MAT 1350F Algebraic Knot Theory
Dror's Open Private Notebook

Hour 1, Friday September 10: Mathematical Course Introduction

on board: Goals.

1. "poly-time computable strong knot invariant w/ good alg. prop"
2. Learn math.
3. Learn computation

Explain knot invariant.

Explain strong.

Explain good algebraic properties

Explain poly-time computable

The YB technique

Hopf algebras

How?

~~Generating functions.~~

~~Go over plan~~

~~Start w/ Kauffman bracket.~~

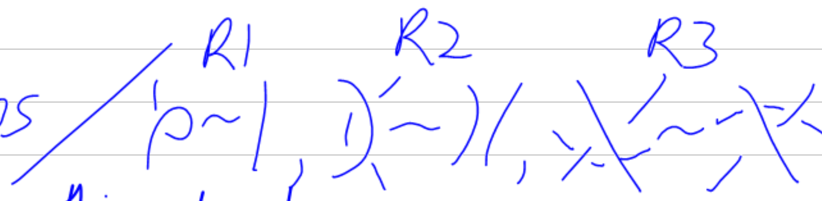
not done

Hour 2, Monday September 13: Administrative Course Introduction, the Jones Polynomial

Please register!!

~~Thm~~ Def

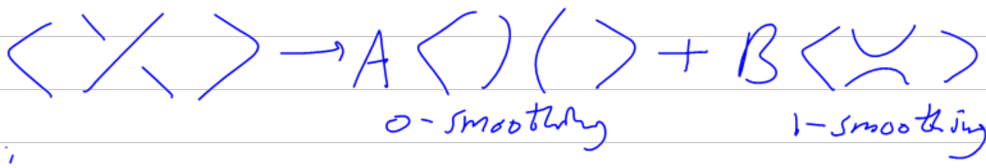
Knots = Knot diagrams /



* Explain why I don't like this chapter

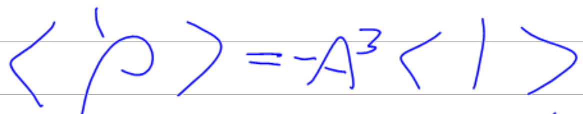
* Explain both sides.

The Kauffman bracket.



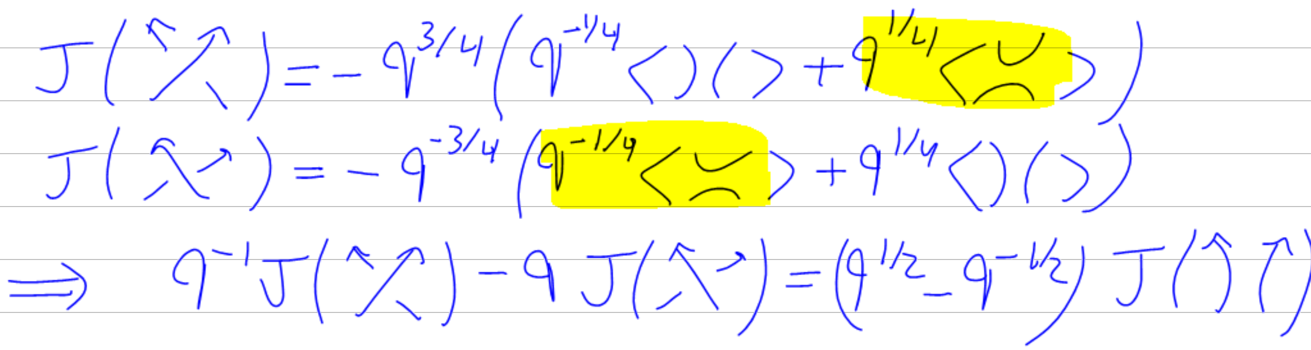
$\langle D \bigcirc \rangle = d \langle D \rangle$

$\longrightarrow B = A^{-1},\ d = -A^2 - A^{-2}$



$J(K) := (-A^3)^{-w(K)} \dfrac{\langle K \rangle}{d} \Big/_{A \to q^{-1/4}}$

The Jones skein relation:



The Kauffman $\langle\cdot\rangle : \left\{\begin{matrix}\text{unoriented}\\ \text{knots/links}\end{matrix}\right\} \longrightarrow \mathbb{Z}[\cancel{A,B,d}]$ $\mathbb{Z}[A,A^{-1}]$ by

$$\langle \times \rangle = A\underset{0\text{-smoothing}}{\langle)(\rangle} + B\underset{1\text{-smoothing}}{\langle \asymp \rangle} \qquad \langle \underbrace{\circ\circ\circ\cdots\circ}_{k} \rangle = d^k$$

R2 necessitates $B=A^{-1}$, $d=-A^2-A^{-2}$

on board.

Prove R3, discuss R1:

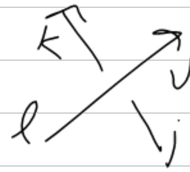
$$\langle \rho \rangle = -A^3 \langle \mid \rangle$$

$$J(K) := (-A^3)^{-w(K)}\frac{\langle K\rangle}{d}\Big/_{A\to q^{-1/4}}$$

on to implementation as in Jones.nb.

Hour 4, Friday September 17: Jones implementation and better implementation. Please register!
HW01 is on web and on Crowdmark!

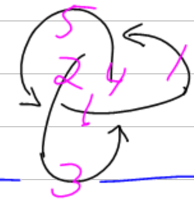
The Kauffman $\langle\cdot\rangle : \left\{\begin{matrix}\text{unoriented}\\ \text{knots/links}\end{matrix}\right\} \longrightarrow \mathbb{Z}[A^{\pm 1}]$ by



X_{ijkl}

$$\langle \times \rangle = A\underbrace{\langle)(\rangle}_{\text{0-smoothing}} + B\underbrace{\langle \asymp \rangle}_{\text{1-smoothing}} \qquad \langle \underbrace{\bigcirc\bigcirc\bigcirc\bigcirc\bigcirc}_{k} \rangle = d^k$$

with $B=A^{-1}$, $d=-A^2-A^{-2}$



PD[X[1, 5, 2, 4], X[5, 3, 6, 2], X[3, 1, 4, 6]]

on board...

Then do Jones2.nb & FasterJones.nb

Hour 5, Monday September 20: Planar algebras and tangles. Please register!

Formal Goals:

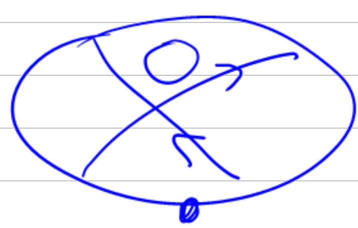
1. "The Kauffman Bracket is a morphism from the planar algebra of framed unoriented tangles to the Temperley-Lieb planar algebra".

2. $\mathrm{Rank}_{\mathbb{Z}[A^{\pm 1}]} TL_{2n} = C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$

Loose Goals. 1. Make the nonsense about "computing half knots" a bit more formal.

2. On beyond zebra! Group, rings, fields, modules, . . .

↑ on board

Tangles  / R1 R2 R3

Unoriented tangles

framed unoriented tangles

planar algebras

planar connection diagrams.

Examples: 1. Unoriented tangles

2. $P_n = V^{\otimes n}$ (f.d. metrized V)

3. TL

Rank.

1. "The Kauffman Bracket is a morphism from the planar algebra of framed unoriented tangles to the Temperley-Lieb planar algebra".

2. $\mathrm{Rank}_{\mathbb{Z}[A^{\pm 1}]} TL_{2n} = C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$

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pf of 2.

Discussion of FastKB

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unknotting number, Gordian distance.

Genus

Ribbon, slice, GST48.

If time: Every knot has a Seifert surface.

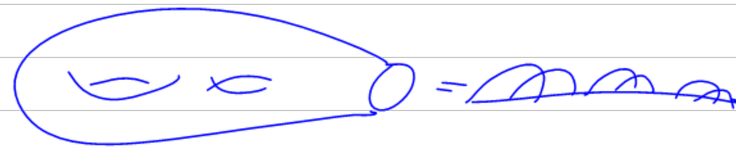
Prep handout:

some Seifert surfaces

a ribbon example

GST48.

Hour 7, Friday September 24: More on surfaces and knots.
HW1 is due by midnight, HW2 will be on web by midnight.

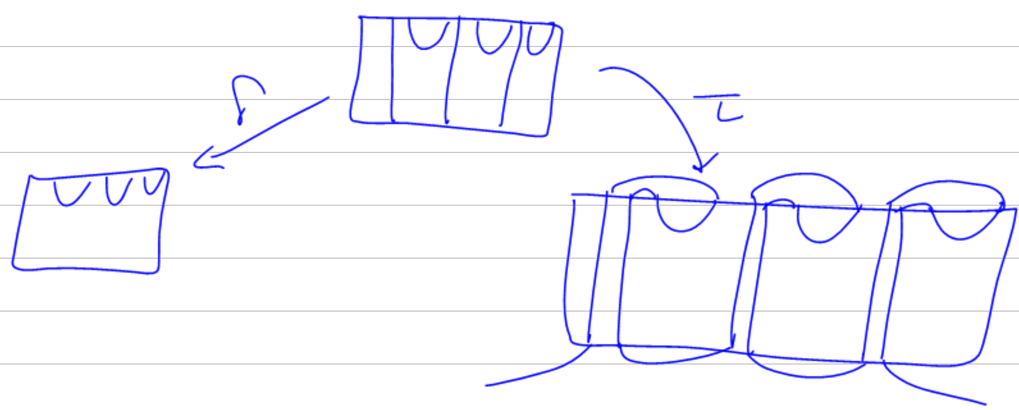
$\Sigma_{1,1} =$  $\Sigma_{g,1} =$ 

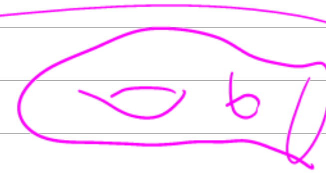
Every thickened graph is a $\Sigma_{g,n}$

Every knot has a Seifert surface.

