

```
In[*]:= Column@Table [
  Coefficient[R1,2, e] → Coefficient[R1,2 R4,3 // m1,4→1, e],
  {e, KBasis@{1}}
]
```

$$a_2 + T d_2 \rightarrow a_2 a_3 + T a_3 d_2 + T a_2 d_3 + T^2 d_2 d_3$$

$$\emptyset \rightarrow \emptyset$$

$$\emptyset \rightarrow \emptyset$$

```
Out[*]:= a_2 - T d_2 → a_2 a_3 - T a_3 d_2 - T a_2 d_3 + T^2 d_2 d_3
∅ → ∅
∅ → ∅
```

$$-((1 - T) kb_2) \rightarrow -a_3 kb_2 + T a_3 kb_2 - T d_3 kb_2 + T^2 d_3 kb_2 - a_2 kb_3 + T a_2 kb_3 + T d_2 kb_3 - T^2 d_2 kb_3$$

$$\emptyset \rightarrow \emptyset$$

```
In[*]:= Column@Table [
  Coefficient[R1,3, e] → Coefficient[R2,3 R1,4 // m3,4→3, e],
  {e, KBasis@{3}}
]
```

$$a_1 + d_1 \rightarrow a_1 a_2 + a_2 d_1 + a_1 d_2 + d_1 d_2$$

$$\emptyset \rightarrow \emptyset$$

$$\emptyset \rightarrow \emptyset$$

```
Out[*]:= T a_1 - T d_1 → T^2 a_1 a_2 - T^2 a_2 d_1 - T^2 a_1 d_2 + T^2 d_1 d_2
∅ → ∅
```

$$-((1 - T) kc_1) \rightarrow -a_2 kc_1 + T a_2 kc_1 - d_2 kc_1 + T d_2 kc_1 - T a_1 kc_2 + T^2 a_1 kc_2 + T d_1 kc_2 - T^2 d_1 kc_2$$

$$\emptyset \rightarrow \emptyset$$

$$\emptyset \rightarrow \emptyset$$

```
In[*]:= Simplify [
  (a_1 a_2 + a_2 d_1 + a_1 d_2 + d_1 d_2 + (T^2 a_1 a_2 - T^2 a_2 d_1 - T^2 a_1 d_2 + T^2 d_1 d_2) / T) / 2
]
```

```
In[*]:= 1/2 (a_1 ((1 + T) a_2 - (-1 + T) d_2) + d_1 (-((-1 + T) a_2) + (1 + T) d_2)) // Expand
```

```
Out[*]:= a_1 a_2 / 2 + 1/2 T a_1 a_2 + a_2 d_1 / 2 - 1/2 T a_2 d_1 + a_1 d_2 / 2 - 1/2 T a_1 d_2 + d_1 d_2 / 2 + 1/2 T d_1 d_2
```

```
In[*]:= E_ // Δi→j_,k_ := Expand [E /. {
  a_i → a_j η_k + γ_j a_k, b_i → b_j η_k + γ_j b_k, c_i → c_j η_k + γ_j c_k, d_i → d_j η_k + γ_j d_k,
  ka_i → ka_j γ_k - η_j a_k, kb_i → kb_j γ_k - η_j b_k, kc_i → kc_j γ_k - η_j c_k, kd_i → kd_j γ_k - η_j d_k}]]
```

```
In[*]:= lhs = KBasis@{1} // Δ1→1,2 // Δ2→2,3
rhs = KBasis@{1} // Δ1→1,3 // Δ1→1,2
lhs - rhs
```

