{lvs⋃lvs*}[lvs*.lvs, Times[
      𝔼[ℰ1s] /. Table[(v : x | p)_i → v_{$@i}, {i, is}],
      𝔼[ℰ2s] /. Table[(v : ξ | π)_i → v_{$@i}, {i, is}]
     ]])
 ]
```

$$[F:\mathcal{E}]_B := \mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\mathbb{e}^{\frac12\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

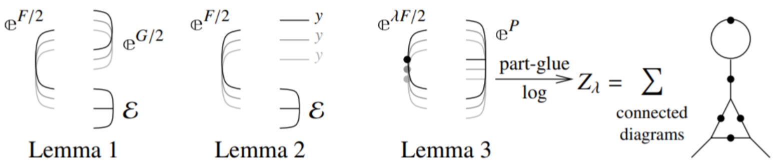
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B = \mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac12\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$




```
In[∘]:= Zip_vs_[ℱ_, ℰ_] := ⟨ℱ, ℰ⟩ // Zip1_vs // Zip2_vs // Zip3_vs
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\mathbb{e}^{\frac12\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B$$

```
In[∘]:= Zip1_{} = Identity;
Zip1_vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PP_Zip1@Module[{ℐ, F, G, u, v},
   ℐ = IdentityMatrix@Length@vs;
   F = Table[∂_{u,v} ℱ, {u, vs*}, {v, vs*}];
   G = Table[∂_{u,v} Q, {u, vs}, {v, vs}];
   CF /@ ⟨vs*.F.Inverse[ℐ - G.F].vs* / 2,
     𝔼[PowerExpand@Factor[ω Det[ℐ - G.F]^(-1/2)], Q - vs.G.vs / 2, P]⟩
  ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F:\mathcal{E}\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B = \mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

```
In[∘]:= Zip2_{} = Identity;
Zip2_vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PP_Zip2@Module[{F, Y, u, v},
   F = Table[∂_{u,v} ℱ, {u, vs*}, {v, vs*}];
   Y = Table[∂_v Q, {v, vs}];
   CF /@ ⟨ℱ, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
  ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda = \sum Z[m]\lambda^m$.

```
In[∘]:= Zip3_vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PP_Zip3@Module[{Z, u, v, m, j},
   Z[0] = P;
   For[m = 0, m < 2 $k, ++m,
    Z[m + 1] = CF[1/(2 (m + 1))
       Sum[∂_{u*,v*} ℱ (∂_{u,v} Z[m] + Sum[(∂_u Z[j]) (∂_v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
   ];
   𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k}] /. Table[v → 0, {v, vs}]]]
  ]
```

Solving for R, C, \$k = 1

```
In[∘]:= $k = 1;
{R_{1,2}, C_1}
unknowns = Cases[{R_{1,2}, R̄_{1,2}, C_1, C̄_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

Out[∘]= $\{\mathbb{E}_{\{\}\to\{1,2\}}[1,\ (-1+T)\ (p_1-p_2)\ x_2,$
$\in\text{Series}[0,\ c_{1,1}+p_1 x_1 c_{1,2}+p_1 x_2 c_{1,3}+p_2 x_1 c_{1,4}+p_2 x_2 c_{1,5}+p_1^2 x_1^2 c_{1,6}+p_1^2 x_1 x_2 c_{1,7}+p_1^2 x_2^2 c_{1,8}+$
$p_1 p_2 x_1^2 c_{1,9}+p_1 p_2 x_1 x_2 c_{1,10}+p_1 p_2 x_2^2 c_{1,11}+p_2^2 x_1^2 c_{1,12}+p_2^2 x_1 x_2 c_{1,13}+p_2^2 x_2^2 c_{1,14}]],$
$\mathbb{E}_{\{\}\to\{1\}}[\sqrt{T},\ 0,\ \in\text{Series}[0,\ e_{1,1}+p_1 x_1 e_{1,2}+p_1^2 x_1^2 e_{1,3}]]\}$

Out[∘]= $\{c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}, c_{1,8}, c_{1,9}, c_{1,10}, c_{1,11}, c_{1,12}, c_{1,13}, c_{1,14}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, e_{1,1}, e_{1,2}, e_{1,3}, f_{1,1}, f_{1,2}, f_{1,3}\}$

```
In[∘]:= Short[errors = {(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})[[3, -1]] -
     (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})[[3, -1]],
    (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2})[[3, -1]],
    (C_1 C̄_2 // m_{1,2→1})[[3, -1]],
    (C_3 R_{1,2} // m_{2,3→2} // m_{2,1→1})[[3, -1]] - (C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})[[3, -1]]},
  10]
```

Out[∘]//Short= $\{\ll 113\gg +$
$p_2^2 x_3^2\ (T^2 c_{1,8}+T^2 c_{1,12}-2T^3 c_{1,12}+T^4 c_{1,12}+T c_{1,13}-2T^2 c_{1,13}+T^3 c_{1,13}+c_{1,14}-2T c_{1,14}+T^2 c_{1,14}),$
$\ll 3\gg\}$