Every Seifert surface is a thickened 

Thm A knot is ribbon iff $\exists$ tangle T (w/ skeleton as below) with $\delta T = U$ & $\tau T = K$

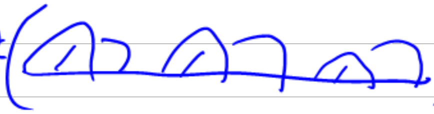


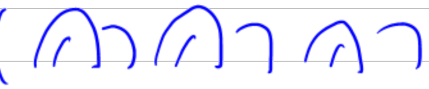
HW. 1. Draw a core in 

2. Prove: If the Gordian distance between two knots is 1, the difference between their genera is at most 1.

3. The tango-only description of ribbon knots w/ no cablings.

Hour 8, Monday September 27: More on surfaces and knots and tangles.

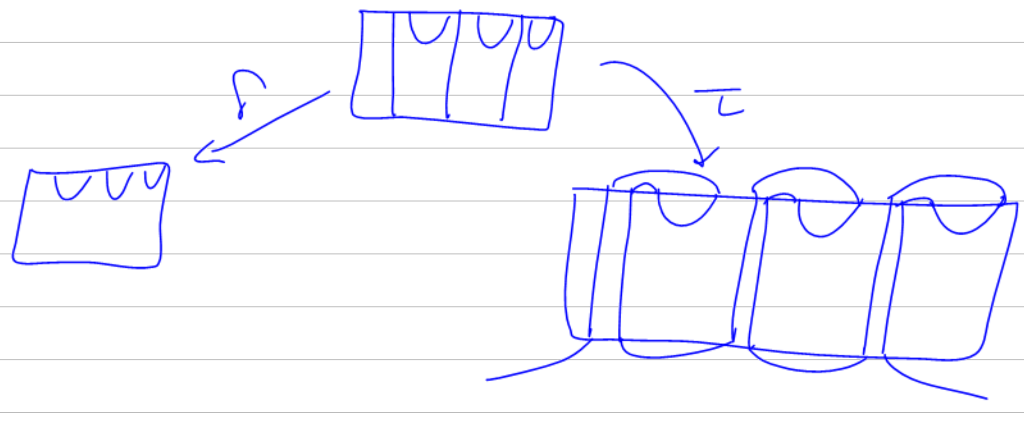
On board. Every genus $\le g$ surface is the boundary of an embedded $\Sigma_{g,1} \simeq$  $=$ Thick()

Thm $T_g : K($  $) \longrightarrow K($  $)$

$$\{K : g(K) \le g\} = \operatorname{im} T_g$$

Morals: Tangles are good, need to know about tangles w/ specifying skel & ops: ___ ___ ___

Thm A knot is ribbon iff $\exists$ tangle T (w/ skeleton as below) with $\delta T = U$ & $\tau T = K$



Intro to YB & IHop algebras.

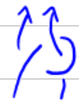
Goal: Find an invariant of knots/links/tangles, compatible w/ PA ops,

$\longrightarrow \quad \rightarrow \quad \rightrightarrows, \quad \leftarrow, \quad \dots\dots$

prep blackboard w/ side for eq'ns!

The YB technique: In an algebra D,  $\longrightarrow R \in D_1 \otimes D_2$ $\quad R = \sum b_\alpha \otimes a_\alpha$

What is needed of D, R?  $\longrightarrow \bar{R} \in D_1 \otimes D_2$ $\quad \bar{R} = \sum \bar{b}_\alpha \otimes \bar{a}_\alpha$

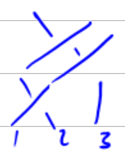
 $\longrightarrow$ multiply along the strands.

Ans 1. D must be an algebra $D \otimes D \xrightarrow{m} A$ s.t. assoc. square commutes.

2. D must be unital: $\eta: \mathbb{Q} \to D$ s.t.

$$\begin{tikzcd} D\otimes\mathbb{Q} \arrow[r, "1\otimes\eta"] \arrow[dr] & D\otimes D \arrow[d, "m"] \\ & D \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathbb{Q}\otimes D \arrow[r, "\eta\otimes 1"] \arrow[dr] & D\otimes D \arrow[d] \\ & D \end{tikzcd}$$

Braid-like relations. R2b  $\sum b_\alpha \bar{b}_\beta \otimes a_\alpha \bar{a}_\beta = 1\otimes 1$ $\quad \bar{R} = R^{-1}$ in $D\otimes D$

R3b  $=$  $\quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ $\quad$ The Yang-Baxter eq'n.

Non-braid-like "cyclic" relations  $\sum \bar{b}_\beta b_\alpha \otimes a_\alpha \bar{a}_\beta = 1\otimes 1$ $\quad \bar{R}$ is the inverse of R in $D^{op}\otimes D$.

R3c & R1' follow!

Doubling: $\Delta: D \to D\otimes D$ $\quad$ co-associative, rel w/ η & m, quasi-triangularity.

strand deletion: $\epsilon: D \to \mathbb{Q}$ s.t. $\quad m /\!/ \epsilon = \epsilon\otimes\epsilon \quad \Delta /\!/ \epsilon\otimes 1 = \Delta /\!/ 1\otimes\epsilon = I$

strand reversal: $S: D \to D$ $\quad S^2 = I$, anti-homo, $\Delta /\!/ S\otimes I /\!/ m = \epsilon /\!/ \eta$ & reverse

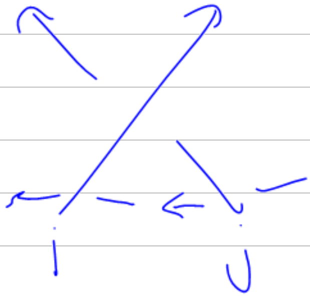
"IHop Algebras"

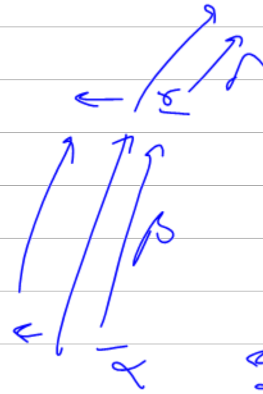
Example For a finite group G, $WG := \langle W(\alpha,\beta) : \alpha,\beta \in G \rangle$ with

$$W(\alpha,\beta)\cdot W(\gamma,\delta) = \delta_{\alpha\beta,\beta\gamma}\, W(\alpha,\beta\delta) \qquad \eta(1) = \sum_\alpha W(\alpha, e) \qquad \epsilon W(\alpha,\beta) = \delta_{\alpha,e}$$

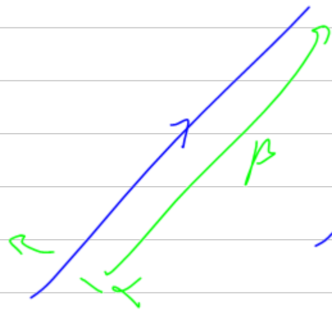
$$\Delta W(\alpha,\beta) = \sum_\gamma W(\gamma,\beta)\otimes W(\alpha\gamma^{-1},\beta) \qquad S W(\alpha,\beta) = W(\beta^{-1}\alpha\beta, \beta^{-1})$$

$$R = \sum_{\alpha,\beta} W(\alpha,1)\otimes W(\beta,\alpha) \qquad \bar{R} = \sum_{\alpha,\beta} W(\alpha,1)\otimes W(\beta,\alpha^{-1})$$


i
j


γ
δ
β
α


β
α


β
α

Hour 10, Friday October 1: IHOP Algebras, R-Elements, and WG.
HW2 due by midnight! HW3 on web by midnight!

Follow the handout!

IHOP Algebras, R-Elements, and WG

IHOP Algebras and R-Elements

Definition. An Involutive Hopf Algebra (IHOP, but only here) is a vector space H that has

- An algebra structure ($m\colon H\otimes H\to H, \eta\colon \mathbb{Q}\to H$) satisfying the usual axioms of an algebra (set $1:=\eta(1)$).
- A co-algebra structure ($\Delta\colon H\to H\otimes H, \varepsilon\colon H\to\mathbb{Q}$) satisfying the "dual" axioms and compatible with the algebra structure in the sense that Δ and ε are morphisms of algebras.
- An "antipode" $S\colon H \to H$ which is an anti-homomorphism of both the algebra structure and the co-algebra structure, which is "involutive", $S^2=I$, and which is a "convolution inverse" of the identity map:

$$\Delta /\!\!/ (I\otimes S) /\!\!/ m = \varepsilon /\!\!/ \eta = \Delta /\!\!/ (S\otimes I) /\!\!/ m.$$

Remember the interpretations!

m: Stitch strands.	η: Insert an empty strand.
Δ: Double a strand.	ε: Delete a strand.
S: Reverse a strand.	R (below): A crossing.

Definition. An "R-element" for H (related to "a quasi-triangular structure") is an invertible $R\in H\otimes H$ such that $\bar R:=R^{-1}$ inverts R also in $H\otimes H^{op}$ and such that

$$(\Delta\otimes I)(R)=R_{23}R_{13} \quad\text{and}\quad (I\otimes\Delta)(R)=R_{12}R_{13},$$
$$(\eta\otimes I)(R)=1=(I\otimes\eta)(R),$$
$$(S\otimes I)(R)=R^{-1}=(I\otimes S)(R).$$

The WG Example

Let G be a finite group with identity element 1 and let $WG:=\mathbb{Q}\langle W(\alpha,\beta)\colon \alpha,\beta\in G\rangle$. Set

$$\begin{aligned}
W(\alpha,\beta)W(\gamma,\delta) &:= \delta_{\alpha\beta,\beta\gamma}W(\alpha,\beta\delta),\\
\eta(1) &:= \sum_\alpha W(\alpha,1),\\
\Delta W(\alpha,\beta) &:= \sum_\gamma W(\gamma,\beta)\otimes W(\alpha\gamma^{-1},\beta),\\
\varepsilon W(\alpha,\beta) &:= \delta_{\alpha,1},\\
SW(\alpha,\beta) &:= W(\beta^{-1}\alpha^{-1}\beta,\beta^{-1}),\\
R &:= \sum_{\alpha,\beta} W(\alpha,1)\otimes W(\beta,\alpha),\\
\bar R &= \sum_{\alpha,\beta} W(\alpha,1)\otimes W(\beta,\alpha^{-1}).
\end{aligned}$$

Proposition. WG is an IHOP algebra and R is an R-element for it.

