May 28 - differential operators, div and j

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On board:

Next meeting June 20, many Videos in between.

$$\begin{tikzcd} 0 \arrow[r] & \langle\text{wheels}\rangle \arrow[r, "\iota"] \arrow[d, equal] & \mathcal{P}\mathcal{A}^w(\uparrow_n) \arrow[r, "\pi"] \arrow[r, bend left, "u"] & \langle\text{trees}\rangle \arrow[r] \arrow[l, bend left, "l"] \arrow[d, equal] & 0 \\ & tr_n & & a_n \oplus tder_n & \end{tikzcd}$$

$$\Rightarrow \mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(tr_n \rtimes (a_n \oplus tder_n))$$

If $D \in \langle\text{trees}\rangle$, $\operatorname{div} D := \iota^{-1}(u-l)(D)$

why "div"? Why "j"? where

$j: TAut_n = \exp(tder_n) \to tr_n$ by $j(e^D) = \frac{e^D - 1}{D}(\operatorname{div} D)$.

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$$\begin{tikzcd} \mathcal{A}^w(\uparrow_n) \arrow[d, "\tau_{\mathfrak{g}}"] \arrow[dr, "T"] & \\ \rho: \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes n} \arrow[r] & TDO(\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}) \end{tikzcd}$$

"invariant universal"

$\mathcal{D} \to$ "vector fields" & functions

$\exp \mathcal{D} \to$ "diffeomorphisms"

$I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$ —

* $\varphi \in \mathfrak{g}^* \xrightarrow{\rho} \hat{\varphi}$, multiplication by $\varphi$.
* $x \in \mathfrak{g} \xrightarrow{\rho}$ derivation in the direction of $-\operatorname{ad} x$;

$$(\rho(x)f)(y) = df_y([y,x]) =$$
$$= \frac{\partial}{\partial\epsilon} f(y + \epsilon[y,x])\Big|_{\epsilon=0}$$
$$= \frac{\partial}{\partial\epsilon} f(e^{-\epsilon x} y e^{\epsilon x})\Big|_{\epsilon=0}$$
$$=: (\partial_x f)(y)$$

[If $f$ is linear, this is $(\rho(x)f)(y) = f([y,x])$]

Claim This extends to $\mathcal{U}(I\mathfrak{g})$.

Proof 1. $\rho(\varphi_1\varphi_2) = \rho(\varphi_1)\rho(\varphi_2)$ ✓

2. $(\rho(\varphi x)f)(y) = \varphi(y) f([x,y])$

$(\rho(x\varphi)f)(y) = (\rho(x)(\varphi f))(y) = \varphi(y) f([x,y]) +$

$+f(y)\,\varphi([x,y])\,)$

$(\rho(x\varphi-\varphi x)f)(y)=f(y)\,\varphi([x,y])$ ✓

$(\rho([x,\varphi])f)(y)=[x,\varphi](y)\,f(y)=-\varphi([x,y])\,f(y)$

3. .....

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$\rho$ maps wheels to functions & trees to v.f. If $D$ is a tree,

<u>claim</u> 1. $\rho(uD)^*=-\rho(lD)$

2. $\rho(\operatorname{div}D)=\operatorname{div}(\rho(u(D)))$

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An Aside on R3 $\quad \tau(|\!\to\!|)=\sum \varphi_i\otimes x^i$

$(\tau(|\!\to\!|)f)(x,y)=\sum \varphi_i(x)(\partial_{(0,x^i)}f)(x,y)$

$=(\partial_{(0,x)}f)(x,y)$

$(\tau(R_{12})f)(x,y)=(\exp(\partial_{(0,x)})f)=$

$=f(x,e^{-x}ye^{x})=:f(x,y^x)$

$=:(f\circ IAd)(x,y)$

$(\tau(R_{12}\cdot R_{13}\cdot R_{23})f)(x,y,z)=$

$(f\circ IAd_{23}\circ IAd_{13}\circ IAd_{12})(x,y,z)$

$(x,y,z)\xmapsto{IAd_{12}}(x,y^x,z)\xmapsto{IAd_{13}}(x,y^x,z^x)\xmapsto{IAd_{23}}$

Other side: $\qquad (x,y^x,(z^x)^{(y^x)})$

$(x,y,z)\xmapsto{23}(x,y,z^y)\xmapsto{13}(x,y,(z^y)^x)\xmapsto{12}(x,y^x,(z^y)^x)$

This is the quandle property!

Claim $\quad j(e^D) := \frac{e^D - 1}{D}(\operatorname{div} D) = \log\left(e^{-lD} e^{uD}\right)$

proof 1 Use the Euler trick; need:

$$\exp\left(\frac{e^{tD}-1}{tD}\right)(\operatorname{div} tD) \overset{?}{=} e^{-ltD} e^{utD}$$

apply $\widetilde{E}_t$ at $t=1$:

$$e^D(\operatorname{div} D) \overset{?}{=} e^{-uD}(-lD)e^{uD} + uD$$

$$\cdots\cdots$$

"proof 2" Interpret $D \rightsquigarrow X = \rho(uD)$, then

$$j(e^D) = \log\left((e^X)^* e^X\right) \sim \log\left(\operatorname{Jac}(e^X)\right)$$

$$\sim \int_0^1 dt\, e^{tX} \operatorname{div} X \sim \frac{e^X - 1}{X} \operatorname{div} X = \frac{e^D - 1}{D} \operatorname{div} D.$$