```
In[∘]:= eqns = Thread[0 == Union @@ (CoefficientRules[#, {x_1, x_2, x_3, p_1, p_2, p_3}][[ ;; , 2]] & /@ errors)]
```

Out[∘]=

$$\{0 == c_{1,4} - T\,c_{1,4},\ 0 == -c_{1,4} + T\,c_{1,4},\ 0 == T\,c_{1,4} - T^2\,c_{1,4},\ 0 == -c_{1,4} + 2\,T\,c_{1,4} - T^2\,c_{1,4},$$

$$0 == -T\,c_{1,4} + T^2\,c_{1,4},\ 0 == T\,c_{1,2} - T^2\,c_{1,2} + c_{1,3} - T\,c_{1,3} + c_{1,5} - T\,c_{1,5},$$

$$0 == -2\,c_{1,6} + 2\,T\,c_{1,6},\ 0 == 2\,T\,c_{1,6} - 2\,T^2\,c_{1,6},\ 0 == c_{1,9} - T\,c_{1,9},$$

$$0 == -c_{1,9} + T\,c_{1,9},\ 0 == 2\,T\,c_{1,9} - 2\,T^2\,c_{1,9},\ 0 == -2\,c_{1,9} + 4\,T\,c_{1,9} - 2\,T^2\,c_{1,9},$$

$$0 == -2\,T\,c_{1,9} + 2\,T^2\,c_{1,9},\ 0 == 2\,T\,c_{1,6} - 2\,T^2\,c_{1,6} - c_{1,9} + 4\,T\,c_{1,9} - 4\,T^2\,c_{1,9} + T^3\,c_{1,9},$$

$$0 == 2\,T\,c_{1,8} - 2\,T^2\,c_{1,8} + T^2\,c_{1,9} - 2\,T^3\,c_{1,9} + T^4\,c_{1,9} + T\,c_{1,10} - 2\,T^2\,c_{1,10} + T^3\,c_{1,10},$$

$$0 == 2\,T\,c_{1,7} - 2\,T^2\,c_{1,7} - c_{1,10} + 4\,T\,c_{1,10} - 3\,T^2\,c_{1,10} + 2\,c_{1,11} - 2\,T\,c_{1,11},$$

$$0 == T^2\,c_{1,9} - T^3\,c_{1,9} + 2\,T\,c_{1,12} - 2\,T^2\,c_{1,12},\ 0 == c_{1,12} - T^2\,c_{1,12},\ 0 == -c_{1,12} + 2\,T\,c_{1,12} - T^2\,c_{1,12},$$

$$0 == c_{1,9} - 2\,T\,c_{1,9} + T^2\,c_{1,9} + c_{1,12} - 2\,T\,c_{1,12} + T^2\,c_{1,12},\ 0 == -2\,T\,c_{1,12} + 2\,T^2\,c_{1,12},$$

$$0 == -4\,T\,c_{1,12} + 8\,T^2\,c_{1,12} - 4\,T^3\,c_{1,12},\ 0 == -2\,c_{1,12} + 6\,T\,c_{1,12} - 6\,T^2\,c_{1,12} + 2\,T^3\,c_{1,12},$$

$$0 == -2\,T^2\,c_{1,12} + 2\,T^3\,c_{1,12},\ 0 == -T^2\,c_{1,12} + 2\,T^3\,c_{1,12} - T^4\,c_{1,12},$$

$$0 == -c_{1,12} + 4\,T\,c_{1,12} - 6\,T^2\,c_{1,12} + 4\,T^3\,c_{1,12} - T^4\,c_{1,12},\ 0 == -2\,T\,c_{1,12} + 6\,T^2\,c_{1,12} - 6\,T^3\,c_{1,12} + 2\,T^4\,c_{1,12},$$

$$0 == 2\,T\,c_{1,13} - 2\,T^2\,c_{1,13},\ 0 == T\,c_{1,13} - T^2\,c_{1,13},\ 0 == 2\,T\,c_{1,12} - 2\,T^2\,c_{1,12} + T\,c_{1,13} - T^2\,c_{1,13},$$

$$0 == 2\,c_{1,8} - 2\,T\,c_{1,8} + c_{1,10} - 2\,T\,c_{1,10} + T^2\,c_{1,10} + c_{1,13} - 2\,T\,c_{1,13} + T^2\,c_{1,13},$$

$$0 == -2\,T\,c_{1,13} + 2\,T^2\,c_{1,13},\ 0 == -2\,T\,c_{1,13} + 4\,T^2\,c_{1,13} - 2\,T^3\,c_{1,13},$$

$$0 == T^2\,c_{1,12} - 2\,T^3\,c_{1,12} + T^4\,c_{1,12} + T\,c_{1,13} - 2\,T^2\,c_{1,13} + T^3\,c_{1,13},\ 0 == -T^2\,c_{1,13} + T^3\,c_{1,13},$$

$$0 == -c_{1,13} + 4\,T\,c_{1,13} - 4\,T^2\,c_{1,13} + T^3\,c_{1,13} + 2\,c_{1,14} - 2\,T\,c_{1,14},\ 0 == 2\,T\,c_{1,14} - 2\,T^2\,c_{1,14},$$

$$0 == T^2\,c_{1,6} - 2\,T^3\,c_{1,6} + T^4\,c_{1,6} + T\,c_{1,7} - 2\,T^2\,c_{1,7} + T^3\,c_{1,7} + c_{1,8} - 4\,T\,c_{1,8} + 3\,T^2\,c_{1,8} + c_{1,11} -$$
$$2\,T\,c_{1,11} + T^2\,c_{1,11} + c_{1,14} - 2\,T\,c_{1,14} + T^2\,c_{1,14},\ 0 == -2\,T\,c_{1,14} + 2\,T^2\,c_{1,14},\ 0 == c_{1,1} + d_{1,1},$$

$$0 == c_{1,2} + d_{1,2} + d_{1,4} - T\,d_{1,4},\ 0 == c_{1,4} + T\,d_{1,4},\ 0 == c_{1,2} - \frac{c_{1,2}}{T} + \frac{c_{1,3}}{T} + d_{1,3} + d_{1,5} - T\,d_{1,5},$$

$$0 == c_{1,4} - \frac{c_{1,4}}{T} + \frac{c_{1,5}}{T} + T\,d_{1,5},\ 0 == c_{1,9} + T\,d_{1,9} + 2\,T\,d_{1,12} - 2\,T^2\,d_{1,12},$$

$$0 == c_{1,12} + T^2\,d_{1,12},\ 0 == c_{1,6} + d_{1,6} + d_{1,9} - T\,d_{1,9} + d_{1,12} - 2\,T\,d_{1,12} + T^2\,d_{1,12},$$

$$0 == 2\,c_{1,9} - \frac{2\,c_{1,9}}{T} + \frac{c_{1,10}}{T} + T\,d_{1,10} + 2\,T\,d_{1,13} - 2\,T^2\,d_{1,13},\ 0 == 2\,c_{1,12} - \frac{2\,c_{1,12}}{T} + \frac{c_{1,13}}{T} + T^2\,d_{1,13},$$

$$0 == 2\,c_{1,6} - \frac{2\,c_{1,6}}{T} + \frac{c_{1,7}}{T} + d_{1,7} + d_{1,10} - T\,d_{1,10} + d_{1,13} - 2\,T\,d_{1,13} + T^2\,d_{1,13},$$

$$0 == c_{1,9} + \frac{c_{1,9}}{T^2} - \frac{2\,c_{1,9}}{T} - \frac{c_{1,10}}{T^2} + \frac{c_{1,10}}{T} + \frac{c_{1,11}}{T^2} + T\,d_{1,11} + 2\,T\,d_{1,14} - 2\,T^2\,d_{1,14},$$

$$0 == c_{1,12} + \frac{c_{1,12}}{T^2} - \frac{2\,c_{1,12}}{T} - \frac{c_{1,13}}{T^2} + \frac{c_{1,13}}{T} + \frac{c_{1,14}}{T^2} + T^2\,d_{1,14},$$

$$0 == c_{1,6} + \frac{c_{1,6}}{T^2} - \frac{2\,c_{1,6}}{T} - \frac{c_{1,7}}{T^2} + \frac{c_{1,7}}{T} + \frac{c_{1,8}}{T^2} + d_{1,8} + d_{1,11} - T\,d_{1,11} + d_{1,14} - 2\,T\,d_{1,14} + T^2\,d_{1,14},$$

$$0 == -\frac{c_{1,3}}{T} + c_{1,4} + \frac{2\,c_{1,8}}{T^2} - 2\,c_{1,12} + e_{1,1} + e_{1,2} - \frac{e_{1,2}}{T} + 2\,e_{1,3} + \frac{2\,e_{1,3}}{T^2} - \frac{4\,e_{1,3}}{T} - f_{1,1},$$

$$0 == e_{1,1} + f_{1,1},\ 0 == e_{1,2} + f_{1,2},\ 0 == c_{1,2} - T\,c_{1,2} - c_{1,3} + \frac{c_{1,3}}{T} + c_{1,4} - T\,c_{1,4} - c_{1,5} + \frac{c_{1,5}}{T} - \frac{2\,c_{1,7}}{T} -$$
$$\frac{4\,c_{1,8}}{T^2} + 2\,T\,c_{1,9} + c_{1,10} - \frac{c_{1,10}}{T} - \frac{2\,c_{1,11}}{T^2} + 4\,T\,c_{1,12} + 2\,c_{1,13} + \frac{e_{1,2}}{T} - \frac{4\,e_{1,3}}{T^2} + \frac{4\,e_{1,3}}{T} - T\,f_{1,2},$$

$$0 == e_{1,3} + f_{1,3},\ 0 == c_{1,6} - T^2\,c_{1,6} + \frac{c_{1,7}}{T} - T\,c_{1,7} - c_{1,8} + \frac{c_{1,8}}{T^2} + c_{1,9} - T^2\,c_{1,9} + \frac{c_{1,10}}{T} -$$

$$\mathrm{T}\, c_{1,10} - c_{1,11} + \frac{c_{1,11}}{\mathrm{T}^2} + c_{1,12} - \mathrm{T}^2 c_{1,12} + \frac{c_{1,13}}{\mathrm{T}} - \mathrm{T}\, c_{1,13} - c_{1,14} + \frac{c_{1,14}}{\mathrm{T}^2} + \frac{e_{1,3}}{\mathrm{T}^2} - \mathrm{T}^2 f_{1,3}\Big\}$$

In[∘]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
$$\Big\{\Big\{c_{1,4}\to 0,\ c_{1,5}\to -\mathrm{T}\,c_{1,2}-c_{1,3},\ c_{1,6}\to 0,\ c_{1,8}\to -\frac{1}{2}\times(1-\mathrm{T})\,c_{1,10},\ c_{1,9}\to 0,$$
$$c_{1,11}\to -\mathrm{T}\,c_{1,7}-\frac{1}{2}\times(-1+3\,\mathrm{T})\,c_{1,10},\ c_{1,12}\to 0,\ c_{1,13}\to 0,\ c_{1,14}\to 0,\ d_{1,1}\to -c_{1,1},\ d_{1,2}\to -c_{1,2},$$
$$d_{1,3}\to -\frac{c_{1,3}}{\mathrm{T}^2},\ d_{1,4}\to 0,\ d_{1,5}\to \frac{c_{1,2}}{\mathrm{T}}+\frac{c_{1,3}}{\mathrm{T}^2},\ d_{1,6}\to 0,\ d_{1,7}\to -\frac{c_{1,7}}{\mathrm{T}}-\frac{(-1+\mathrm{T})\,c_{1,10}}{\mathrm{T}^2},$$
$$d_{1,8}\to -\frac{(1-\mathrm{T})\,c_{1,10}}{2\,\mathrm{T}^3},\ d_{1,9}\to 0,\ d_{1,10}\to -\frac{c_{1,10}}{\mathrm{T}^2},\ d_{1,11}\to \frac{c_{1,7}}{\mathrm{T}^2}-\frac{(-1-\mathrm{T})\,c_{1,10}}{2\,\mathrm{T}^3},\ d_{1,12}\to 0,$$
$$d_{1,13}\to 0,\ d_{1,14}\to 0,\ e_{1,1}\to \frac{c_{1,3}}{2\,\mathrm{T}},\ e_{1,2}\to -\frac{c_{1,10}}{\mathrm{T}},\ e_{1,3}\to 0,\ f_{1,1}\to -\frac{c_{1,3}}{2\,\mathrm{T}},\ f_{1,2}\to \frac{c_{1,10}}{\mathrm{T}},\ f_{1,3}\to 0\Big\}\Big\}$$

In[∘]:=
```
sol /. (a_ → b_) ⧴ (a = b)
```

Out[∘]=
$$\Big\{0,\ -\mathrm{T}\,c_{1,2}-c_{1,3},\ 0,\ -\frac{1}{2}\times(1-\mathrm{T})\,c_{1,10},\ 0,\ -\mathrm{T}\,c_{1,7}-\frac{1}{2}\times(-1+3\,\mathrm{T})\,c_{1,10},\ 0,\ 0,$$
$$0,\ -c_{1,1},\ -c_{1,2},\ -\frac{c_{1,3}}{\mathrm{T}^2},\ 0,\ \frac{c_{1,2}}{\mathrm{T}}+\frac{c_{1,3}}{\mathrm{T}^2},\ 0,\ -\frac{c_{1,7}}{\mathrm{T}}-\frac{(-1+\mathrm{T})\,c_{1,10}}{\mathrm{T}^2},\ -\frac{(1-\mathrm{T})\,c_{1,10}}{2\,\mathrm{T}^3},$$
$$0,\ -\frac{c_{1,10}}{\mathrm{T}^2},\ \frac{c_{1,7}}{\mathrm{T}^2}-\frac{(-1-\mathrm{T})\,c_{1,10}}{2\,\mathrm{T}^3},\ 0,\ 0,\ 0,\ \frac{c_{1,3}}{2\,\mathrm{T}},\ -\frac{c_{1,10}}{\mathrm{T}},\ 0,\ -\frac{c_{1,3}}{2\,\mathrm{T}},\ \frac{c_{1,10}}{\mathrm{T}},\ 0\Big\}$$

In[∘]:=
$$\{R_{1,2},\ \overline{R}_{1,2},\ C_1,\ \overline{C}_1\}$$

Out[∘]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ (-1+\mathrm{T})\,(p_1-p_2)\,x_2,$$
$$\in\mathrm{Series}\Big[0,\ c_{1,1}+p_1\,x_1\,c_{1,2}+p_2\,x_2\,(-\mathrm{T}\,c_{1,2}-c_{1,3})+p_1\,x_2\,c_{1,3}+p_1^2\,x_1\,x_2\,c_{1,7}+$$
$$p_1\,p_2\,x_1\,x_2\,c_{1,10}-\frac{1}{2}\times(1-\mathrm{T})\,p_1^2\,x_2^2\,c_{1,10}+p_1\,p_2\,x_2^2\left(-\mathrm{T}\,c_{1,7}-\frac{1}{2}\times(-1+3\,\mathrm{T})\,c_{1,10}\right)\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ \left(-1+\frac{1}{\mathrm{T}}\right)\,(p_1-p_2)\,x_2,\ \in\mathrm{Series}\Big[0,\ -c_{1,1}-p_1\,x_1\,c_{1,2}-\frac{p_1\,x_2\,c_{1,3}}{\mathrm{T}^2}+$$
$$p_2\,x_2\left(\frac{c_{1,2}}{\mathrm{T}}+\frac{c_{1,3}}{\mathrm{T}^2}\right)-\frac{p_1\,p_2\,x_1\,x_2\,c_{1,10}}{\mathrm{T}^2}-\frac{(1-\mathrm{T})\,p_1^2\,x_2^2\,c_{1,10}}{2\,\mathrm{T}^3}+$$
$$p_1\,p_2\,x_2^2\left(\frac{c_{1,7}}{\mathrm{T}^2}-\frac{(-1-\mathrm{T})\,c_{1,10}}{2\,\mathrm{T}^3}\right)+p_1^2\,x_1\,x_2\left(-\frac{c_{1,7}}{\mathrm{T}}-\frac{(-1+\mathrm{T})\,c_{1,10}}{\mathrm{T}^2}\right)\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{\mathrm{T}},\ 0,\ \in\mathrm{Series}\Big[0,\ \frac{c_{1,3}}{2\,\mathrm{T}}-\frac{p_1\,x_1\,c_{1,10}}{\mathrm{T}}\Big]\Big],\ \mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{\mathrm{T}}},\ 0,\ \in\mathrm{Series}\Big[0,\ -\frac{c_{1,3}}{2\,\mathrm{T}}+\frac{p_1\,x_1\,c_{1,10}}{\mathrm{T}}\Big]\Big]\Big\}$$

In[∘]:= $c_{1,1} = c_{1,2} = c_{1,3} = c_{1,7} = 0;\ c_{1,10} = 1;$
$\{R_{1,2}, \overline{R}_{1,2}, C_1, \overline{C}_1\}$

Out[∘]= $\{\mathbb{E}_{\{\}\to\{1,2\}}[1, (-1+T)(p_1-p_2)x_2, \epsilon\text{Series}[0, p_1 p_2 x_1 x_2 + \frac{1}{2}\times(-1+T)p_1^2 x_2^2 + \frac{1}{2}\times(1-3T)p_1 p_2 x_2^2]],$
$\mathbb{E}_{\{\}\to\{1,2\}}[1, \left(-1+\frac{1}{T}\right)(p_1-p_2)x_2, \epsilon\text{Series}[0, -\frac{(-1+T)p_1^2 x_1 x_2}{T^2} - \frac{p_1 p_2 x_1 x_2}{T^2} - \frac{(1-T)p_1^2 x_2^2}{2T^3} - \frac{(-1-T)p_1 p_2 x_2^2}{2T^3}]],$
$\mathbb{E}_{\{\}\to\{1\}}[\sqrt{T}, 0, \epsilon\text{Series}[0, -\frac{p_1 x_1}{T}]], \mathbb{E}_{\{\}\to\{1\}}[\frac{1}{\sqrt{T}}, 0, \epsilon\text{Series}[0, \frac{p_1 x_1}{T}]]\}$

In[∘]:= $\{(R_{1,2} R_{4,3} R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}) \equiv (R_{2,3} R_{4,5} R_{1,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}),$
$(R_{1,2}\overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}) \equiv (\eta_1\eta_2),$
$(C_1\overline{C}_2\ //\ m_{1,2\to1}) \equiv \eta_1,$
$(C_3 R_{1,2}\ //\ m_{2,3\to2}\ //\ m_{2,1\to1}) \equiv (\overline{C}_3 R_{1,2}\ //\ m_{1,3\to1}\ //\ m_{1,2\to1})\}$

Out[∘]= {True, True, True, True}

Solving for *R*, *C*, $k = 2

In[∘]:= $\$k = 2;$
$\text{Short}[\#, 10]\ \&\ [$
$\{(R_{1,2} R_{4,3} R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}) \equiv (R_{2,3} R_{4,5} R_{1,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}),$
$(R_{1,2}\overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}) \equiv (\eta_1\eta_2),$
$(C_1\overline{C}_2\ //\ m_{1,2\to1}) \equiv \eta_1,$
$(C_3 R_{1,2}\ //\ m_{2,3\to2}\ //\ m_{2,1\to1}) \equiv (\overline{C}_3 R_{1,2}\ //\ m_{1,3\to1}\ //\ m_{1,2\to1})\}]$

Out[∘]//Short= $\{(-1+T)p_1^2 p_3 x_1 x_2 x_3 - 3T p_1 p_2 p_3 x_1 x_2 x_3 +$
$\ll 114 \gg + p_1^2 p_3 x_3^3 (T^2 - 4T^3 + 3T^4 + T c_{2,22} - 2T^2 c_{2,22} + 2T^3 c_{2,22} +$
$2T c_{2,26} - 4T^2 c_{2,26} + 2T^3 c_{2,26} + 3T c_{2,30} - 6T^2 c_{2,30} + 3T^3 c_{2,30}) ==$
$3c_{2,1} + 2p_1 x_1 c_{2,2} + \ll 143 \gg + p_2^2 p_3 x_3^3 (T^3 c_{2,22} + 3T c_{2,30} - 6T^2 c_{2,30} + 3T^3 c_{2,30}),$
$\ll 2 \gg, \ll 1 \gg == \ll 1 \gg\}$

In[∘]:= $\text{unknowns} = \text{Cases}[\{R_{1,2}, \overline{R}_{1,2}, C_1, \overline{C}_1\}, (c\ |\ d\ |\ e\ |\ f)_{\$k,\_}, \infty]\ //\ \text{Union}$

Out[∘]= $\{c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}, c_{2,5}, c_{2,6}, c_{2,7}, c_{2,8}, c_{2,9}, c_{2,10}, c_{2,11}, c_{2,12}, c_{2,13}, c_{2,14},$
$c_{2,15}, c_{2,16}, c_{2,17}, c_{2,18}, c_{2,19}, c_{2,20}, c_{2,21}, c_{2,22}, c_{2,23}, c_{2,24}, c_{2,25}, c_{2,26}, c_{2,27},$
$c_{2,28}, c_{2,29}, c_{2,30}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{2,11},$
$d_{2,12}, d_{2,13}, d_{2,14}, d_{2,15}, d_{2,16}, d_{2,17}, d_{2,18}, d_{2,19}, d_{2,20}, d_{2,21}, d_{2,22}, d_{2,23}, d_{2,24},$
$d_{2,25}, d_{2,26}, d_{2,27}, d_{2,28}, d_{2,29}, d_{2,30}, e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}\}$

