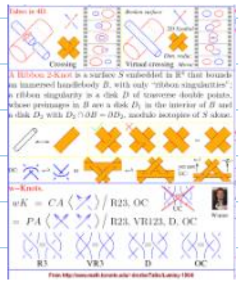
presented by Zsuzsi

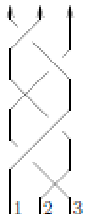
120118_Handout.png



2.1.2 The "abstract" way.

2.1.2. *The "Abstract" Way.* The relations (2) and (6) that govern the behaviour of virtual crossings precisely say that virtual crossings really are "virtual" — if a piece of strand is routed within a braid so that there are only virtual crossings around it, it can be rerouted in any other "virtual only" way, provided the ends remain fixed (this is Kauffman's "detour move" [Ka2, KL]). Since a v-braid B is independent of the routing of virtual pieces of strand, we may as well never supply this routing information.

Thus for example, a perfectly fair verbal description of the (pure!) v-braid on the right is "strand 1 goes over strand 3 by a positive crossing then likewise positively over strand 2 then negatively over 3 then 2 goes positively over 1". We don't need to specify how strand 1 got to be near strand 3 so it can go over it — it got there by means of virtual crossings, and it doesn't matter how. Hence we arrive at the following "abstract" presentation of PvB_n and vB_n:



Proposition 2.3. *(E.g.* [Ba]*)*

(1) *The group PvB_n of pure v-braids is isomorphic to the group generated by symbols σ_{ij} for $1 \le i \ne j \le n$ (meaning "strand i crosses over strand j at a positive crossing"[9]), subject to the third Reidemeister move and to locality in space (compare with* (3) *and* (4)*):*

$$\sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \qquad \textit{whenever} \qquad |\{i,j,k\}| = 3,$$
$$\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \qquad \textit{whenever} \qquad |\{i,j,k,l\}| = 4.$$

[9]The inverse, σ_{ij}^{-1}, is "strand i crosses over strand j at a negative crossing"

7

(2) *If $\tau \in S_n$, then with the action $\sigma_{ij}^\tau := \sigma_{\tau i, \tau j}$ we recover the semi-direct product decomposition $vB_n = PvB_n \rtimes S_n$.* □

2.2. On to w-Braids.

To define w-braids, we break the symmetry between over crossings and under crossings by imposing one of the "forbidden moves" virtual knot theory, but not the other:

$$\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}, \qquad \text{yet} \qquad s_i\sigma_{i+1}\sigma_i \ne \sigma_{i+1}\sigma_i s_{i+1}. \tag{9}$$

Alternatively,

$$\sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}, \qquad \text{yet} \qquad \sigma_{ik}\sigma_{jk} \ne \sigma_{jk}\sigma_{ik}.$$

In pictures, this is

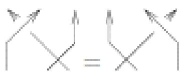
(10)

The relation we have just imposed may be called the "unforbidden relation", or, perhaps more appropriately, the "overcrossings commute" relation (OC). Ignoring the non-crossings[10] ╳, the OC relation says that it is the same if strand i first crosses over strand j and then over strand k, or if it first crosses over strand k and then over strand j. The "undercrossings commute" relation UC, the one we do not impose in (9), would say the same except with "under" replacing "over".

Definition 2.4. The group of w-braids is $wB_n := vB_n/OC$. Note that ς descends to wB_n and hence we can define the group of pure w-braids to be $PwB_n := \ker \varsigma : wB_n \to S_n$. We still have a split exact sequence as at (7) and a semi-direct product decomposition $wB_n = PwB_n \rtimes S_n$.