Proof. Think about homomorphisms from the fundamental group of the complement of a tangle to G.

An Implementation of WG

```
DeclareGroup[S_k_] := Module[{α, β, e, γs},
  Clear[G, n, g, ι, m, inv];
  G = PermutationCycles /@ (Permutations@Range@k);
  n = Length[G];
  Do[g[α] = e = G[[α]]; ι[e] = α, {α, n} ];
  m[] = ι[Cycles[{}]];
  Do[m[α, β] = ι[g[α]~PermutationProduct~g[β]],
   {α, n}, {β, n} ];
  m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
  Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
 ]
```

```
DeclareGroup[S3];
Table[m[i, j], {i, n}, {j, n}] // MatrixForm
```

$$\begin{pmatrix} 1&2&3&4&5&6\\ 2&1&4&3&6&5\\ 3&5&1&6&2&4\\ 4&6&2&5&1&3\\ 5&3&6&1&4&2\\ 6&4&5&2&3&1 \end{pmatrix}$$

```
Basis[] = {1};
Basis[i_, is___] :=
 Flatten@Table[W_i[α, β] Basis[is], {α, n}, {β, n}]
```

```
Basis[1, 2]
```

```
{W_1[1, 1] W_2[1, 1], W_1[1, 1] W_2[1, 2],
 W_1[1, 1] W_2[1, 3], W_1[1, 1] W_2[1, 4],
 W_1[1, 1] W_2[1, 5], W_1[1, 1] W_2[1, 6],
 W_1[1, 1] W_2[2, 1], W_1[1, 1] W_2[2, 2],
 W_1[1, 1] W_2[2, 3], ... 1278 ..., W_1[6, 6] W_2[5, 4],
 W_1[6, 6] W_2[5, 5], W_1[6, 6] W_2[5, 6],
 W_1[6, 6] W_2[6, 1], W_1[6, 6] W_2[6, 2],
 W_1[6, 6] W_2[6, 3], W_1[6, 6] W_2[6, 4],
 W_1[6, 6] W_2[6, 5], W_1[6, 6] W_2[6, 6]}
```

large output | show less | show more | show all | set size limit...

```
m_{i_,j_→k_}[𝒮_] :=
  Expand[𝒮] /. W_i[α_, β_] W_j[γ_, δ_] ↦
    If[m[α, β] == m[β, γ], W_k[α, m[β, δ]], 0];
η_{i_}[𝒮_] := Expand[𝒮 Sum[W_i[α, m[]], {α, n}]];
```

```
Δ_{i_→j_,k_}[𝒮_] :=
  Expand[
   𝒮 /. W_i[α_, β_] ↦
     Sum[W_j[γ, β] W_k[m[α, inv[γ]], β], {γ, n}]];
ε_{i_}[𝒮_] :=
  Expand[𝒮 /. W_i[α_, β_] ↦ If[α == m[], 1, 0]];
```

```
S_{i_}[𝒮_] :=
  Expand[
   𝒮 /. W_i[α_, β_] ↦ W_i[m[inv[β], inv[α], β],
      inv[β]]];
```

```
R_{i_,j_} := Sum[W_i[α, m[]] W_j[β, α], {α, n}, {β, n}];
R̄_{i_,j_} := Sum[W_i[α, m[]] W_j[β, inv@α], {α, n},
   {β, n}];
```

```
b = Basis[1, 2, 3];
(b // m_{1,2→1} // m_{1,3→1}) == (b // m_{2,3→2} // m_{1,2→1})
True
b = Basis[1]; (b // η_2 // m_{1,2→1}) == b == (b // η_2 // m_{1,2→1})
True
b = Basis[1];
(b // Δ_{1→1,2} // Δ_{2→2,3}) == (b // Δ_{1→1,3} // Δ_{1→1,2})
True
b = Basis[1]; (b // Δ_{1→1,2} // ε_2) == b == (b // Δ_{1→2,1} // ε_2)
True
b = Basis[1, 2];
(b // ε_1 // ε_2) == (b // m_{1,2→1} // ε_1)
True
b = Basis[1, 3];
(b // Δ_{1→1,2} // Δ_{3→3,4} // m_{1,3→1} // m_{2,4→2}) ==
 (b // m_{1,3→1} // Δ_{1→1,2})
True
b = Basis[1]; (b // S_1 // S_1) == b
True
b = Basis[1]; (b // Δ_{1→1,2} // S_2 // m_{1,2→1}) ==
 (b // ε_1 // η_1) == (b // Δ_{1→1,2} // S_1 // m_{1,2→1})
True
(R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) == (1 // η_1 // η_2) ==
 (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{4,2→2})
True
(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ==
 (R_{2,3} R_{1,4} R_{5,6} // m_{1,5→1} // m_{2,6→2} // m_{3,4→3})
True
{(R_{1,3} // Δ_{1→1,2}) == (R_{2,3} R_{1,4} // m_{3,4→3}),
 (R_{1,2} // Δ_{2→2,3}) == (R_{0,2} R_{1,3} // m_{0,1→1})}
{True, True}
{(R_{1,2} // ε_1) == (1 // η_2), (R_{1,2} // ε_2) == (1 // η_1)}
{True, True}
(R_{1,2} // S_1) == R̄_{1,2} == (R_{1,2} // S_2)
True
```

Does R1 hold?

```
{R_{1,2} // m_{1,2→1}, 1 // η_1}
{W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] +
  W_1[5, 5] + W_1[6, 6], W_1[1, 1] + W_1[2, 1] +
  W_1[3, 1] + W_1[4, 1] + W_1[5, 1] + W_1[6, 1]}
```

```
Ks = {PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]],
   PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4],
    X[2, 7, 3, 8]],
   PD[X[1, 6, 2, 7], X[3, 8, 4, 9], X[5, 10, 6, 1],
    X[7, 2, 8, 3], X[9, 4, 10, 5]],
   PD[X[1, 4, 2, 5], X[3, 8, 4, 9], X[5, 10, 6, 1],
    X[9, 6, 10, 7], X[7, 2, 8, 3]]};
Z[pd_PD] := Module[{z},
  z =
   Expand[Times @@ pd /.
     x : X[i_, j_, k_, l_] ↦
      If[PositiveQ@x, R_{i,l}, R̄_{j,i}]];
  Do[z = z // m_{1,k→1}, {k, 2 Length@pd}];
  z]
Table[K → Echo[Timing[Z[K]]], {K, Ks}]
{1.01563, W_1[1, 1] + 3 W_1[2, 2] +
  3 W_1[3, 3] + W_1[4, 1] + W_1[5, 1] + 3 W_1[6, 6]}
$Aborted
```

Hour 11, Monday October 4: The WG Algebra: testing, knots, optimization.
On board:

$WG := \mathbb{Q}\langle W(\alpha,\beta)\colon \alpha,\beta\in G\rangle$. Set

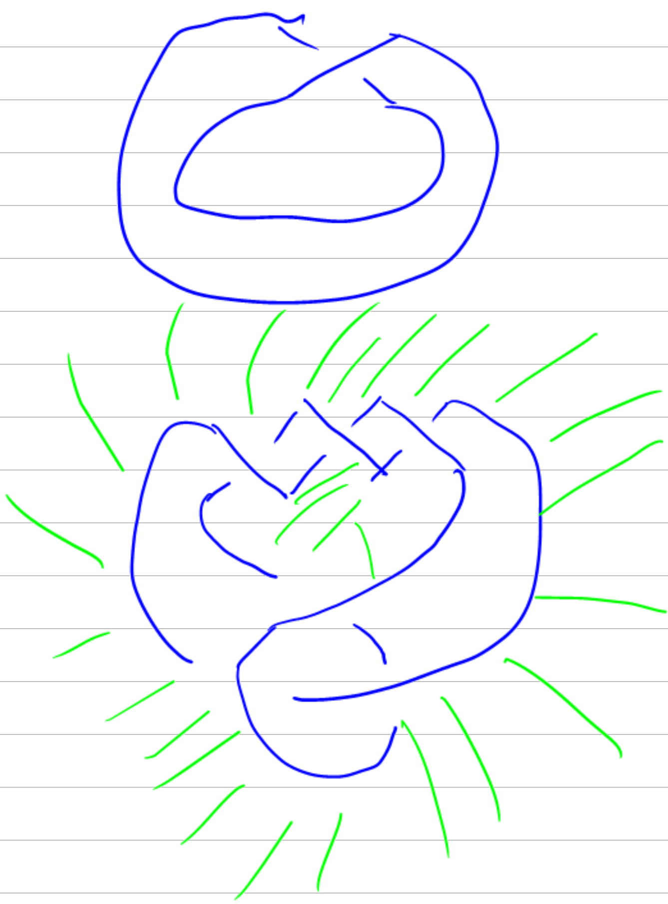
$$\begin{aligned}
W(\alpha,\beta)W(\gamma,\delta) &:= \delta_{\alpha\beta,\beta\gamma}W(\alpha,\beta\delta),\\
\eta(1) &:= \sum_\alpha W(\alpha,1),\\
\Delta W(\alpha,\beta) &:= \sum_\gamma W(\gamma,\beta)\otimes W(\alpha\gamma^{-1},\beta),\\
\varepsilon W(\alpha,\beta) &:= \delta_{\alpha,1},\\
SW(\alpha,\beta) &:= W(\beta^{-1}\alpha^{-1}\beta,\beta^{-1}),\\
R &:= \sum_{\alpha,\beta} W(\alpha,1)\otimes W(\beta,\alpha),\\
\bar{R} &= \sum_{\alpha,\beta} W(\alpha,1)\otimes W(\beta,\alpha^{-1}).
\end{aligned}$$

Assoc, co-assoc, unital, co-unital, (m,Δ) compatible.
S: anti-morphism, $\Delta /\!/ {1\otimes S \atop S\otimes 1} /\!/ m = I$, $S^2=I$

R: compatible w/ (Δ,m), ϵ, S
Satisfies YB (R3)

Then go over WG2.nb.

Scratch:



Hour 12, Wednesday October 6: Meta-IHOPs, virtual tangles, Hopf algebras, meta-Hopf-algebras, rotational virtual tangles.
On board: HW2P2 is false!