```
In[•]:= Short[errors = CF@{(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]] -
          (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]],
        (R1,2 R̄3,4 // m1,3→1 // m2,4→2)[[3, -1]],
        (C1 C̄2 // m1,2→1)[[3, -1]],
        (C3 R1,2 // m2,3→2 // m2,1→1)[[3, -1]] - (C̄3 R1,2 // m1,3→1 // m1,2→1)[[3, -1]]},
     10]
```

Out[•]//Short=

$$\left\{p_3\, x_1\, (c_{2,4} - T\, c_{2,4}) + p_1\, x_2\, (c_{2,4} - T\, c_{2,4}) + p_2\, x_1\, (-c_{2,4} + T\, c_{2,4}) + \ll 131 \gg + p_1^2\, p_3\, x_3^3\, \left(T^2 - 4\,T^3 + 3\,T^4 - 2\,T^2\, c_{2,22} + 2\,T^3\, c_{2,22} + 2\,T\, c_{2,26} - 4\,T^2\, c_{2,26} + 2\,T^3\, c_{2,26} + 3\,T\, c_{2,30} - 6\,T^2\, c_{2,30} + 3\,T^3\, c_{2,30}\right) + p_2\, p_3^2\, x_3^3\, \left(-3\,T^2\, c_{2,30} + 3\,T^3\, c_{2,30}\right), \ll 1 \gg, \ll 1 \gg, \frac{\ll 1 \gg}{T^4} + \frac{\ll 1 \gg}{T^4} + \frac{\ll 1 \gg}{2 \ll 1 \gg} + \frac{p_1^3 \ll 1 \gg (-1 + \ll 55 \gg)}{2\,T^3}\right\}$$

