Jan 11 - Introduction, v-Braids

January-05-12
5:33 AM

Subject Introduction.

top objects.

braid group

Alg. structure

projectivization machine

Is there a "homomorphic expansion"?

comb. objects

chord-diagram algebra

Alg. structure

$\xrightarrow{\text{given } \mathfrak{g}}$ "$\mathcal{U}(\mathfrak{g})$"

Topology — High Algebra — Combinatorics — Low Algebra

	u-Knots →	v-Knots →	w-Knots
Topology	Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc.	Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization.	Ribbon knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute".
Combinatorics	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Like v, but also with "tails commute". Only "two in one out" internal vertices.
Low Algebra	Finite dimensional metrized Lie algebras, representations, and associated spaces.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces.
High Algebra	The Drinfel'd theory of associators.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras.

figs

Why bother? 1. Knot theory is an excuse.

2. Better understanding of the algebra.

3. "Algebraic knot theory".

Course Introduction. A class, a seminar, an experiment.

* Not a standard UofT class: * No grades/credits
  * Our own time frame.

I need your help!

Dissonances.

1. Private / scientific record.
2. Should be complete / Motivation to complete
(this also has a technical side: how do I refer to numbered items?)
3. Needs time & love / high season administratively.

Location. Today only: exile next time: BA 4010
Eventually: My office I hope for lunches!

Section 2.1 – v-braids.

2.1.1 The "planar" way.

$$vB_n = \left\langle \begin{array}{l} \sigma_i = \\ s_i = \end{array} \middle| \begin{array}{ll} s_i^2=1,\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1},\ |i-j|>1 \Rightarrow s_is_j=s_js_i & \text{"Symmetric group" } \langle s_i \rangle \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ |i-j|>1 \Rightarrow \sigma_i\sigma_j = \sigma_j\sigma_i & \text{"braid part"} \\ s_i\sigma_{i+1}^{\pm1}s_i = s_{i+1}\sigma_i^{\pm1}s_{i+1},\ |i-j|>1 \Rightarrow \sigma_is_j = s_j\sigma_i & \\ \quad \text{"mixed relations"} & \end{array} \right\rangle$$

There is an obvious "Skeleton map":

$$\sigma : vB_n \longrightarrow S_n$$

<u>Def</u> $PvB_n := \ker(\sigma)$

<u>Claim</u> we have a split sequence

$$1 \longrightarrow PvB_n \hookrightarrow vB_n \overset{\sigma}{\rightleftarrows} S_n \longrightarrow 0$$

& hence $vB_n = PvB_n \rtimes S_n$

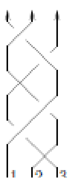
So studying vB_n is +/– the same as studying PvB_n.

done line

2.1.2 The "abstract" way.

2.1.2. *The "Abstract" Way.* The relations (2) and (6) that govern the behaviour of virtual crossings precisely say that virtual crossings really are "virtual" — if a piece of strand is routed within a braid so that there are only virtual crossings around it, it can be rerouted in any other "virtual only" way, provided the ends remain fixed (this is Kauffman's "detour move" [Ka2, KL]). Since a v-braid B is independent of the routing of virtual pieces of strand, we may as well never supply this routing information.

Thus for example, a perfectly fair verbal description of the (pure!) v-braid on the right is "strand 1 goes over strand 3 by a positive crossing then likewise positively over strand 2 then negatively over 3 then 2 goes positively over 1". We don't need to specify how strand 1 got to be near strand 3 so it can go over it — it got there by means of virtual crossings, and it doesn't matter how. Hence we arrive at the following "abstract" presentation of PvB_n and vB_n:



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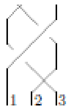

Proposition 2.3. *(E.g.* [Ba]*)*

(1) *The group PvB_n of pure v-braids is isomorphic to the group generated by symbols σ_{ij} for $1 \leq i \neq j \leq n$ (meaning "strand i crosses over strand j at a positive crossing"*[9]*), subject to the third Reidemeister move and to locality in space (compare with* (3) *and* (4)*):*

$$\sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \qquad \textit{whenever} \qquad |\{i,j,k\}| = 3,$$
$$\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \qquad \textit{whenever} \qquad |\{i,j,k,l\}| = 4.$$

[9] The inverse, σ_{ij}^{-1}, is "strand i crosses over strand j at a negative crossing"