$$\begin{aligned}
 \text{Out}[*]= & \{ a_1 a_2 a_3 + a_1 a_3 d_2 + a_1 a_2 d_3 + a_1 d_2 d_3 + a_2 a_3 ka_1 + a_2 d_3 ka_1 + a_1 a_3 ka_2 + a_1 d_3 ka_2 + \\
 & a_3 ka_1 ka_2 + a_2 a_3 kd_1 + a_2 d_3 kd_1 + a_3 ka_2 kd_1 + a_1 a_3 kd_2 + a_1 d_3 kd_2 + a_3 ka_1 kd_2 + a_3 kd_1 kd_2, \\
 & a_2 a_3 b_1 + a_3 b_1 d_2 + a_2 b_1 d_3 + b_1 d_2 d_3 + a_3 b_2 ka_1 + b_2 d_3 ka_1 + a_3 b_1 ka_2 + b_1 d_3 ka_2 + \\
 & b_3 ka_1 ka_2 + a_3 b_2 kd_1 + b_2 d_3 kd_1 + b_3 ka_2 kd_1 + a_3 b_1 kd_2 + b_1 d_3 kd_2 + b_3 ka_1 kd_2 + b_3 kd_1 kd_2, \\
 & a_2 a_3 c_1 + a_3 c_1 d_2 + a_2 c_1 d_3 + c_1 d_2 d_3 + a_3 c_2 ka_1 + c_2 d_3 ka_1 + a_3 c_1 ka_2 + c_1 d_3 ka_2 + \\
 & c_3 ka_1 ka_2 + a_3 c_2 kd_1 + c_2 d_3 kd_1 + c_3 ka_2 kd_1 + a_3 c_1 kd_2 + c_1 d_3 kd_2 + c_3 ka_1 kd_2 + c_3 kd_1 kd_2, \\
 & a_2 a_3 d_1 + a_3 d_1 d_2 + a_2 d_1 d_3 + d_1 d_2 d_3 + a_3 d_2 ka_1 + d_2 d_3 ka_1 + a_3 d_1 ka_2 + d_1 d_3 ka_2 + \\
 & d_3 ka_1 ka_2 + a_3 d_2 kd_1 + d_2 d_3 kd_1 + d_3 ka_2 kd_1 + a_3 d_1 kd_2 + d_1 d_3 kd_2 + d_3 ka_1 kd_2 + d_3 kd_1 kd_2, \\
 & -a_1 a_2 a_3 - a_2 a_3 d_1 - a_1 a_2 d_3 - a_2 d_1 d_3 - a_2 a_3 ka_1 - a_3 d_2 ka_1 - a_2 d_3 ka_1 - d_2 d_3 ka_1 - a_1 a_3 ka_2 - \\
 & a_3 d_1 ka_2 + ka_1 ka_2 ka_3 - a_1 a_3 kd_2 - a_3 d_1 kd_2 + ka_1 ka_3 kd_2 + ka_1 ka_2 kd_3 + ka_1 kd_2 kd_3, \\
 & -a_1 a_3 b_2 - a_3 b_2 d_1 - a_1 b_2 d_3 - b_2 d_1 d_3 - a_1 b_3 ka_2 - b_3 d_1 ka_2 - a_2 a_3 kb_1 - a_3 d_2 kb_1 - a_2 d_3 kb_1 - \\
 & d_2 d_3 kb_1 + ka_2 ka_3 kb_1 - a_1 b_3 kd_2 - b_3 d_1 kd_2 + ka_3 kb_1 kd_2 + ka_2 kb_1 kd_3 + kb_1 kd_2 kd_3, \\
 & -a_1 a_3 c_2 - a_3 c_2 d_1 - a_1 c_2 d_3 - c_2 d_1 d_3 - a_1 c_3 ka_2 - c_3 d_1 ka_2 - a_2 a_3 kc_1 - a_3 d_2 kc_1 - a_2 d_3 kc_1 - \\
 & d_2 d_3 kc_1 + ka_2 ka_3 kc_1 - a_1 c_3 kd_2 - c_3 d_1 kd_2 + ka_3 kc_1 kd_2 + ka_2 kc_1 kd_3 + kc_1 kd_2 kd_3, \\
 & -a_1 a_3 d_2 - a_3 d_1 d_2 - a_1 d_2 d_3 - d_1 d_2 d_3 - a_1 d_3 ka_2 - d_1 d_3 ka_2 - a_2 a_3 kd_1 - a_3 d_2 kd_1 - a_2 d_3 kd_1 - \\
 & d_2 d_3 kd_1 + ka_2 ka_3 kd_1 - a_1 d_3 kd_2 - d_1 d_3 kd_2 + ka_3 kd_1 kd_2 + ka_2 kd_1 kd_3 + kd_1 kd_2 kd_3 \}
 \end{aligned}$$