```
In[•]:= Short[#, 10] &[eqns =
      Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, p1, p2, p3}][[;; , 2]] & /@ errors)]]
```

Out[•]//Short=

$$\left\{0 == c_{2,4} - T\, c_{2,4},\ 0 == -c_{2,4} + T\, c_{\ll 1 \gg},\ \ll 169 \gg,\ 0 == \ll 1 \gg + \ll 1 \gg,\ 0 == \frac{1}{2} - \frac{1}{2\,T^3} + \frac{1}{2\,T^2} - \frac{T}{2} + c_{2,15} - T^3\, c_{2,15} + \frac{c_{2,16}}{T} - T^2\, c_{2,16} + \frac{c_{2,17}}{T^2} - T\, c_{2,17} - c_{2,18} + \frac{c_{2,18}}{T^3} + c_{2,19} - T^3\, c_{2,19} + \frac{c_{2,20}}{T} - T^2\, c_{2,20} + \ll 12 \gg + \frac{c_{2,25}}{T^2} - T\, c_{2,25} - c_{2,26} + \frac{c_{2,26}}{T^3} + c_{2,27} - T^3\, c_{2,27} + \frac{c_{2,28}}{T} - T^2\, c_{2,28} + \frac{c_{2,29}}{T^2} - T\, c_{2,29} - c_{2,30} + \frac{c_{2,30}}{T^3} + \frac{e_{2,4}}{T^3} - T^3\, f_{2,4}\right\}$$

In[∘]:= `{sol} = Solve[eqns, unknowns]`

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]= $\Big\{\Big\{c_{2,4}\to 0,\ c_{2,5}\to -T\,c_{2,2}-c_{2,3},\ c_{2,6}\to 0,\ c_{2,8}\to -\frac{1}{2}\times(1-T)\,c_{2,10},\ c_{2,9}\to 0,$

$c_{2,11}\to -\frac{1}{2}-T\,c_{2,7}-\frac{1}{2}\times(-1+3T)\,c_{2,10},\ c_{2,12}\to 0,\ c_{2,13}\to 0,\ c_{2,14}\to 0,\ c_{2,15}\to 0,$

$c_{2,17}\to -((-1+T)\,c_{2,16}),\ c_{2,18}\to -\frac{-1+4T-3T^2}{6T},\ c_{2,19}\to 0,\ c_{2,20}\to -\frac{1}{2T},$

$c_{2,21}\to -\frac{1-3T}{2T},\ c_{2,22}\to -\frac{1-11T+16T^2}{6T}-\left(T-T^2\right)c_{2,16},\ c_{2,23}\to 0,\ c_{2,24}\to 0,$

$c_{2,25}\to -\frac{1}{2},\ c_{2,26}\to \frac{1}{6}\times(-1+7T)-T^2\,c_{2,16},\ c_{2,27}\to 0,\ c_{2,28}\to 0,\ c_{2,29}\to 0,\ c_{2,30}\to 0,$

$d_{2,1}\to -c_{2,1},\ d_{2,2}\to -c_{2,2},\ d_{2,3}\to -\frac{c_{2,3}}{T^2},\ d_{2,4}\to 0,\ d_{2,5}\to \frac{c_{2,2}}{T}+\frac{c_{2,3}}{T^2},\ d_{2,6}\to 0,$

$d_{2,7}\to -\frac{1-T}{T^3}-\frac{c_{2,7}}{T}-\frac{(-1+T)\,c_{2,10}}{T^2},\ d_{2,8}\to -\frac{-1+T}{2T^4}-\frac{(1-T)\,c_{2,10}}{2T^3},\ d_{2,9}\to 0,$

$d_{2,10}\to \frac{1}{T^3}-\frac{c_{2,10}}{T^2},\ d_{2,11}\to -\frac{1}{2T^4}+\frac{c_{2,7}}{T^2}-\frac{(-1-T)\,c_{2,10}}{2T^3},\ d_{2,12}\to 0,\ d_{2,13}\to 0,\ d_{2,14}\to 0,$

$d_{2,15}\to 0,\ d_{2,16}\to -\frac{-1+T}{2T^3}-\frac{c_{2,16}}{T},\ d_{2,17}\to -\frac{3-4T+T^2}{2T^4}-\frac{(-1+T)\,c_{2,16}}{T^2},\ d_{2,18}\to -\frac{-3+4T-T^2}{6T^5},$

$d_{2,19}\to 0,\ d_{2,20}\to -\frac{1}{2T^3},\ d_{2,21}\to \frac{2}{T^4},\ d_{2,22}\to -\frac{4+T+T^2}{6T^5}-\frac{(1-T)\,c_{2,16}}{T^3},\ d_{2,23}\to 0,\ d_{2,24}\to 0,$

$d_{2,25}\to -\frac{1}{2T^4},\ d_{2,26}\to -\frac{-1+T}{6T^5}+\frac{c_{2,16}}{T^3},\ d_{2,27}\to 0,\ d_{2,28}\to 0,\ d_{2,29}\to 0,\ d_{2,30}\to 0,\ e_{2,1}\to \frac{c_{2,3}}{2T},$

$e_{2,2}\to -\frac{c_{2,10}}{T},\ e_{2,3}\to 0,\ e_{2,4}\to 0,\ f_{2,1}\to -\frac{c_{2,3}}{2T},\ f_{2,2}\to -\frac{1}{T^2}+\frac{c_{2,10}}{T},\ f_{2,3}\to 0,\ f_{2,4}\to 0\Big\}\Big\}$

In[∘]:= **sol /. (a_ → b_) :→ (a = b)**

Out[∘]= $\{0, -T c_{2,2} - c_{2,3}, 0, -\frac{1}{2}\times(1-T)\, c_{2,10}, 0, -\frac{1}{2} - T c_{2,7} - \frac{1}{2}\times(-1+3T)\, c_{2,10}, 0, 0, 0, 0, -((-1+T)\, c_{2,16}), -\frac{-1+4T-3T^2}{6T}, 0, -\frac{1}{2T}, -\frac{1-3T}{2T}, -\frac{1-11T+16T^2}{6T} - (T-T^2)\, c_{2,16}, 0, 0, -\frac{1}{2}, \frac{1}{6}\times(-1+7T) - T^2 c_{2,16}, 0, 0, 0, 0, -c_{2,1}, -c_{2,2}, -\frac{c_{2,3}}{T^2}, 0, \frac{c_{2,2}}{T} + \frac{c_{2,3}}{T^2}, 0, -\frac{1-T}{T^3} - \frac{c_{2,7}}{T} - \frac{(-1+T)\, c_{2,10}}{T^2}, -\frac{-1+T}{2T^4} - \frac{(1-T)\, c_{2,10}}{2T^3}, 0, \frac{1}{T^3} - \frac{c_{2,10}}{T^2}, -\frac{1}{2T^4} + \frac{c_{2,7}}{T^2} - \frac{(-1-T)\, c_{2,10}}{2T^3}, 0, 0, 0, 0, -\frac{-1+T}{2T^3} - \frac{c_{2,16}}{T}, -\frac{3-4T+T^2}{2T^4} - \frac{(-1+T)\, c_{2,16}}{T^2}, -\frac{-3+4T-T^2}{6T^5}, 0, -\frac{1}{2T^3}, \frac{2}{T^4}, -\frac{4+T+T^2}{6T^5} - \frac{(1-T)\, c_{2,16}}{T^3}, 0, 0, -\frac{1}{2T^4}, -\frac{-1+T}{6T^5} + \frac{c_{2,16}}{T^3}, 0, 0, 0, 0, \frac{c_{2,3}}{2T}, -\frac{c_{2,10}}{T}, 0, 0, -\frac{c_{2,3}}{2T}, -\frac{1}{T^2} + \frac{c_{2,10}}{T}, 0, 0\}$

In[∘]:= **$c_{2,1} = c_{2,2} = c_{2,3} = c_{2,7} = c_{2,10} = c_{2,16} = 0$;**
$\{R_{1,2}, \overline{R}_{1,2}, C_1, \overline{C}_1\}$

Out[∘]= $\{\mathbb{E}_{\{\}\to\{1,2\}}[1, (-1+T)(p_1-p_2)\, x_2, \epsilon\text{Series}[0, p_1 p_2 x_1 x_2 + \frac{1}{2}\times(-1+T)\, p_1^2 x_2^2 + \frac{1}{2}\times(1-3T)\, p_1 p_2 x_2^2, -\frac{p_1^2 p_2 x_1^2 x_2}{2T} - \frac{1}{2} p_1 p_2 x_2^2 - \frac{(1-3T)\, p_1^2 p_2 x_1 x_2^2}{2T} - \frac{1}{2} p_1 p_2^2 x_1 x_2^2 - \frac{(-1+4T-3T^2)\, p_1^3 x_2^3}{6T} - \frac{(1-11T+16T^2)\, p_1^2 p_2 x_2^3}{6T} + \frac{1}{6}\times(-1+7T)\, p_1 p_2^2 x_2^3]], \mathbb{E}_{\{\}\to\{1,2\}}[1, (-1+\frac{1}{T})(p_1-p_2)\, x_2, \epsilon\text{Series}[0, -\frac{(-1+T)\, p_1^2 x_1 x_2}{T^2} - \frac{p_1 p_2 x_1 x_2}{T^2} - \frac{(1-T)\, p_1^2 x_2^2}{2T^3} - \frac{(-1-T)\, p_1 p_2 x_2^2}{2T^3}, -\frac{(1-T)\, p_1^2 x_1 x_2}{T^3} + \frac{p_1 p_2 x_1 x_2}{T^3} - \frac{(-1+T)\, p_1^3 x_1^2 x_2}{2T^3} - \frac{p_1^2 p_2 x_1^2 x_2}{2T^3} - \frac{(-1+T)\, p_1^2 x_2^2}{2T^4} - \frac{p_1 p_2 x_2^2}{2T^4} - \frac{(3-4T+T^2)\, p_1^3 x_1 x_2^2}{2T^4} + \frac{2 p_1^2 p_2 x_1 x_2^2}{T^4} - \frac{p_1 p_2^2 x_1 x_2^2}{2T^4} - \frac{(-3+4T-T^2)\, p_1^3 x_2^3}{6T^5} - \frac{(4+T+T^2)\, p_1^2 p_2 x_2^3}{6T^5} - \frac{(-1+T)\, p_1 p_2^2 x_2^3}{6T^5}]], \mathbb{E}_{\{\}\to\{1\}}[\sqrt{T}, 0, \epsilon\text{Series}[0, -\frac{p_1 x_1}{T}, 0]], \mathbb{E}_{\{\}\to\{1\}}[\frac{1}{\sqrt{T}}, 0, \epsilon\text{Series}[0, \frac{p_1 x_1}{T}, -\frac{p_1 x_1}{T^2}]]\}$

In[∘]:= **$\{(R_{1,2} R_{4,3} R_{5,6} \,//\, m_{1,4\to1} \,//\, m_{2,5\to2} \,//\, m_{3,6\to3}) \equiv (R_{2,3} R_{4,5} R_{1,6} \,//\, m_{1,4\to1} \,//\, m_{2,5\to2} \,//\, m_{3,6\to3}),$**
$(R_{1,2} \overline{R}_{3,4} \,//\, m_{1,3\to1} \,//\, m_{2,4\to2}) \equiv (\eta_1 \eta_2),$
$(C_1 \overline{C}_2 \,//\, m_{1,2\to1}) \equiv \eta_1,$
$(C_3 R_{1,2} \,//\, m_{2,3\to2} \,//\, m_{2,1\to1}) \equiv (\overline{C}_3 R_{1,2} \,//\, m_{1,3\to1} \,//\, m_{1,2\to1})\}$

Out[∘]= {True, True, True, True}

Some Knot Theory

In[∘]:=
```
$k = 1
```

Out[∘]= 1

In[∘]:=
```
NewBit[K_] := Module[{Alex = Alexander[K][T]},
  T^3 Alex^2/(T - 1) ZF[K][[3, 2]] // Factor]
```

In[∘]:=
```
NewBit /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]= $\left\{2 - T + T^2,\ (1 + T) \times \left(1 - 3 T + T^2\right),\ \frac{4 - 3 T + 5 T^2 - 3 T^3 + 3 T^4 - T^5 + T^6}{T^2},\ 9 - 11 T + 7 T^2 - T^3\right\}$