RI':  = |

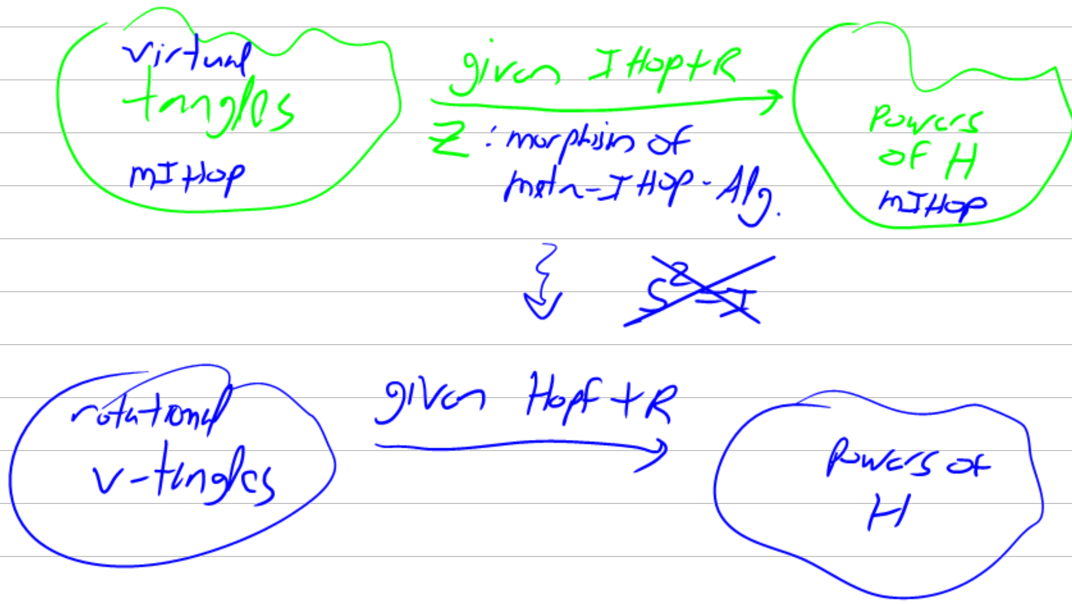
$(R_{12}\bar{R}_{43})//m_1^{12}//m_1^{13}//m_1^{14}$

on board: green

HW2P2 is false
wrong proof...
& indeed,

$u(6_2)=1$ $g(6_2)=2$

1. slow
2. Not enough examples
3. Fuzzy.



Def Meta monoid. $M_X, \sqcup, \sigma^i_j, m^{ij}_k, \eta_i$

Def Meta IHop algebra: add Δ, ϵ, S

Def v-tangles

MM⟨crossings⟩ / (R1', R2, R3, good old planar relations)

extends to a mIHop

done line

other type of v-knots: checkerboard, Alexander, face-centric.

Rotation numbers & rotational v-knots.

Hour 13, Friday October 8: Rotational virtual tangles, Hopf+R, the Kerler(?) Algebra.
Happy Thanksgiving! No class Monday.

virtual tangles mIHop $\xrightarrow[Z:\text{ morphism of meta-IHOP-Alg.}]{\text{given IHop+R}}$ Powers of H mIHop

$\rightsquigarrow$ rotational v-tangles $\xrightarrow{\text{given Hopf+R}}$ Powers of H

<u>Def</u> v-tangles: $MM\langle \nearrow\!\!\!\!\nwarrow, \nwarrow\!\!\!\!\nearrow \rangle / (R1'), R2, R3$ extends to a mIHop

pure

on board

<u>comment:</u> v-tangles know knots & links [but don't know the plane].

other type of v-knots: checker-board, Alexander, face-centric.

<u>Rotation numbers & rotational v-knots.</u>

$$rv\text{-tangles} = MM\langle \nearrow\!\!\!\!\nwarrow, \nwarrow\!\!\!\!\nearrow, \circlearrowleft, \circlearrowright \rangle / R1', R2, R3$$


$= \nwarrow$

$$K = \langle a, b, c, d, k \rangle / \ldots$$

$$\eta_i = a_i + d_i \qquad C_i^{\pm 1} = T^{\mp 1/2}(ka_i + kd_i)$$

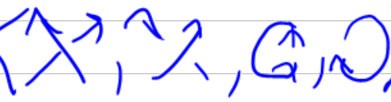
$$R_{ij}^{\pm 1} = T^{\mp 1/2}\left(a_i k a_j - T^{\pm 1} a_i k d_j + (T^{\pm 1} - 1) b_j k c_i + d_i k a_j + T^{\pm 1} d_i k d_j\right)$$

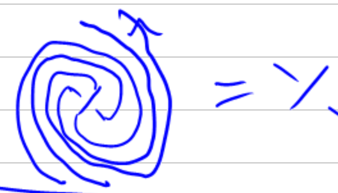
HW4:

1. Turn M_X:=M_{XxX}(bbZ) into a (traced)-meta-IHOP + R computing linking numbers.

rotational v-tangles $\xrightarrow[\text{mtbpf morphism}]{\text{given Hopf}+R}$ Powers of H

board line

rv-tangles = MM⟨  ⟩ / R1', R2b, R2c, R3, W

W: 

$K = \langle a, b, c, d, k \rangle / \ldots$

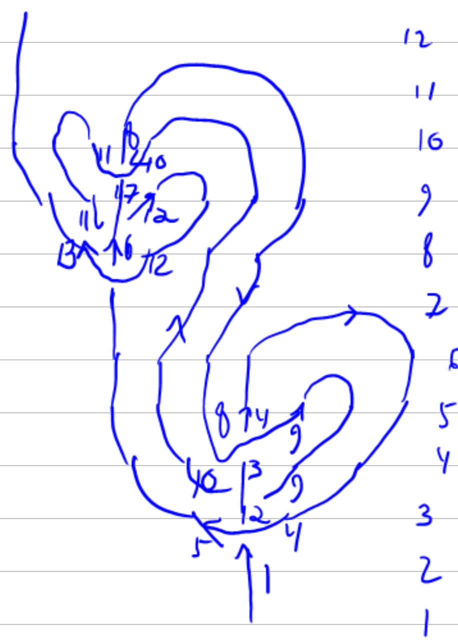
$\eta_i = a_i + d_i$ $\qquad C_i^{\pm 1} = T^{\mp 1/2}(ka_i + kd_i)$

$$R_{ij}^{\pm 1} = T^{\mp 1/2}\left(a_i ka_j - T^{\pm 1} a_i kd_j + (T^{\pm 1} - 1) b_j kc_i + d_i ka_j + T^{\pm 1} d_i kd_j\right)$$

Continue as in common.nb & kerler.nb.

$$\{H_X\}, \sqcup, m^{ij}_k, \sigma^i_j, \eta_i, \Delta^i_{jk}, \epsilon^i, S_i, R_{ij}$$

$$m^{ij}_k : H_{Y\cup\{i,j\}} \to H_{Y\cup\{k\}}$$

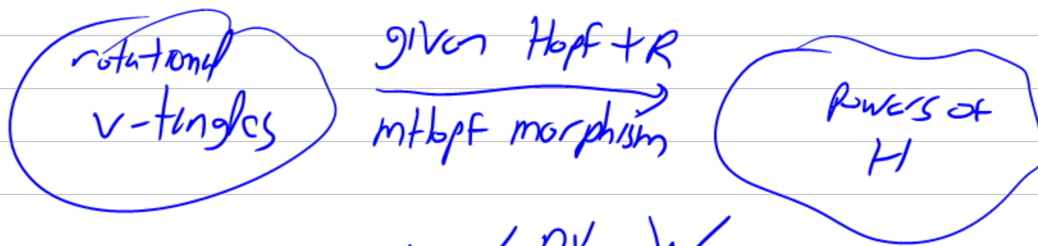


$X_{14\,2\,5}\, X_{7,10,8,11}\, X_{3\,9\,4\,8}\, X_{9\,3,10,2}\, X_{5\,12\,6\,1}\, X_{11\,6\,12\,7}$

000000000000

The structure of this class – a buildup to something hard.

0. An outline of a dream.
1. F.D. IHopf+R; Example: WG
2. F.D. Hopf+R; Example: Fail [Explain]
3. ∞-D Heisenberg enveloping algebra
4. Heis at $\epsilon > 0$
5. sl_{2+}^0
6. sl_{2+}^ϵ



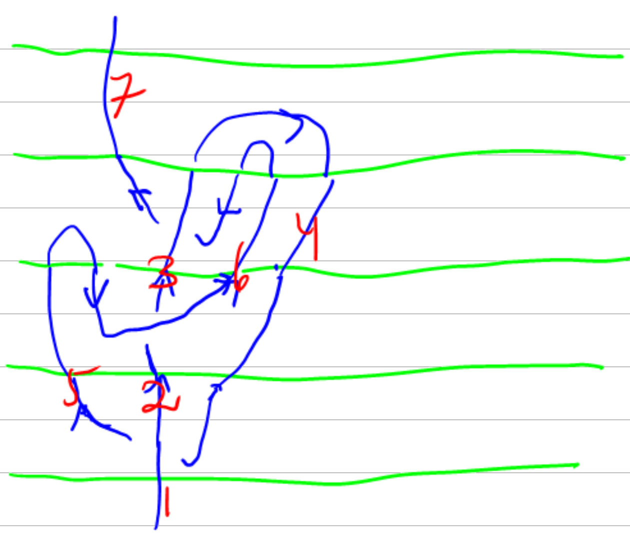
rv-tangles = MM⟨ ... ⟩ / R1', W, R2b R2c R3

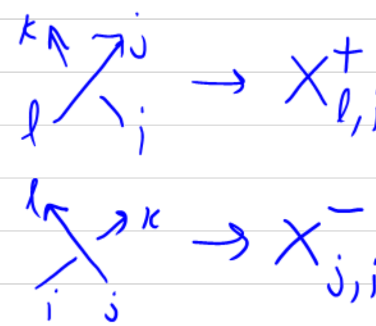
board line

Hopf algebras; $S^2 \neq I$

On to common.nb & 4DAlexander.nb

Write meta-code for RVK & Z





$X_{4251} X_{2635} X_{6473}$

Go over common.nb.