$$\begin{aligned}
 \text{Out}[*]= & \{ -a_2 a_3 d_1 - a_3 d_1 d_2 + a_1 a_2 d_3 + a_1 d_2 d_3 + a_2 a_3 ka_1 + \\
 & a_2 d_3 ka_1 + a_3 ka_1 ka_2 + a_2 a_3 kd_1 + a_2 d_3 kd_1 + a_3 ka_2 kd_1 + a_3 ka_1 kd_2 + a_3 kd_1 kd_2, \\
 & a_2 a_3 b_1 - a_1 a_2 b_3 - a_2 b_3 d_1 + a_3 b_1 d_2 - a_1 b_3 d_2 - b_3 d_1 d_2 + a_2 b_1 d_3 + b_1 d_2 d_3 + a_3 b_2 ka_1 + \\
 & b_2 d_3 ka_1 + b_3 ka_1 ka_2 + a_3 b_2 kd_1 + b_2 d_3 kd_1 + b_3 ka_2 kd_1 + b_3 ka_1 kd_2 + b_3 kd_1 kd_2, \\
 & a_2 a_3 c_1 - a_1 a_2 c_3 - a_2 c_3 d_1 + a_3 c_1 d_2 - a_1 c_3 d_2 - c_3 d_1 d_2 + a_2 c_1 d_3 + c_1 d_2 d_3 + a_3 c_2 ka_1 + \\
 & c_2 d_3 ka_1 + c_3 ka_1 ka_2 + a_3 c_2 kd_1 + c_2 d_3 kd_1 + c_3 ka_2 kd_1 + c_3 ka_1 kd_2 + c_3 kd_1 kd_2, \\
 & a_2 a_3 d_1 + a_3 d_1 d_2 - a_1 a_2 d_3 - a_1 d_2 d_3 + a_3 d_2 ka_1 + d_2 d_3 ka_1 + d_3 ka_1 ka_2 + \\
 & a_3 d_2 kd_1 + d_2 d_3 kd_1 + d_3 ka_2 kd_1 + d_3 ka_1 kd_2 + d_3 kd_1 kd_2, \\
 & -a_1 a_2 a_3 - a_2 a_3 d_1 - a_1 a_3 d_2 - a_3 d_1 d_2 - a_2 a_3 ka_1 - a_3 d_2 ka_1 - a_1 a_2 ka_3 - a_2 d_1 ka_3 + ka_1 ka_2 ka_3 - \\
 & a_2 a_3 kd_1 - a_3 d_2 kd_1 + ka_1 ka_3 kd_2 - a_1 a_2 kd_3 - a_2 d_1 kd_3 + ka_1 ka_2 kd_3 + ka_1 kd_2 kd_3, \\
 & -a_1 a_2 b_3 - a_2 b_3 d_1 - a_1 b_3 d_2 - b_3 d_1 d_2 - a_2 b_3 ka_1 - b_3 d_2 ka_1 - a_1 b_2 ka_3 - b_2 d_1 ka_3 + ka_2 ka_3 kb_1 - \\
 & a_2 b_3 kd_1 - b_3 d_2 kd_1 + ka_3 kb_1 kd_2 - a_1 b_2 kd_3 - b_2 d_1 kd_3 + ka_2 kb_1 kd_3 + kb_1 kd_2 kd_3, \\
 & -a_1 a_2 c_3 - a_2 c_3 d_1 - a_1 c_3 d_2 - c_3 d_1 d_2 - a_2 c_3 ka_1 - c_3 d_2 ka_1 - a_1 c_2 ka_3 - c_2 d_1 ka_3 + ka_2 ka_3 kc_1 - \\
 & a_2 c_3 kd_1 - c_3 d_2 kd_1 + ka_3 kc_1 kd_2 - a_1 c_2 kd_3 - c_2 d_1 kd_3 + ka_2 kc_1 kd_3 + kc_1 kd_2 kd_3, \\
 & -a_1 a_2 d_3 - a_2 d_1 d_3 - a_1 d_2 d_3 - d_1 d_2 d_3 - a_2 d_3 ka_1 - d_2 d_3 ka_1 - a_1 d_2 ka_3 - d_1 d_2 ka_3 - a_2 d_3 kd_1 - \\
 & d_2 d_3 kd_1 + ka_2 ka_3 kd_1 + ka_3 kd_1 kd_2 - a_1 d_2 kd_3 - d_1 d_2 kd_3 + ka_2 kd_1 kd_3 + kd_1 kd_2 kd_3 \}
 \end{aligned}$$