In[∘]:=
```
(*Two knots with equal Alexander, new bit does not agree*)
Alexander[Knot[6, 1]] == Alexander[Knot[9, 46]]
Timing[NewBit[Knot[6, 1]] == NewBit[Knot[9, 46]]]
```

Out[∘]= True

Out[∘]= $\left\{37.4375,\ 5 - 11 T - T^2 + 3 T^3 == 7 - 21 T + 9 T^2 + T^3\right\}$

In[∘]:=
```
equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
Length@Union[ZF /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[∘]= 2

In[∘]:=
```
ZF /@ equiv
```

Out[∘]= $\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{T^3}{1 - 4 T + 9 T^2 - 15 T^3 + 17 T^4 - 15 T^5 + 9 T^6 - 4 T^7 + T^8},\ 0,\ \epsilon\text{Series}\Big[0,\ \left(-3 + 20 T - 69 T^2 + 161 T^3 - 272 T^4 + 328 T^5 - 225 T^6 - 92 T^7 + 548 T^8 - 952 T^9 + 1113 T^{10} - 980 T^{11} + 668 T^{12} - 349 T^{13} + 135 T^{14} - 36 T^{15} + 5 T^{16}\right) / \left(T - 8 T^2 + 34 T^3 - 102 T^4 + 235 T^5 - 436 T^6 + 669 T^7 - 860 T^8 + 935 T^9 - 860 T^{10} + 669 T^{11} - 436 T^{12} + 235 T^{13} - 102 T^{14} + 34 T^{15} - 8 T^{16} + T^{17}\right)\Big]\Big],$

$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{T^2}{1 - 4 T + 9 T^2 - 15 T^3 + 17 T^4 - 15 T^5 + 9 T^6 - 4 T^7 + T^8},\ 0,\ \epsilon\text{Series}\Big[0,\ \left(-3 + 20 T - 69 T^2 + 161 T^3 - 272 T^4 + 328 T^5 - 225 T^6 - 92 T^7 + 548 T^8 - 952 T^9 + 1113 T^{10} - 980 T^{11} + 668 T^{12} - 349 T^{13} + 135 T^{14} - 36 T^{15} + 5 T^{16}\right) / \left(T - 8 T^2 + 34 T^3 - 102 T^4 + 235 T^5 - 436 T^6 + 669 T^7 - 860 T^8 + 935 T^9 - 860 T^{10} + 669 T^{11} - 436 T^{12} + 235 T^{13} - 102 T^{14} + 34 T^{15} - 8 T^{16} + T^{17}\right)\Big]\Big]\Big\}$

In[∘]:=
```
equiv =
  {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
Length@Union[ZF /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

Out[∘]= 1

In[∘]:=
```
$k = 2
```

Out[∘]= 2

In[∘]:=
```
equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
Length@Union[ZF /@ equiv]
```

Out[∘]= 2

In[∘]:=
```
equiv =
  {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
Length@Union[ZF /@ equiv]
```

Out[∘]= 1

Solving for *R*, *C*, $k = 3

In[∘]:= `$k = 3;`
`Short[#, 10] & [`

$\{(R_{1,2}\, R_{4,3}\, R_{5,6} \mathbin{//} m_{1,4\to1} \mathbin{//} m_{2,5\to2} \mathbin{//} m_{3,6\to3}) \equiv (R_{2,3}\, R_{4,5}\, R_{1,6} \mathbin{//} m_{1,4\to1} \mathbin{//} m_{2,5\to2} \mathbin{//} m_{3,6\to3}),$