Go over 4DAlexander2.nb excluding Z.

— Some discussion of ribbon knots & Fox-Milnor.

Go over Z (if time)

The Heisenberg Algebra following http://drorbn.net/cat20

<u>Def</u> $\mathbb{H} := \left(A\langle p,x\rangle / [p,x]=1\right)$

$C_i = e^{t/2}$ $\quad R_{ij} = e^{t/2} e^{t(p_i - p_j)x_j}$ in $\mathbb{H}[[t]]$

<u>Thm</u> (PBW) $\mathbb{O}_{px}: S(p,x) \longrightarrow \mathbb{H}$ is an iso of v.s.

<u>Claim</u> $R_{ij} = \mathbb{O}_{px}\left(e^{(e^t-1)(p_i-p_j)x_j}\right)$

Proof Let $\Phi_1 := \mathbb{e}^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{O}_{p_jx_j}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right) =: \mathbb{O}(\Psi)$. We show that $\Phi_1 = \Phi_2$ in $(\mathfrak{h}_i \otimes \mathfrak{h}_j)[[t]]$ by showing that both solve the ODE $\partial_t\Phi = (p_i - p_j)x_j\Phi$ with $\Phi|_{t=0} = 1$. For Φ_1 this is trivial. $\Phi_2|_{t=0} = 1$ is trivial, and

$$\partial_t\Phi_2 = \mathbb{O}(\partial_t\Psi) = \mathbb{O}(\mathbb{e}^t(p_i - p_j)x_j\Psi)$$

$$(p_i - p_j)x_j\Phi_2 = (p_i-p_j)x_j\mathbb{O}(\Psi) = (p_i-p_j)\mathbb{O}(x_j\Psi - \partial_{p_j}\Psi)$$
$$= \mathbb{O}\Big((p_i-p_j)(x_j\Psi + (\mathbb{e}^t - 1)x_j\Psi)\Big) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi) \qquad \square$$

Def $H := \dfrac{A\langle p, x\rangle}{[p,x]=1}$ $\qquad C_i = e^{t/2} \in H_i[[t]]$ $\qquad R_{ij} = e^{t/2} e^{t(p_i-p_j)x_j} \in (H_i \otimes H_j)[[t]]$

Thm (PBW) $\mathbb{O}_{px}: S(p,x) \longrightarrow H$ is an iso of v.s.

Claim $R_{ij} = \mathbb{O}_{px}\left(e^{(e^t-1)(p_i-p_j)x_j}\right)$

board line

Proof Let $\Phi_1 := \mathbb{e}^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{O}_{p_j x_j}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right) =: \mathbb{O}(\Psi)$. We show that $\Phi_1 = \Phi_2$ in $(\mathfrak{h}_i \otimes \mathfrak{h}_j)[[t]]$ by showing that both solve the ODE $\partial_t \Phi = (p_i - p_j)x_j\Phi$ with $\Phi|_{t=0} = 1$. For Φ_1 this is trivial. $\Phi_2|_{t=0} = 1$ is trivial, and

$$\partial_t \Phi_2 = \mathbb{O}(\partial_t \Psi) = \mathbb{O}(\mathbb{e}^t (p_i - p_j)x_j \Psi)$$

$$(p_i - p_j)x_j\Phi_2 = (p_i-p_j)x_j\mathbb{O}(\Psi) = (p_i-p_j)\mathbb{O}(x_j\Psi - \partial_{p_j}\Psi)$$
$$= \mathbb{O}\left((p_i-p_j)(x_j\Psi + (\mathbb{e}^t-1)x_j\Psi)\right) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi) \qquad \square$$

Generating fnctns.

$\mathcal{F}(a_n) := \sum a_n x^n$

e.g. $f_{0,1} = 1$, $f_n = f_{n-1} + f_{n-2}$ $\qquad \mathcal{F}(f_n) = \dfrac{1}{1-x-x^2}$

exercise: what is $\mathcal{G}(f_n)$?

Sometimes "exp gen fnctns", $\mathcal{G}(a_n) = \sum \dfrac{a_n}{n!} x^n$

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $\qquad (p,x)^* = (\pi, \xi)$

The Generating Series $\mathcal{G}$: $\mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \to \mathbb{Q}[[\zeta_A, z_B]]$.

Claim. $L \in \mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][[\zeta_A]] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A} \frac{\zeta_A^n}{n!} L(z_A^n) = L\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

done line

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

~~**Claim.** If $L \in \mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$, $M \in \mathrm{Hom}(\mathbb{Q}[z_B] \to \mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.~~

delay

Further examples:

$p(x) \mapsto p(x+1)$

$p \mapsto p'$

$p \mapsto \int_0^x p(t)\,dt$

Examples. • $\mathcal{G}(id: \mathbb{Q}[p,x] \to \mathbb{Q}[p,x]) = \mathbb{e}^{\pi p + \xi x}$.

• Consider $R_{ij} \in (\mathfrak{h}_i \otimes \mathfrak{h}_j)[[t]] \cong \mathrm{Hom}\left(\mathbb{Q}[] \to \mathbb{Q}[p_i, x_i, p_j, x_j]\right)[[t]]$.

Then $\mathcal{G}(R_{ij}) = \mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j} = \mathbb{e}^{(T-1)(p_i-p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p, x\rangle/([p,x] = 1)$, let $\mathbb{O}_i: \mathbb{Q}[p_i, x_i] \to \mathfrak{h}_i$ is the "p before x" PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i, x_i, p_j, x_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathfrak{h}_i \otimes \mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k, x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{-\xi_i\pi_j + (\pi_i+\pi_j)p_k + (\xi_i+\xi_j)x_k}$.

Proof. Recall the "Weyl CCR" $\mathbb{e}^{\xi x}\mathbb{e}^{\pi p} = \mathbb{e}^{-\xi\pi}\mathbb{e}^{\pi p}\mathbb{e}^{\xi x}$, and find

$$\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{\pi_i p_i + \xi_i x_i + \pi_j p_j + \xi_j x_j} /\!/ \mathbb{O}_i \otimes \mathbb{O}_j /\!/ m_k^{ij} /\!/ \mathbb{O}_k^{-1}$$
$$= \mathbb{e}^{\pi_i p_i}\mathbb{e}^{\xi_i x_i}\mathbb{e}^{\pi_j p_j}\mathbb{e}^{\xi_j x_j} /\!/ m_k^{ij} /\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{\pi_i p_k}\mathbb{e}^{\xi_i x_k}\mathbb{e}^{\pi_j p_k}\mathbb{e}^{\xi_j x_k} /\!/ \mathbb{O}_k^{-1}$$
$$= \mathbb{e}^{-\xi_i\pi_j}\mathbb{e}^{(\pi_i+\pi_j)p_k}\mathbb{e}^{(\xi_i+\xi_j)x_k} /\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{-\xi_i\pi_j + (\pi_i+\pi_j)p_k + (\xi_i+\xi_j)x_k}.$$

$R_{ij} = \mathbb{O}_{px}\left(e^{(e^t-1)(p_i-p_j)x_j}\right)$

on board: $\mathcal{G} : \mathrm{Hom}_{v.s.}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \xrightarrow{\sim} \mathbb{Q}[z_B][[\zeta_A]]$

by

$L \mapsto \sum_{n\in\mathbb{N}^A} \frac{\zeta_A^n}{n!} L(z_A^n) = L\left(e^{\sum \zeta_a z_a}\right) = \mathcal{L}$ = greek $\mathcal{L}$atin

board line

E.g. $\mathcal{G}(R_{ij}) = e^{(T-1)(p_i-p_j)x_j}$

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p, x\rangle/([p,x]=1)$, let $\mathbb{O}_i : \mathbb{Q}[p_i, x_i] \to \mathfrak{h}_i$ is the "p before x" PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i,x_i,p_j,x_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathfrak{h}_i\otimes\mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k,x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$.

Proof. Recall the "Weyl CCR" $\mathbb{e}^{\xi x}\mathbb{e}^{\pi p} = \mathbb{e}^{-\xi\pi}\mathbb{e}^{\pi p}\mathbb{e}^{\xi x}$, and find

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= \mathbb{e}^{\pi_i p_i+\xi_i x_i+\pi_j p_j+\xi_j x_j} /\!\!/ \mathbb{O}_i\otimes\mathbb{O}_j /\!\!/ m_k^{ij} /\!\!/ \mathbb{O}_k^{-1} \\ &= \mathbb{e}^{\pi_i p_i}\mathbb{e}^{\xi_i x_i}\mathbb{e}^{\pi_j p_j}\mathbb{e}^{\xi_j x_j} /\!\!/ m_k^{ij} /\!\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{\pi_i p_k}\mathbb{e}^{\xi_i x_k}\mathbb{e}^{\pi_j p_k}\mathbb{e}^{\xi_j x_k} /\!\!/ \mathbb{O}_k^{-1} \\ &= \mathbb{e}^{-\xi_i\pi_j}\mathbb{e}^{(\pi_i+\pi_j)p_k}\mathbb{e}^{(\xi_i+\xi_j)x_k} /\!\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}.\end{aligned}$$

Also prove the Weyl CCR!