Exercise 2.5. Show that the OC relation is equivalent to the relation

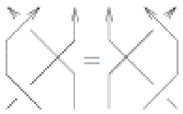
$$\sigma_i^{-1}s_{i+1}\sigma_i = \sigma_{i+1}s_i\sigma_{i+1}^{-1} \qquad \text{or}$$



skip

Exercise 2.5. Show that the OC relation is equivalent to the relation

$$\sigma_i^{-1}s_{i+1}\sigma_i = \sigma_{i+1}s_i\sigma_{i+1}^{-1} \qquad \text{or}$$



skip

While mostly in this paper the pictorial / algebraic definition of w-braids (and other w-knotted objects) will suffice, we ought describe at least briefly 2-3 further interpretations of wB_n:

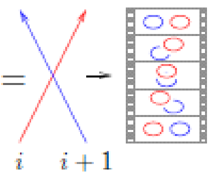
2.2.1. *The group of flying rings.* Let X_n be the space of all placements of n numbered disjoint geometric circles in $\mathbb{R}^3$, such that all circles are parallel to the xy plane. Such placements will be called horizontal[11]. A horizontal placement is determined by the centers in $\mathbb{R}^3$ of the n circles and by n radii, so $\dim X_n = 3n + n = 4n$. The permutation group S_n acts on X_n by permuting the circles, and one may think of the quotient $\tilde{X}_n := X_n/S_n$ as the space of all horizontal placements of n unmarked circles in $\mathbb{R}^3$. The fundamental group $\pi_1(\tilde{X}_n)$ is a group of paths traced by n disjoint horizontal circles (modulo homotopy), so it is fair to think of it as "the group of flying rings".

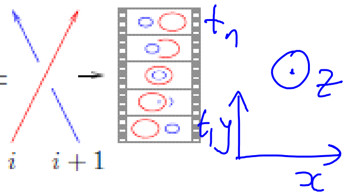
[10] Why this is appropriate was explained in the previous section.

[11] For the group of non-horizontal flying rings see Section 2.5.4

Theorem 2.6. *The group of pure w-braids PwB_n is isomorphic to the group of flying rings $\pi_1(X_n)$. The group wB_n is isomorphic to the group of unmarked flying rings $\pi_1(\tilde{X}_n)$.*

For the proof of this theorem, see [Gol, Sa] and especially [BH]. Here we will contend ourselves with pictures describing the images of the generators of wB_n in $\pi_1(\tilde{X}_n)$ and a few comments:

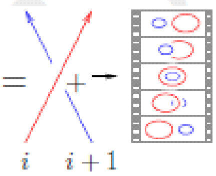
$s_i =$ 

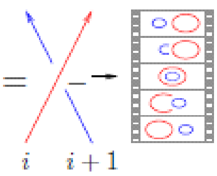
$\sigma_i =$ 

Thus we map the permutation s_i to the movie clip in which ring number i trades its place with ring number $i+1$ by having the two flying around each other. This acrobatic feat is performed in $\mathbb{R}^3$ and it does not matter if ring number i goes "above" or "below" or "left" or "right" of ring number $i+1$ when they trade places, as all of these possibilities are homotopic. More interestingly, we map the braiding σ_i to the movie clip in which ring $i+1$ shrinks a bit and flies through ring i. It is a worthwhile exercise for the reader to verify that the relations in the definition of wB_n become homotopies of movie clips. Of these relations it is most interesting to see why the "overcrossings commute" relation $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$ holds, yet the "undercrossings commute" relation $\sigma_i^{-1}\sigma_{i+1}^{-1}s_i = s_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ doesn't.

Exercise 2.7. To be perfectly precise, we have to specify the fly-through direction. In our notation, σ_i means that the ring corresponding to the under-strand approaches the bigger ring representing the over-strand from below, flies through it and exists above. For σ_i^{-1} we are "playing the movie backwards", i.e., the ring of the under-strand comes from above and exits below the ring of the over-strand.