$(R_{1,2}\, \overline{R}_{3,4} \mathbin{//} m_{1,3\to1} \mathbin{//} m_{2,4\to2}) \equiv (\eta_1\, \eta_2),$

$(C_1\, \overline{C}_2 \mathbin{//} m_{1,2\to1}) \equiv \eta_1,$

$(C_3\, R_{1,2} \mathbin{//} m_{2,3\to2} \mathbin{//} m_{2,1\to1}) \equiv (\overline{C}_3\, R_{1,2} \mathbin{//} m_{1,3\to1} \mathbin{//} m_{1,2\to1})\}]$

Out[∘]//Short= { <<1>> }

In[∘]:= `unknowns = Cases[{`$R_{1,2}, \overline{R}_{1,2}, C_1, \overline{C}_1$`}, (c | d | e | f)`$_{\$k,\_}$`, ∞] // Union`

Out[∘]= $\{c_{3,1}, c_{3,2}, c_{3,3}, c_{3,4}, c_{3,5}, c_{3,6}, c_{3,7}, c_{3,8}, c_{3,9}, c_{3,10}, c_{3,11}, c_{3,12}, c_{3,13}, c_{3,14}, c_{3,15}, c_{3,16}, c_{3,17}, c_{3,18}, c_{3,19}, c_{3,20}, c_{3,21}, c_{3,22}, c_{3,23}, c_{3,24}, c_{3,25}, c_{3,26}, c_{3,27}, c_{3,28}, c_{3,29}, c_{3,30}, c_{3,31}, c_{3,32}, c_{3,33}, c_{3,34}, c_{3,35}, c_{3,36}, c_{3,37}, c_{3,38}, c_{3,39}, c_{3,40}, c_{3,41}, c_{3,42}, c_{3,43}, c_{3,44}, c_{3,45}, c_{3,46}, c_{3,47}, c_{3,48}, c_{3,49}, c_{3,50}, c_{3,51}, c_{3,52}, c_{3,53}, c_{3,54}, c_{3,55}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, d_{3,5}, d_{3,6}, d_{3,7}, d_{3,8}, d_{3,9}, d_{3,10}, d_{3,11}, d_{3,12}, d_{3,13}, d_{3,14}, d_{3,15}, d_{3,16}, d_{3,17}, d_{3,18}, d_{3,19}, d_{3,20}, d_{3,21}, d_{3,22}, d_{3,23}, d_{3,24}, d_{3,25}, d_{3,26}, d_{3,27}, d_{3,28}, d_{3,29}, d_{3,30}, d_{3,31}, d_{3,32}, d_{3,33}, d_{3,34}, d_{3,35}, d_{3,36}, d_{3,37}, d_{3,38}, d_{3,39}, d_{3,40}, d_{3,41}, d_{3,42}, d_{3,43}, d_{3,44}, d_{3,45}, d_{3,46}, d_{3,47}, d_{3,48}, d_{3,49}, d_{3,50}, d_{3,51}, d_{3,52}, d_{3,53}, d_{3,54}, d_{3,55}, e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, f_{3,5}\}$

In[∘]:=
```
Short[errors = CF@{(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]] -
    (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]],
   (R1,2 R̄3,4 // m1,3→1 // m2,4→2)[[3, -1]],
   (C1 C̄2 // m1,2→1 )[[3, -1]],
   (C3 R1,2 // m2,3→2 // m2,1→1)[[3, -1]] - (C̄3 R1,2 // m1,3→1 // m1,2→1)[[3, -1]]},
 10]
```

Out[∘]//Short=

$$\{p_3 x_1 (c_{3,4} - T c_{3,4}) + p_1 x_2 (c_{3,4} - T c_{3,4}) + p_2 x_1 (-c_{3,4} + T c_{3,4}) + p_2 x_3 (T c_{3,4} - T^2 c_{3,4}) + \ll 1 \gg + \ll 347 \gg + p_2^3 p_3 x_3^4 (-4 T c_{3,55} + 12 T^2 c_{3,55} - 12 T^3 c_{3,55} + 4 T^4 c_{3,55}) + p_2 p_3^3 x_3^4 (-4 T^3 c_{3,55} + 4 T^4 c_{3,55}) + \frac{1}{24} p_1^2 p_3^2 x_3^4 (7 T^2 - 97 T^3 + 329 T^4 - 239 T^5 - 48 T^3 c_{3,45} + 48 T^4 c_{3,45} + 72 T^2 c_{3,50} - 144 T^3 c_{3,50} + 72 T^4 c_{3,50} + 144 T^2 c_{3,55} - 288 T^3 c_{3,55} + 144 T^4 c_{3,55}) + \frac{1}{12 T} p_1^4 x_3^4 (5 - 19 T + 13 T^2 + 38 T^3 - 89 T^4 + 77 T^5 - 25 T^6 + 12 T^5 c_{3,31} - 48 T^6 c_{3,31} + 72 T^7 c_{3,31} - 48 T^8 c_{3,31} + 12 T^9 c_{3,31} + \ll 38 \gg + 72 T^3 c_{3,45} - 48 T^4 c_{3,45} + 12 T^5 c_{3,45} + 12 T c_{3,50} - 48 T^2 c_{3,50} + 72 T^3 c_{3,50} - 48 T^4 c_{3,50} + 12 T^5 c_{3,50} + 12 T c_{3,55} - 48 T^2 c_{3,55} + 72 T^3 c_{3,55} - 48 T^4 c_{3,55} + 12 T^5 c_{3,55}), \ll 1 \gg, \ll 1 \gg, \frac{\ll 1 \gg}{\ll 1 \gg} + \ll 4 \gg\}$$

In[∘]:=
```
Short[#, 10] &[eqns =
  Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, p1, p2, p3}][[ ;; , 2]] & /@ errors)]]
```

Out[∘]//Short=

$$\{0 == c_{3,4} - T c_{3,4},\ 0 == -c_{3,4} + T c_{3,4},\ 0 == T c_{3,4} - T^2 c_{3,4},\ \ll 415 \gg,\ 0 == e_{3,5} + f_{3,5},$$
$$0 == \frac{3}{4} + \frac{5}{12 T^5} - \frac{3}{4 T^4} - \frac{1}{6 T^3} - \frac{5}{12 T} + \frac{T}{6} + c_{3,31} - T^4 c_{3,31} + \frac{c_{3,32}}{T} - T^3 c_{3,32} + \frac{c_{3,33}}{T^2} - T^2 c_{3,33} + \frac{c_{3,34}}{T^3} - T c_{3,34} - c_{3,35} + \frac{c_{3,35}}{T^4} + c_{3,36} - T^4 c_{3,36} + \frac{c_{3,37}}{T} - T^3 c_{3,37} + \frac{c_{3,38}}{T^2} - T^2 c_{3,38} + \frac{c_{3,39}}{T^3} - T c_{3,39} - c_{3,40} + \frac{c_{3,40}}{T^4} + c_{3,41} - T^4 c_{3,41} + \frac{c_{3,42}}{T} - T^3 c_{3,42} + \frac{c_{3,43}}{T^2} - T^2 c_{3,43} + \frac{c_{3,44}}{T^3} - T c_{3,44} - c_{3,45} + \frac{c_{3,45}}{T^4} + c_{3,46} - T^4 c_{3,46} + \frac{c_{3,47}}{T} - T^3 c_{3,47} + \frac{c_{3,48}}{T^2} - T^2 c_{3,48} + \frac{c_{3,49}}{T^3} - T c_{3,49} - c_{3,50} + \frac{c_{3,50}}{T^4} + c_{3,51} - T^4 c_{3,51} + \frac{c_{3,52}}{T} - T^3 c_{3,52} + \frac{c_{3,53}}{T^2} - T^2 c_{3,53} + \frac{c_{3,54}}{T^3} - T c_{3,54} - c_{3,55} + \frac{c_{3,55}}{T^4} + \frac{e_{3,5}}{T^4} - T^4 f_{3,5}\}$$

In[∘]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=

$$\{\{c_{3,4} \to 0,\ c_{3,5} \to -T c_{3,2} - c_{3,3},\ c_{3,6} \to 0,\ c_{3,8} \to -\frac{1}{2} \times (1 - T)\ c_{3,10},\ c_{3,9} \to 0,\ c_{3,11} \to -T c_{3,7} - \frac{1}{2} \times (-1 + 3 T)\ c_{3,10},\ c_{3,12} \to 0,\ c_{3,13} \to 0,\ c_{3,14} \to 0,\ c_{3,15} \to 0,\ c_{3,17} \to -((-1 + T)\ c_{3,16}),\ c_{3,18} \to -\frac{1 - T}{6 T},\ c_{3,19} \to 0,\ c_{3,20} \to 0,\ c_{3,21} \to \frac{1}{2 T},$$

$c_{3,22} \to -\frac{-2+5T}{2T} - (T - T^2)\, c_{3,16},\ c_{3,23} \to 0,\ c_{3,24} \to 0,\ c_{3,25} \to 0,\ c_{3,26} \to \frac{5}{6} - T^2 c_{3,16},$