Use $H \circlearrowright \mathbb{Q}[x]$ w/ $p \mapsto \partial_x$

Note that both are Gaussian!

done line

GDO := The category with objects finite sets and

$$\mathrm{mor}(A\to B) = \left\{\mathcal{L} = \omega\mathbb{e}^{Q}\right\} \subset \mathbb{Q}[[\zeta_A, z_B]],$$

where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A \cup z_B$.
• Compositions: $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

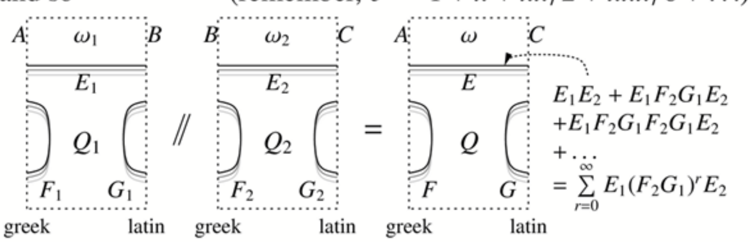
Claim. If $L \in \mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M \in \mathrm{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Compositions. In $\mathrm{mor}(A\to B)$,

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij}z_i z_j,$$


R. Feynman

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^T G_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1/2}$

$$g(R_{ij}) = e^{(\overset{e^t}{T}-1)(p_i-p_j)x_j} \qquad m^{ij}_k = g(hm^{ij}_k) = e^{(\pi_i+\pi_j)p_k + (\xi_i+\xi_j)x_k - \xi_i\pi_j}$$

both are Gaussians!

border line

The category Gen

GDO := The category with objects finite sets and
$$\operatorname{mor}(A \to B) = \left\{\mathcal{L} = \omega e^{Q}\right\} \subset \mathbb{Q}[\![\zeta_A, z_B]\!],$$
where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A \cup z_B$.
• Compositions: $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i \to \partial_{\zeta_i}} \mathcal{M}\right)_{\zeta_i=0}$.

Comment below

done line

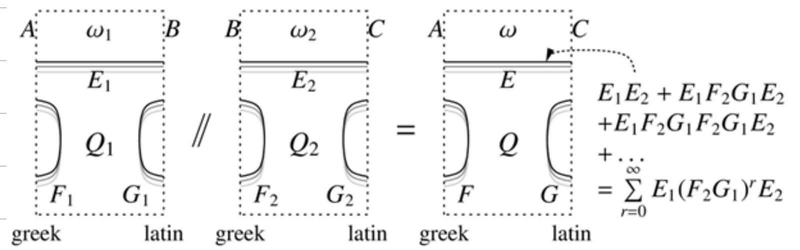
$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a \to \partial_{\zeta_a}} \mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. If $L \in \operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$, $M \in \operatorname{Hom}(\mathbb{Q}[z_B] \to \mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b \to \partial_{\zeta_b}} \mathcal{G}(M)\right)_{\zeta_b=0}$.

Compositions. In $\operatorname{mor}(A \to B)$,
$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j,$$

R. Feynman

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



where • $E = E_1(I - F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1/2}$

* when gluing exp(connected), the result is exp(connected)
* In the result, every diagram must be divided by the order of its group of automorphisms.

Continue as in GDO-Heisenberg@.nb

That was a failure. It took a whole class to go over what should have been simple intuitive material. I clearly haven't figured how to convey my intuition.

Yet see comments on the next hour.

Hour 20, October 27: Gaussians and compositions (2)

HP $\quad \mathrm{mor}_{A,B} = (\mathrm{Hom}(\mathbb{Q}[A] \to \mathbb{Q}[B])$

$\downarrow g$

Gen $\quad \mathrm{mor}_{A,B} = \mathbb{Q}[z_B][[z_A]] \qquad \mathcal{L}/\!/\mathcal{M} = \left(\mathcal{L}\big|_{z_b \to \partial_{\zeta_b}} \mathcal{M}\right)\Big|_{\zeta_b = 0}$

$\uparrow$

GDO $\quad \mathrm{mor}_{AB} = \{W e^{Q}\}$ $\qquad$ Composition ?

$\circ$

must be small

$$Q = \sum_{i\in A,\, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j$$

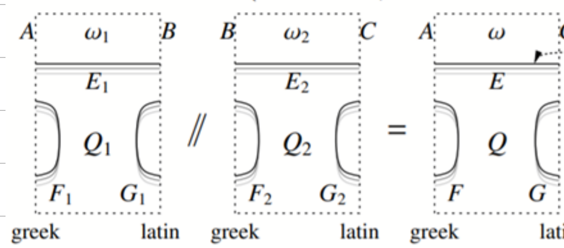
board line

$\circ$

* when gluing exp(connected), the result is exp(connected)

* In the result, every diagram must be divided by the order of its group of automorphisms.

and so $\qquad$ (remember, $e^x = 1 + x + xx/2 + xxx/6 + \ldots$)



$E_1E_2 + E_1F_2G_1E_2 + E_1F_2G_1F_2G_1E_2 + \ldots = \sum_{r=0}^{\infty} E_1(F_2G_1)^rE_2$

where • $E = E_1(I - F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$ • $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1/2}$

done line.

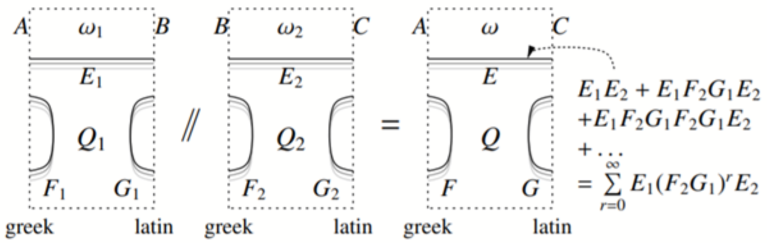
Continue as in GDO-Heisenberg@.nb

Moral: This is hard material. Perhaps next time I should give it more time.

On board: Today: Implementing, testing, using GDO.

Monday: An "analytical" proof of the composition formula.

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$ • $G = G_2 + E_2^T G_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2 \det(I-F_2G_1)^{-1/2}$

Then on w/ GDO-Heisenberg2@.nb.

$${}_A[W_1 e^{Q_1}]_B \,//\, {}_B[W_2 e^{Q_2}]_C = {}_A[W e^{Q}]_C$$ works, produces invariants too loose, too complicated.

$$W e^{Q} = e^{\sum \partial_{z_b} \partial_{\zeta_b}} \left(W_1 W_2 e^{Q_1 + Q_2} \right)\Big|_{z_b = \zeta_b = 0}$$ (assuming no name clashes)

* Prob. compute $e^{\frac{1}{2}\sum F_{ij} \partial_{z_i} \partial_{z_j}} \mathcal{E} \Big|_{z_i = 0} =: \langle F : \mathcal{E} \rangle_B$ in general, where F is symmetric

* Def. $\Psi = [\lambda F : \mathcal{E}]_B = e^{\frac{\lambda}{2}\sum F_{ij} \partial_{z_i} \partial_{z_j}} \mathcal{E}$

* $\psi = \log \Psi =: \{\lambda F, E\}_B$ If $E = \log \mathcal{E}$ makes sense.

* Claim $\Psi|_{\lambda=0} = \mathcal{E}$ $\quad \partial_\lambda \Psi = \frac{1}{2} \sum F_{ij} \partial_{z_i z_j} \Psi$

"The synthesis eq'n"

* $\psi|_{\lambda=0} = E$ $\quad \partial_\lambda \psi = \frac{1}{2} \sum_{i,j} F_{ij} \left(\partial_{z_i z_j} \psi + (\partial_{z_i} \psi)(\partial_{z_j} \psi) \right)$

a comment about existence & uniqueness.

* Thm $[F : \mathcal{E} e^{\frac{1}{2} G_{ij} z_i z_j}] = \det(1 - FG)^{-\frac{1}{2}} e^{\frac{1}{2}(G(I - FG)^{-1})_{ij} z_i z_j}$ $\cdot [F(1 - GF)^{-1} : \mathcal{E}] \Big|_{z_B \to (1-FG)^{-1} z_B}$

(G symmetric)

* $\underbrace{\{\lambda F : \tfrac{1}{2} z_B G z_B + E\}}_{\psi} = \underbrace{-\tfrac{1}{2} \operatorname{tr} \log (I - \lambda FG)}_{\psi_1} + \underbrace{\tfrac{1}{2} z_B G (I - \lambda FG)^{-1} z_B}_{\psi_2} + \{\lambda F(1 - \lambda GF)^{-1} : E\} \Big|_{z_B \to (1 - \lambda FG)^{-1} z_B}$

* Pf @ $E = 0$: $\partial_\lambda \psi_1 = \frac{1}{2} \operatorname{tr}(FG(1 - \lambda FG)^{-1})$

done line.

* $\partial_\lambda \psi_2 = -\frac{1}{2} z_B G (1 - \lambda FG)^{-1} FG (1 - \lambda FG)^{-1} z_B$

Aside
$(A + \epsilon B)^{-1} = [A(1 + \epsilon A^{-1} B)]^{-1}$
$= A^{-1} - \epsilon A^{-1} B A^{-1}$

$$\partial_{z_i}\Psi_1 = 0$$

* $$\partial_{z_i}\Psi_2 = \sum (G(1-\lambda FG)^{-1})_{ij} z_j$$

* $$\partial_{z_i z_j}\Psi_2 = (G(1-\lambda FG)^{-1})_{ij}$$

Second pass: combinatorial interp. & Feynman diagrams

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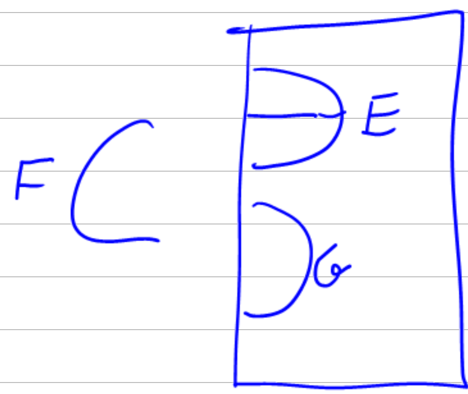
private.