$$\begin{aligned}
\text{Out}[*]= & \{ a_1 a_2 a_3 + a_2 a_3 d_1 + a_1 a_3 d_2 + a_3 d_1 d_2 + a_1 a_3 k a_2 + a_1 d_3 k a_2 + a_1 a_3 k d_2 + a_1 d_3 k d_2, \\
& a_1 a_2 b_3 + a_2 b_3 d_1 + a_1 b_3 d_2 + b_3 d_1 d_2 + a_3 b_1 k a_2 + b_1 d_3 k a_2 + a_3 b_1 k d_2 + b_1 d_3 k d_2, \\
& a_1 a_2 c_3 + a_2 c_3 d_1 + a_1 c_3 d_2 + c_3 d_1 d_2 + a_3 c_1 k a_2 + c_1 d_3 k a_2 + a_3 c_1 k d_2 + c_1 d_3 k d_2, \\
& a_1 a_2 d_3 + a_2 d_1 d_3 + a_1 d_2 d_3 + d_1 d_2 d_3 + a_3 d_1 k a_2 + d_1 d_3 k a_2 + a_3 d_1 k d_2 + d_1 d_3 k d_2, \\
& a_1 a_3 d_2 + a_3 d_1 d_2 - a_1 a_2 d_3 - a_2 d_1 d_3 - a_2 d_3 k a_1 - d_2 d_3 k a_1 - a_1 a_3 k a_2 - a_3 d_1 k a_2 + \\
& a_1 a_2 k a_3 + a_2 d_1 k a_3 + a_2 a_3 k d_1 + a_3 d_2 k d_1 - a_1 a_3 k d_2 - a_3 d_1 k d_2 + a_1 a_2 k d_3 + a_2 d_1 k d_3, \\
& - a_1 a_3 b_2 + a_1 a_2 b_3 - a_3 b_2 d_1 + a_2 b_3 d_1 + a_1 b_3 d_2 + b_3 d_1 d_2 - a_1 b_2 d_3 - b_2 d_1 d_3 + a_2 b_3 k a_1 + \\
& b_3 d_2 k a_1 - a_1 b_3 k a_2 - b_3 d_1 k a_2 + a_1 b_2 k a_3 + b_2 d_1 k a_3 - a_2 a_3 k b_1 - a_3 d_2 k b_1 - \\
& a_2 d_3 k b_1 - d_2 d_3 k b_1 + a_2 b_3 k d_1 + b_3 d_2 k d_1 - a_1 b_3 k d_2 - b_3 d_1 k d_2 + a_1 b_2 k d_3 + b_2 d_1 k d_3, \\
& - a_1 a_3 c_2 + a_1 a_2 c_3 - a_3 c_2 d_1 + a_2 c_3 d_1 + a_1 c_3 d_2 + c_3 d_1 d_2 - a_1 c_2 d_3 - c_2 d_1 d_3 + a_2 c_3 k a_1 + \\
& c_3 d_2 k a_1 - a_1 c_3 k a_2 - c_3 d_1 k a_2 + a_1 c_2 k a_3 + c_2 d_1 k a_3 - a_2 a_3 k c_1 - a_3 d_2 k c_1 - \\
& a_2 d_3 k c_1 - d_2 d_3 k c_1 + a_2 c_3 k d_1 + c_3 d_2 k d_1 - a_1 c_3 k d_2 - c_3 d_1 k d_2 + a_1 c_2 k d_3 + c_2 d_1 k d_3, \\
& - a_1 a_3 d_2 - a_3 d_1 d_2 + a_1 a_2 d_3 + a_2 d_1 d_3 + a_2 d_3 k a_1 + d_2 d_3 k a_1 - a_1 d_3 k a_2 - d_1 d_3 k a_2 + \\
& a_1 d_2 k a_3 + d_1 d_2 k a_3 - a_2 a_3 k d_1 - a_3 d_2 k d_1 - a_1 d_3 k d_2 - d_1 d_3 k d_2 + a_1 d_2 k d_3 + d_1 d_2 k d_3 \}
\end{aligned}$$