$c_{3,27} \to 0,\ c_{3,28} \to 0,\ c_{3,29} \to 0,\ c_{3,30} \to 0,\ c_{3,31} \to 0,\ c_{3,33} \to -\frac{3}{2} \times (-1+T)\, c_{3,32},$

$c_{3,34} \to -\left((-1+2T-T^2)\, c_{3,32}\right),\ c_{3,35} \to -\frac{1-12T+27T^2-16T^3}{24T^2},$

$c_{3,36} \to 0,\ c_{3,37} \to \frac{1}{6T^2},\ c_{3,38} \to -\frac{-1+3T}{4T^2},\ c_{3,39} \to -\frac{-1+11T-16T^2}{6T^2},$

$c_{3,40} \to -\frac{-1+31T-131T^2+125T^3}{24T^2} - (T-2T^2+T^3)\, c_{3,32},\ c_{3,41} \to 0,\ c_{3,42} \to 0,\ c_{3,43} \to \frac{1}{T},$

$c_{3,44} \to -\frac{-5+23T}{6T},\ c_{3,45} \to -\frac{-5+69T-142T^2}{24T} + \frac{3}{2} \times (-1+T)\, T^2 c_{3,32},\ c_{3,46} \to 0,\ c_{3,47} \to 0,$

$c_{3,48} \to 0,\ c_{3,49} \to \frac{1}{6},\ c_{3,50} \to \frac{1}{24} \times (1-15T) - T^3 c_{3,32},\ c_{3,51} \to 0,\ c_{3,52} \to 0,\ c_{3,53} \to 0,$

$c_{3,54} \to 0,\ c_{3,55} \to 0,\ d_{3,1} \to -c_{3,1},\ d_{3,2} \to -c_{3,2},\ d_{3,3} \to -\frac{c_{3,3}}{T^2},\ d_{3,4} \to 0,\ d_{3,5} \to \frac{c_{3,2}}{T} + \frac{c_{3,3}}{T^2},$

$d_{3,6} \to 0,\ d_{3,7} \to -\frac{-1+T}{T^4} - \frac{c_{3,7}}{T} - \frac{(-1+T)\, c_{3,10}}{T^2},\ d_{3,8} \to -\frac{1-T}{2T^5} - \frac{(1-T)\, c_{3,10}}{2T^3},\ d_{3,9} \to 0,$

$d_{3,10} \to -\frac{1}{T^4} - \frac{c_{3,10}}{T^2},\ d_{3,11} \to \frac{1}{2T^5} + \frac{c_{3,7}}{T^2} - \frac{(-1-T)\, c_{3,10}}{2T^3},\ d_{3,12} \to 0,\ d_{3,13} \to 0,\ d_{3,14} \to 0,\ d_{3,15} \to 0,$

$d_{3,16} \to -\frac{1-T}{T^4} - \frac{c_{3,16}}{T},\ d_{3,17} \to -\frac{-7+9T-2T^2}{2T^5} - \frac{(-1+T)\, c_{3,16}}{T^2},\ d_{3,18} \to -\frac{7-9T+2T^2}{6T^6},\ d_{3,19} \to 0,$

$d_{3,20} \to \frac{1}{T^4},\ d_{3,21} \to -\frac{9-T}{2T^5},\ d_{3,22} \to \frac{3}{2T^6} - \frac{(1-T)\, c_{3,16}}{T^3},\ d_{3,23} \to 0,\ d_{3,24} \to 0,\ d_{3,25} \to \frac{1}{T^5},$

$d_{3,26} \to -\frac{1}{3T^6} + \frac{c_{3,16}}{T^3},\ d_{3,27} \to 0,\ d_{3,28} \to 0,\ d_{3,29} \to 0,\ d_{3,30} \to 0,\ d_{3,31} \to 0,\ d_{3,32} \to -\frac{-1+T}{6T^4} - \frac{c_{3,32}}{T},$

$d_{3,33} \to -\frac{2-3T+T^2}{T^5} - \frac{3 \times (-1+T)\, c_{3,32}}{2T^2},\ d_{3,34} \to -\frac{-16+27T-12T^2+T^3}{6T^6} - \frac{(1-2T+T^2)\, c_{3,32}}{T^3},$

$d_{3,35} \to -\frac{16-27T+12T^2-T^3}{24T^7},\ d_{3,36} \to 0,\ d_{3,37} \to -\frac{1}{6T^4},\ d_{3,38} \to -\frac{-3+T}{T^5},$

$d_{3,39} \to \frac{3 \times (-3+T)}{2T^6},\ d_{3,40} \to -\frac{-27+5T-T^2-T^3}{24T^7} - \frac{(-1+2T-T^2)\, c_{3,32}}{T^4},\ d_{3,41} \to 0,$

$d_{3,42} \to 0,\ d_{3,43} \to -\frac{1}{T^5},\ d_{3,44} \to \frac{2}{T^6},\ d_{3,45} \to -\frac{12-T-5T^2}{24T^7} + \frac{3 \times (-1+T)\, c_{3,32}}{2T^4},$

$d_{3,46} \to 0,\ d_{3,47} \to 0,\ d_{3,48} \to 0,\ d_{3,49} \to -\frac{1}{6T^6},\ d_{3,50} \to -\frac{-1-T}{24T^7} + \frac{c_{3,32}}{T^4},\ d_{3,51} \to 0,$

$d_{3,52} \to 0,\ d_{3,53} \to 0,\ d_{3,54} \to 0,\ d_{3,55} \to 0,\ e_{3,1} \to \frac{c_{3,3}}{2T},\ e_{3,2} \to -\frac{c_{3,10}}{T},\ e_{3,3} \to 0,$

$e_{3,4} \to 0,\ e_{3,5} \to 0,\ f_{3,1} \to -\frac{c_{3,3}}{2T},\ f_{3,2} \to \frac{1}{T^3} + \frac{c_{3,10}}{T},\ f_{3,3} \to 0,\ f_{3,4} \to 0,\ f_{3,5} \to 0 \}\}$

In[ ]:= `sol /. (a_ → b_) :→ (a = b)`

Out[ ]= $\left\{0, -T c_{3,2} - c_{3,3}, 0, -\frac{1}{2} \times (1-T)\, c_{3,10}, 0, -T c_{3,7} - \frac{1}{2} \times (-1+3T)\, c_{3,10}, 0, 0, 0, 0, -((-1+T)\, c_{3,16}), -\frac{1-T}{6T}, 0, 0, \frac{1}{2T}, -\frac{-2+5T}{2T} - (T-T^2)\, c_{3,16}, 0, 0, 0, \frac{5}{6} - T^2 c_{3,16}, 0, 0, 0, 0, 0, -\frac{3}{2} \times (-1+T)\, c_{3,32}, -((-1+2T-T^2)\, c_{3,32}), -\frac{1-12T+27T^2-16T^3}{24T^2}, 0, \frac{1}{6T^2}, -\frac{-1+3T}{4T^2}, -\frac{-1+11T-16T^2}{6T^2}, -\frac{-1+31T-131T^2+125T^3}{24T^2} - (T-2T^2+T^3)\, c_{3,32}, 0, 0, \frac{1}{T}, -\frac{-5+23T}{6T}, -\frac{-5+69T-142T^2}{24T} + \frac{3}{2} \times (-1+T)\, T^2 c_{3,32}, 0, 0, 0, \frac{1}{6}, \frac{1}{24} \times (1-15T) - T^3 c_{3,32}, 0, 0, 0, 0, 0, -c_{3,1}, -c_{3,2}, -\frac{c_{3,3}}{T^2}, 0, \frac{c_{3,2}}{T} + \frac{c_{3,3}}{T^2}, 0, -\frac{-1+T}{T^4} - \frac{c_{3,7}}{T} - \frac{(-1+T)\, c_{3,10}}{T^2}, -\frac{1-T}{2T^5} - \frac{(1-T)\, c_{3,10}}{2T^3}, 0, -\frac{1}{T^4} - \frac{c_{3,10}}{T^2}, \frac{1}{2T^5} + \frac{c_{3,7}}{T^2} - \frac{(-1-T)\, c_{3,10}}{2T^3}, 0, 0, 0, 0, -\frac{1-T}{T^4} - \frac{c_{3,16}}{T}, -\frac{-7+9T-2T^2}{2T^5} - \frac{(-1+T)\, c_{3,16}}{T^2}, -\frac{7-9T+2T^2}{6T^6}, 0, \frac{1}{T^4}, -\frac{9-T}{2T^5}, \frac{3}{2T^6} - \frac{(1-T)\, c_{3,16}}{T^3}, 0, 0, \frac{1}{T^5}, -\frac{1}{3T^6} + \frac{c_{3,16}}{T^3}, 0, 0, 0, 0, 0, -\frac{-1+T}{6T^4} - \frac{c_{3,32}}{T}, -\frac{2-3T+T^2}{T^5} - \frac{3 \times (-1+T)\, c_{3,32}}{2T^2}, -\frac{-16+27T-12T^2+T^3}{6T^6} - \frac{(1-2T+T^2)\, c_{3,32}}{T^3}, -\frac{16-27T+12T^2-T^3}{24T^7}, 0, -\frac{1}{6T^4}, -\frac{-3+T}{T^5}, \frac{3 \times (-3+T)}{2T^6}, -\frac{-27+5T-T^2-T^3}{24T^7} - \frac{(-1+2T-T^2)\, c_{3,32}}{T^4}, 0, 0, -\frac{1}{T^5}, \frac{2}{T^6}, -\frac{12-T-5T^2}{24T^7} + \frac{3 \times (-1+T)\, c_{3,32}}{2T^4}, 0, 0, 0, -\frac{1}{6T^6}, -\frac{-1-T}{24T^7} + \frac{c_{3,32}}{T^4}, 0, 0, 0, 0, 0, \frac{c_{3,3}}{2T}, -\frac{c_{3,10}}{T}, 0, 0, 0, -\frac{c_{3,3}}{2T}, \frac{1}{T^3} + \frac{c_{3,10}}{T}, 0, 0, 0\right\}$