Testing in 2020-02/Testing123.nb

Hour 23, November 3: Contracting Gaussians (2), \Gamma-calculus.

compute $e^{\frac{1}{2}\sum F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E}\Big|_{z_i=0} =: \langle F:\mathcal{E}\rangle_B$

$\Psi = [\lambda F:\mathcal{E}]_B = e^{\frac{\lambda}{2}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E}$ $\quad \psi = \log\Psi =: \{\lambda F, E\}$

w/ $E = \log\mathcal{E}$

<u>Claim</u> "The synthesis eq'n"

$\psi|_{\lambda=0} = E \qquad \partial_\lambda\psi = \frac{1}{2}\sum_{i,j}F_{ij}\left(\partial_{z_i z_j}\psi + (\partial_{z_i}\psi)(\partial_{z_j}\psi)\right)$

<u>Example</u> $\{F:\sum y_i z_i\} = \frac{1}{2}\sum F_{ij}y_iy_j + \sum y_i z_i$

<u>Claim</u> $\underbrace{\{\lambda F : \tfrac{1}{2}z\,Gz\}}_{\psi} = \underbrace{-\tfrac{1}{2}\operatorname{tr}\log(I-\lambda FG)}_{\psi_1} + \underbrace{\tfrac{1}{2}z^T G(I-\lambda FG)^{-1}z}_{\psi_2}$

<u>Pf</u> $\partial_\lambda\psi_1 = \frac{1}{2}\operatorname{tr}\left(FG(1-\lambda FG)^{-1}\right)$

board line

$\partial_\lambda\psi_2 = -\frac{1}{2}z^T G(1-\lambda FG)^{-1}FG(1-\lambda FG)^{-1}z$

$= -\frac{1}{2}\left(G(1-\lambda FG)^{-1}z\right)^T F\left(G(1-\lambda FG)^{-1}z\right)$

Aside
$(A+\epsilon B)^{-1} = [A(1+\epsilon A^{-1}B)]^{-1}$
$= A^{-1} - \epsilon A^{-1}BA^{-1}$

$\partial_{z_i}\psi_1 = 0$

$\partial_{z_i}\psi_2 = \sum\left(G(1-\lambda FG)^{-1}\right)_{ij}z_j \qquad \partial_{z_i z_j}\psi_2 = \left(G(1-\lambda FG)^{-1}\right)_{ij}$

done line

Second pass: Combinatorial interp. & Feynman diagrams

Now allow extras:

1. $\{F : E + \sum y_i z_i\} = \frac{1}{2}\sum F_{ij}y_iy_j + \sum y_i z_i + \{F:E\}\Big|_{z\to z+Fy}$

2. $\{\lambda F : \frac{1}{2}z\,Gz + E\} = -\frac{1}{2}\operatorname{tr}\log(I-\lambda FG) + \frac{1}{2}z\,G(I-\lambda FG)^{-1}z\ +$

$+\ \{\lambda F(1-\lambda GF)^{-1} : E\}\Big|_{z\to(1-\lambda FG)^{-1}z}$

Meta-Thm $\exists$ invt of tangles that any student of LinAlg can understand!

$$R_{ij} = e^{(T-1)(p_i-p_j)x_j} \qquad T = e^t \qquad m^{ij}_k = e^{(\pi_i+\pi_j)p_k + (\xi_i+\xi_j)x_k - \xi_i\pi_j}$$

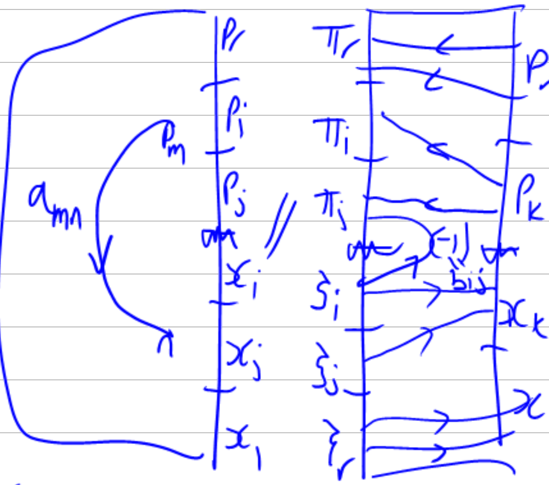
board line

Explain meta-Theorem 1, then 2. Use "balanced quadratics" $\begin{pmatrix} p & \xi \\ x & \pi \end{pmatrix}$

Meta-Theorem ~~1~~ 2 Let $R=\mathbb{Z}(T)$. There exists an invariant of tangles w/ n components w/ values in $R \times M_{\cancel{2n\times 2n}\ n\times n}(R)$ that any LinAlg ~~A~~ B-student can understand.

$$R^{\pm 1}_{ij} = \begin{array}{c|cc} 1 & x_i & x_j \\ \hline p_i & 0 & T^{\pm 1}-1 \\ p_j & 0 & 1-T^{\pm 1} \end{array} \qquad \eta_i = \begin{array}{c|c} 1 & x_i \\ \hline p_i & 0 \end{array} \qquad T_i \to \begin{array}{c|c} W_i & \cdots \\ \hline \vdots & A_i \end{array}$$

$$T_1 T_2 \longrightarrow \cdots$$



(W, A) $\qquad m^{ij}_k$

$$\begin{array}{c|ccc} W & x_i & x_j & x_s \\ \hline p_i & \alpha & \beta & \theta \\ p_j & \gamma & \delta & \epsilon \\ p_r & \phi & \psi & \Xi \end{array} \rightarrow \begin{array}{c|cc} (1+\gamma)W & x_k & x_s \\ \hline p_k & 1+\beta-\frac{(1-\alpha)(1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\epsilon}{1+\gamma} \\ p_r & \psi+\frac{(1-\delta)\phi}{1+\gamma} & \Xi-\frac{\phi\epsilon}{1+\gamma} \end{array}$$

$p_r \to x_s:\ a_{rs} + a_{rp}\frac{-1}{1+\gamma}a_{ps}$

$$p_k \to x_k:\ a_{ij} + a_{ii}\left(1 + b_{ij}(a_{ji}b_{ij})^*(a_{ji}+a_{jj})\right) + (a_{ji}b_{ij})^*(a_{ji}+a_{jj})$$
$$= \beta + \alpha\left(1-(1+\gamma)^{-1}(\gamma+\delta)\right) + (1+\gamma)^{-1}(\gamma+\delta)$$
$$= \beta + \alpha\left(\frac{1-\delta}{1+\gamma}\right) + \frac{\delta-1}{1+\gamma} + 1$$
$$= 1 + \beta - \frac{(1-\alpha)(1-\delta)}{(1+\gamma)}$$

Better: $R^{\pm 1}_{ij} \longrightarrow \begin{pmatrix} 1 & 1-T^{\pm 1} \\ 0 & T^{\pm 1} \end{pmatrix}$

$$\underbrace{\begin{array}{c|ccc} W & x_i & x_j & x_r \\ \hline p_i & \alpha & \beta & \theta \\ p_j & \gamma & \delta & \epsilon \\ p_r & \phi & \psi & \Xi \end{array}}_{A} \longrightarrow \begin{array}{c|cc} (1-\gamma)W & x_k & x_r \\ \hline p_k & \beta+\frac{\alpha\delta}{1-\gamma} & \theta+\frac{\alpha\epsilon}{1-\gamma} \\ p_r & \psi+\frac{\delta\phi}{1-\gamma} & \Xi+\frac{\phi\epsilon}{1-\gamma} \end{array}$$

A remove γ's r+c, add $c \cdot r/_{1-\gamma}$

Estimate complexity...

runs on knots w/ 1,000 xings!

Fine structure constant $\frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c}$

Motivation:

QED: $\int \mathcal{D}A\mathcal{D}\psi\, e^{i\mathcal{L}}$ $\quad \mathcal{L} = (\partial A)^2 + \psi\partial\psi - \psi^2 + \frac{1}{137}A\psi^2$

A, ψ: functions

CS: $\int e^{i\int AdA + \frac{2}{3}A^3}$ $\quad \mathcal{L}/\!/\mathcal{M} = \mathbb{e}^{\sum \partial_{z_b}\partial_{\zeta_b}}(\mathcal{L}\cdot\mathcal{M})\big|_{z_b=\zeta_b=0} \propto \int \mathbb{e}^{-\sum_b z_b\zeta_b}(\mathcal{L}\cdot\mathcal{M})\prod_{b\in B} dz_b d\zeta_b.$

A: Field

board line

So we want $\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j + \frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k}$

<u>Calc 1</u> $\int_{\mathbb{R}^2} e^{-\frac{\lambda x^2}{2}}dx$, $\lambda > 0$, $= \int_0^{2\pi} d\theta \int_0^\infty dr\, r\cdot e^{-\frac{\lambda r^2}{2}} = 2\pi\left[-\frac{1}{\lambda}e^{-\lambda r^2/2}\right]_0^\infty = \frac{2\pi}{\lambda}$

<u>Claim 2</u> $\int_{\mathbb{R}} e^{-\frac{\lambda x^2}{2}}dx = \sqrt{\frac{2\pi}{\lambda}}$

<u>Claim 3</u> $\int dx\, e^{-\frac{1}{2}\lambda_{ij}x^ix^j} = \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})} =: C$

For symmetric $\mathbb{R}^n$ Pos def $\Lambda = (\lambda_{ij})$

<u>Claim 4</u> $\int_{\mathbb{R}^n} P(x)e^{-\frac{1}{2}x^T\Lambda x}dx = P\left(\frac{\partial}{\partial y_i}\right)e^{\frac{1}{2}y^T\Lambda^{-1}y}\Big|_{y=0}$

$\|$ $\qquad\qquad\qquad\qquad\qquad\qquad \|$

$P\left(\frac{\partial}{\partial y}\right)\int_{\mathbb{R}^n} dx\, e^{-\frac{1}{2}x^T\Lambda x + y\cdot x}\Big|_{y=0} = \ldots$

$\downarrow$

$-\frac{1}{2}x^T\Lambda x + yx = -\frac{1}{2}(x-\Lambda^{-1}y)^T\Lambda(x-\Lambda^{-1}y) + \frac{1}{2}y^T\Lambda^{-1}y$

Justify the composition formula.

done line

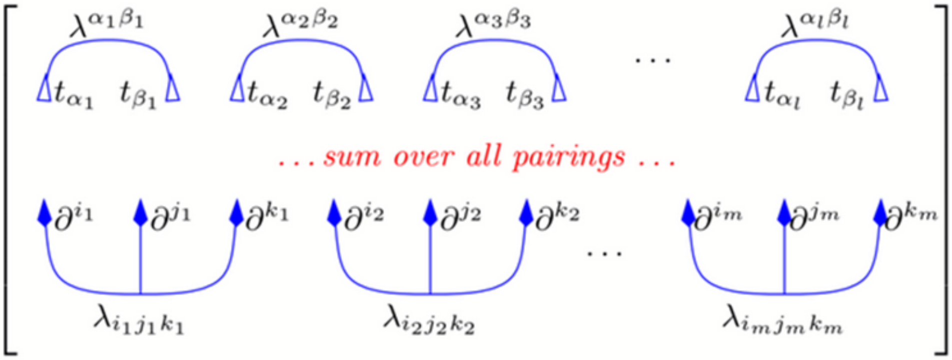
continue as next page.