Let "the signed w braid group", swB_n, be the group of horizontal flying rings where both fly-through directions are allowed. This introduces a "sign" for each crossing σ_i:

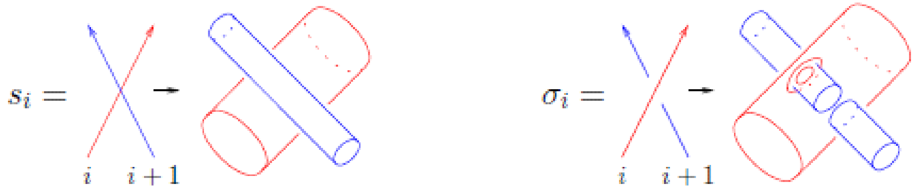
$\sigma_{i+} =$ 

$\sigma_{i-} =$ 

In other words, swB_n is generated by s_i, σ_{i+} and σ_{i-}, for $i = 1, ..., n$. Check that in swB_n $\sigma_{i-} = s_i\sigma_{i+}^{-1}s_i$, and this, along with the other obvious relations implies $swB_n \cong wB_n$.

2.2.2. *Certain ribbon tubes in $\mathbb{R}^4$*. With time as the added dimension, a flying ring in $\mathbb{R}^3$ traces a tube (an annulus) in $\mathbb{R}^4$, as shown in the picture below:

2.2.2. Certain ribbon tubes in $\mathbb{R}^4$. With time as the added dimension, a flying ring in $\mathbb{R}^3$ traces a tube (an annulus) in $\mathbb{R}^4$, as shown in the picture below:



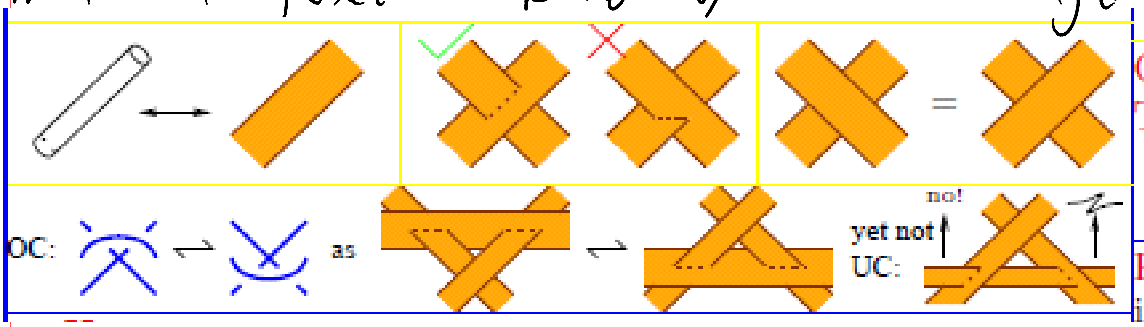
here z is the "fourth" dimension.

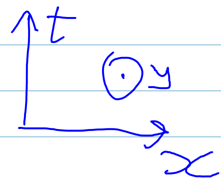
9

Note that we adopt here the drawing conventions of Carter and Saito [CS] — we draw surfaces as if they were projected from $\mathbb{R}^4$ to $\mathbb{R}^3$, and we cut them open whenever they are "hidden" by something with a higher t coordinate.

Note also that the tubes we get in $\mathbb{R}^4$ always bound natural 3D "solids" — their "insides", in the pictures above. These solids are disjoint in the case of s_i and have a very specific kind of intersection in the case of σ_i — these are transverse intersections with no triple points, and their inverse images are a meridional disk on the "thin" solid tube and an interior disk on the "thick" one. By analogy with the case of ribbon knots and ribbon singularities in $\mathbb{R}^3$ (e.g. [Ka1, Chapter V]) and following Satoh [Sa], we call this kind if intersections of solids in $\mathbb{R}^4$ "ribbon singularities" and thus our tubes in $\mathbb{R}^4$ are always "ribbon tubes".

Not in text: "band w/ ribbon singularities"





z not shown.

From http://www.math.toronto.edu/drorbn/Talks/Luminy-1004/