In[∘]:= $c_{3,1} = c_{3,2} = c_{3,3} = c_{3,7} = c_{3,10} = c_{3,16} = c_{3,32} = 0;$
$\{R_{1,2}, \overline{R}_{1,2}, C_1, \overline{C}_1\}$

Out[∘]=
$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ (-1+T)\ (p_1-p_2)\ x_2,\ \epsilon\text{Series}\Big[0,\ p_1 p_2 x_1 x_2 + \frac{1}{2}\times(-1+T)\ p_1^2 x_2^2 + \frac{1}{2}\times(1-3T)\ p_1 p_2 x_2^2,$$
$$-\frac{p_1^2 p_2 x_1^2 x_2}{2T} - \frac{1}{2} p_1 p_2 x_2^2 - \frac{(1-3T)\ p_1^2 p_2 x_1 x_2^2}{2T} - \frac{1}{2} p_1 p_2^2 x_1 x_2^2 -$$
$$\frac{(-1+4T-3T^2)\ p_1^3 x_2^3}{6T} - \frac{(1-11T+16T^2)\ p_1^2 p_2 x_2^3}{6T} + \frac{1}{6}\times(-1+7T)\ p_1 p_2^2 x_2^3,$$
$$\frac{p_1^3 p_2 x_1^3 x_2}{6T^2} + \frac{p_1^2 p_2 x_1 x_2^2}{2T} - \frac{(-1+3T)\ p_1^3 p_2 x_1^2 x_2^2}{4T^2} + \frac{p_1^2 p_2^2 x_1^2 x_2^2}{T} - \frac{(1-T)\ p_1^3 x_2^3}{6T} -$$
$$\frac{(-2+5T)\ p_1^2 p_2 x_2^3}{2T} + \frac{5}{6} p_1 p_2^2 x_2^3 - \frac{(-1+11T-16T^2)\ p_1^3 p_2 x_1 x_2^3}{6T^2} - \frac{(-5+23T)\ p_1^2 p_2^2 x_1 x_2^3}{6T} +$$
$$\frac{1}{6} p_1 p_2^3 x_1 x_2^3 - \frac{(1-12T+27T^2-16T^3)\ p_1^4 x_2^4}{24T^2} - \frac{(-1+31T-131T^2+125T^3)\ p_1^3 p_2 x_2^4}{24T^2} -$$
$$\frac{(-5+69T-142T^2)\ p_1^2 p_2^2 x_2^4}{24T} + \frac{1}{24}\times(1-15T)\ p_1 p_2^3 x_2^4\Big]\Big],\ \mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ \left(-1+\frac{1}{T}\right)\ (p_1-p_2)\ x_2,$$
$$\epsilon\text{Series}\Big[0,\ -\frac{(-1+T)\ p_1^2 x_1 x_2}{T^2} - \frac{p_1 p_2 x_1 x_2}{T^2} - \frac{(1-T)\ p_1^2 x_2^2}{2T^3} - \frac{(-1-T)\ p_1 p_2 x_2^2}{2T^3},$$
$$-\frac{(1-T)\ p_1^2 x_1 x_2}{T^3} + \frac{p_1 p_2 x_1 x_2}{T^3} - \frac{(-1+T)\ p_1^3 x_1^2 x_2}{2T^3} - \frac{p_1^2 p_2 x_1^2 x_2}{2T^3} - \frac{(-1+T)\ p_1^2 x_2^2}{2T^4} - \frac{p_1 p_2 x_2^2}{2T^4} -$$
$$\frac{(3-4T+T^2)\ p_1^3 x_1 x_2^2}{2T^4} + \frac{2 p_1^2 p_2 x_1 x_2^2}{T^4} - \frac{p_1 p_2^2 x_1 x_2^2}{2T^4} - \frac{(-3+4T-T^2)\ p_1^3 x_2^3}{6T^5} - \frac{(4+T+T^2)\ p_1^2 p_2 x_2^3}{6T^5} -$$
$$\frac{(-1+T)\ p_1 p_2^2 x_2^3}{6T^5},\ -\frac{(-1+T)\ p_1^2 x_1 x_2}{T^4} - \frac{p_1 p_2 x_1 x_2}{T^4} - \frac{(1-T)\ p_1^3 x_1^2 x_2}{T^4} + \frac{p_1^2 p_2 x_1^2 x_2}{T^4} -$$
$$\frac{(-1+T)\ p_1^4 x_1^3 x_2}{6T^4} - \frac{p_1^3 p_2 x_1^3 x_2}{6T^4} - \frac{(1-T)\ p_1^2 x_2^2}{2T^5} + \frac{p_1 p_2 x_2^2}{2T^5} - \frac{(-7+9T-2T^2)\ p_1^3 x_1 x_2^2}{2T^5} -$$
$$\frac{(9-T)\ p_1^2 p_2 x_1 x_2^2}{2T^5} + \frac{p_1 p_2^2 x_1 x_2^2}{T^5} - \frac{(2-3T+T^2)\ p_1^4 x_1^2 x_2^2}{T^5} - \frac{(-3+T)\ p_1^3 p_2 x_1^2 x_2^2}{T^5} - \frac{p_1^2 p_2^2 x_1^2 x_2^2}{T^5} -$$
$$\frac{(7-9T+2T^2)\ p_1^3 x_2^3}{6T^6} + \frac{3 p_1^2 p_2 x_2^3}{2T^6} - \frac{p_1 p_2^2 x_2^3}{3T^6} - \frac{(-16+27T-12T^2+T^3)\ p_1^4 x_1 x_2^3}{6T^6} +$$
$$\frac{3\times(-3+T)\ p_1^3 p_2 x_1 x_2^3}{2T^6} + \frac{2 p_1^2 p_2^2 x_1 x_2^3}{T^6} - \frac{p_1 p_2^3 x_1 x_2^3}{6T^6} - \frac{(16-27T+12T^2-T^3)\ p_1^4 x_2^4}{24T^7} -$$
$$\frac{(-27+5T-T^2-T^3)\ p_1^3 p_2 x_2^4}{24T^7} - \frac{(12-T-5T^2)\ p_1^2 p_2^2 x_2^4}{24T^7} - \frac{(-1-T)\ p_1 p_2^3 x_2^4}{24T^7}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\ 0,\ \epsilon\text{Series}\Big[0,\ -\frac{p_1 x_1}{T},\ 0,\ 0\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\ 0,\ \epsilon\text{Series}\Big[0,\ \frac{p_1 x_1}{T},\ -\frac{p_1 x_1}{T^2},\ \frac{p_1 x_1}{T^3}\Big]\Big]\Big\}$$

In[∘]:= $\{(R_{1,2}\, R_{4,3}\, R_{5,6}\ //\ m_{1,4\to 1}\ //\ m_{2,5\to 2}\ //\ m_{3,6\to 3}) \equiv (R_{2,3}\, R_{4,5}\, R_{1,6}\ //\ m_{1,4\to 1}\ //\ m_{2,5\to 2}\ //\ m_{3,6\to 3}),$
$(R_{1,2}\, \overline{R}_{3,4}\ //\ m_{1,3\to 1}\ //\ m_{2,4\to 2}) \equiv (\eta_1\, \eta_2),$
$(C_1\, \overline{C}_2\ //\ m_{1,2\to 1}) \equiv \eta_1,$
$(C_3\, R_{1,2}\ //\ m_{2,3\to 2}\ //\ m_{2,1\to 1}) \equiv (\overline{C}_3\, R_{1,2}\ //\ m_{1,3\to 1}\ //\ m_{1,2\to 1})\}$

Out[∘]= {True, True, True, True}