Gaussian Integration. (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

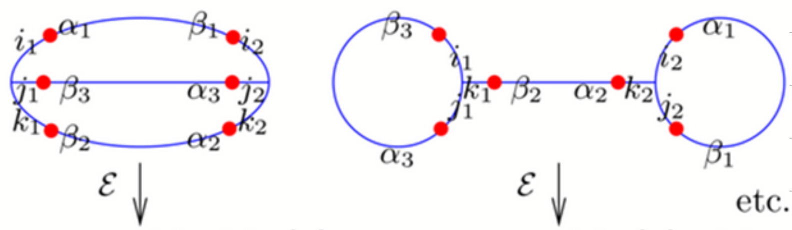
$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k} = \sum_{m\geq 0}\frac{\epsilon^m}{6^m m!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m e^{-\frac{1}{2}\lambda_{ij}x^ix^j}$$

$$= \sum_{m\geq 0}\frac{C\epsilon^m}{6^m m!}(\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left[\ \cdots\ \right]$$



$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$



etc.

$\lambda_{i_1j_1k_1}\lambda_{i_2j_2k_2}\lambda^{i_1i_2}\lambda^{j_1j_2}\lambda^{k_1k_2}$ $\lambda_{i_1j_1k_1}\lambda_{i_2j_2k_2}\lambda^{i_1j_1}\lambda^{k_1k_2}\lambda^{i_2j_2}$

$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m]\times[(S_2)^l\rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

Hour 26, November 17:

Finish F.D. (make handout?)
Rel. to $\langle F:G\rangle$
Connected diagrams.

Hour 27, November 19: impromptu review

$R_\epsilon = \exp\left[(T-1)(p_i - p_j)x_j + R'\right] \qquad R' = \sum_{k=1} \epsilon^k R^{(k)}$?

$${}_A\mathcal{L}_B \,//\, {}_B\mathcal{M}_C = e^{\sum_{i\in B}\partial_{\zeta_i}\partial_{z_i}}(\mathcal{L}\cdot\mathcal{M})\big|_{\zeta_i=z_i=0}$$

$$\langle F:\mathcal{E}\rangle_B = e^{\frac{1}{2}F_{uv}\partial^u\partial^v}\mathcal{E}\big|_{z=0} \qquad [F:\mathcal{E}]_B = e^{\frac{1}{2}F_{uv}\partial^u\partial^v}\mathcal{E}$$

$Z_\lambda := \log[\lambda F : e^P]$ satisfies

"synthesis eqn"

$$Z_0 = P \qquad \partial_\lambda Z_\lambda = \frac{1}{2}F_{uv}\left(\partial_u\partial_v Z_\lambda + (\partial_u Z_\lambda)(\partial_v Z_\lambda)\right)$$

board line

1. How solve the synthesis eq'n?
2. Are there manageable subspaces in which to look for R' [likewise, for C']? * Balanced * Docile?

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1} : \mathcal{E}\right\rangle_B.$$

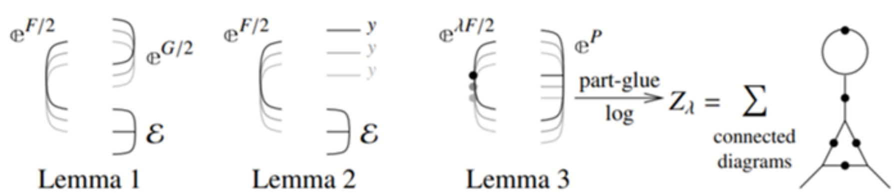
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F : \mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F : \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Continue as in PerturbedHeisenberg.nb.

Hour 29, November 24: Perturbing the Heisenberg R-Element (2)

Follow PerturbedHeisenberg.nb

Hour 30, November 26: Computations with perturbed Heisenberg.

Follow PerturbedHeisenberg2@.nb

Hour 31, November 29: CU and QU.

HW5 will be assigned tomorrow or on Wednesday and will be due a week later.
HW6 will be assigned on or near Wednesday Dec 8 and will be due a week later.

Then follow CUQU.html...

Review the CUQU handout.

Go over prop 7 & its $\frac{1}{2}$proof as there.

NTS. lhs & rhs fall in the same subset, on which ρ is injective.

Δ is well-def. on $U(\mathfrak{g})$

$\Phi : \mathbb{Q}[x_i] \longrightarrow U(\mathfrak{g})$ is a iso-morphism of co-algs.

primitive & group-like in a co-alg. $\Delta x = x\otimes 1 + 1\otimes x$ $\Delta X = X \otimes X$ (?)

group like elements form a group

primitives in $\mathbb{Q}[\mathfrak{g}]$ & in $U(\mathfrak{g})$

group like elements in $U(\mathfrak{g})$

group like elements in $U(\mathfrak{g})[[\hbar]]$; e.g. $e^{\hbar x}$

group like $\Longleftrightarrow$ exp of primitive

Proof $\Leftarrow$ easy

$\Rightarrow$ Assume $\Delta X = X \otimes X$ & $X = e^{x_k}$ to $\hbar^k$ (x_k primitive). Then

$X e^{-x_k} = 1 + \hbar^{k+1} y$ & y is primitive. set $x_{k+1} = x_k + \hbar^k y$

let $x = \lim x_k$.

if ρ is faithful on $\mathfrak{g}$, it is also faithful ~~on~~ as

$\rho : \left\{ \begin{matrix} \text{group} \\ \text{like} \end{matrix} \right\} \longrightarrow M_{n\times n}[[\hbar]]$

Remaining classes:

Friday: Finish CM + a word on the relationship between sl_{2+}^{ϵ} & sl_2

Monday: OU & the Drinfel'd double.

Wednesday: Wrap-up.

$QU, CU, Cm_k^{ij} = e^{\Lambda}$ [pf late today] //

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i - \epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right) y_k + \left(\beta_i + \beta_j + \frac{\log(1+\epsilon\eta_j\xi_i)}{\epsilon}\right) b_k + (\alpha_i + \alpha_j + \log(1+\epsilon\eta_j\xi_i))\, a_k + \left(\frac{e^{-\alpha_j - \epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i} + \xi_j\right) x_k$$

$$(a_k(\alpha_i+\alpha_j) + y_k(\eta_i + e^{-\alpha_i}\eta_j) + b_k(\beta_i+\beta_j+\eta_j\xi_i) + x_k(e^{-\alpha_j}\xi_i + \xi_j)) + \left(a_k\eta_j\xi_i - \frac{1}{2}b_k\eta_j^2\xi_i^2 - e^{-\alpha_i}y_k\eta_j(\beta_i+\eta_j\xi_i) - e^{-\alpha_j}x_k\xi_i(\beta_j+\eta_j\xi_i)\right)\epsilon + \left(-\frac{1}{2}a_k\eta_j^2\xi_i^2 + \frac{1}{3}b_k\eta_j^3\xi_i^3 + \frac{1}{2}e^{-\alpha_i}y_k\eta_j(\beta_i^2 + 2\beta_i\eta_j\xi_i + 2\eta_j^2\xi_i^2) + \frac{1}{2}e^{-\alpha_j}x_k\xi_i(\beta_j^2+2\beta_j\eta_j\xi_i+2\eta_j^2\xi_i^2)\right)\epsilon^2 + O[\epsilon]^3$$

$= \omega e^{L+Q+P}$ where $wt\begin{pmatrix} y & b & a & x \\ \eta & \beta & \alpha & \xi\end{pmatrix} = \begin{pmatrix}1&0&2&1\\1&2&0&1\end{pmatrix}$ "scalar" := wt 0.

L: a wt 2 quadratic in wt ≠ 0 variables w/ $\mathbb{Q}$-coeffs.

Q: a —||— wt 1 —||— w/ scalar coefs.

P a weight-bounded perturbation: $P = \sum \epsilon^k p^{(k)}$, $wt\, p^{(k)} \le 2k+2$

ω: a scalar

We need to contract it $\langle F : \rangle$, where F is wt 2 w/ $\mathbb{Q}$-coeffs

From the perspective of F_{11}: $\underbrace{\omega e^L}_{\text{scalar}} e^{Q+P}$ zippable over scalars

From the perspective of F_{02}: $e^{L+Q+\log\omega+P}$ zippable over $\mathbb{Q}$!

we can restrict scalars to rational fncts in $\alpha, e^{\alpha}, L, e^b$...

on board!

primitive & group-like in a co-alg. $\Delta x = x\otimes 1 + 1\otimes x$, $\Delta X = X \otimes X$

primitive elements form a Lie algebra!

group like elements form a group (!)

done live

$\mathbb{O} : \mathbb{Q}[x_i] \longrightarrow \mathcal{U}(\mathfrak{g})$ is a iso-morphism of co-algs.

$\Phi: \mathbb{Q}[x_i] \longrightarrow \mathcal{U}(\mathfrak{g})$ is a iso-morphism of co-algs.

primitives in $\mathbb{Q}[\mathfrak{g}]$ & in $\mathcal{U}(\mathfrak{g})$

group like elements in $\mathcal{U}(\mathfrak{g})$

group like elements in $\mathcal{U}(\mathfrak{g})[[\hbar]]$; e.g. $e^{\hbar x}$

group like $\Longleftrightarrow$ exp of primitive

Proof $\Leftarrow$ easy

$\Rightarrow$ Assume $\Delta X = X \otimes X$ & $X = e^{x_k}$ to $\hbar^k$ (primitive). Then

$X e^{-x_k} = 1 + \hbar^{k+1} y$ & y is primitive. set $x_{k+1} = x_k + \hbar^k y$

let $x = \lim x_k$.

if ρ is faithful on $\mathfrak{g}$, it is also faithful as

$\rho: \left\{ \begin{matrix} \text{group} \\ \text{like} \end{matrix} \right\} \longrightarrow M_{n \times n}[[\hbar]]$