April 19, 9-11 am - MAT257 office hours.

Q1. Spivak $\to$ $\omega(v, w) = \det \begin{pmatrix} v \\ w \\ n(x) \end{pmatrix}$

generalize: $\omega(v_1 \ldots v_{n-1}) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ n(x) \end{pmatrix} =$

$= \sum_{i=1}^{n} (-1)^{i ?} n_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$

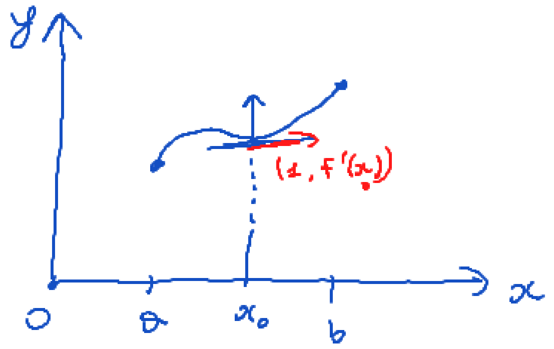
Q4. Idea: graph is a 1-mfld (with boundary ~~as~~ endpoints), therefore, we have dV (volume element).

$dV = \underline{g_1 \, dx + g_2 \, dy}$

Want to find g_1, g_2.

But (g_1, g_2) - normal vector

$$\underbrace{dV}_{\text{Volume element on } \Gamma_f.} = \underbrace{?}_{\text{find!}}\, dx + \underbrace{?}_{\text{find!}}\, dy$$



Immediately:

$(-f'(x_0), 1)$ ← not normalized!

$\frac{1}{\sqrt{1+f'(x_0)^2}}(-f'(x_0), 1)$ ← normalized!

$$n_1 = -\frac{f'(x_0)}{\sqrt{1+f'(x_0)^2}}$$

$$n_2 = \frac{1}{\sqrt{1+f'(x_0)^2}}$$

$$dV \overset{?}{=} n_2\, dx - n_1\, dy$$

computing by definition!

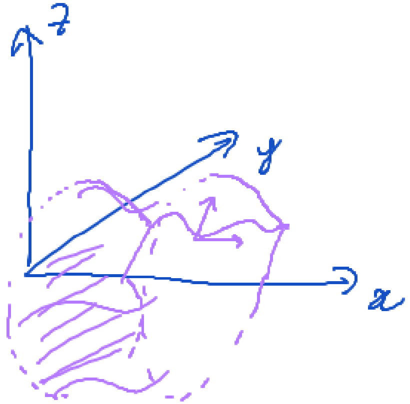
$$\int_M dV = \int_M \frac{dx}{\sqrt{1+f'^2}} + \frac{f'}{\sqrt{1+f'^2}}\, dy \overset{y=f(x)}{=} \int_a^b \frac{dx}{\sqrt{1+f'^2}} + \frac{f'\, df}{\sqrt{1+f'^2}} =$$

$$= \int_a^b \frac{dx}{\sqrt{1+f'^2}} + \frac{f'^2\, dx}{\sqrt{1+f'^2}} = \int_a^b \frac{1+f'^2}{\sqrt{1+f'^2}}\, dx.$$

b) Write down the Riemann partitions corresp. to $\int_0^1 \sqrt{1+f'^2}\, dt$.

Q2: two parameters: x, φ

φ angle



rotations take normal vectors to normal vectors

$(-f'(x_0), 1, 0)$

$(x, f(x)\cos\varphi, f(x)\sin\varphi)$ — param. the manifold.

tgt vectors are (at (x_0, φ_0)):

$(1, f'(x_0)\cos\varphi, f'(x_0)\sin\varphi)$

$(0, -f(x_0)\sin\varphi_0, f(x_0)\cos\varphi_0)$

lin ind.

normal vector of

$(-f'(x_0), \cos\varphi_0, \sin\varphi_0)$

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then we use Q1.

and integrate.

Q3a) Polarization identity

Q3b) $$\int_{TM} \underbrace{dV}_{\text{vol. elem on } TM} = \int_M T^*(dV) = \int_M dV$$

Remark: dV is defined uniquely by orient., a compatible coll. of metrics on each T_xM. (only works for Riemannian mfolds)

require $dV(\xi_1, \dots, \xi_n) = 1$ for every orthonormal ξ_i

$$(T^* dV)(\xi_i) = dV(\underbrace{T\xi_i}_{\text{also orthonormal}}) = 1.$$

Remark: this equality works for orient. preserving T, but it general. to any T.

Q5) Hint: skip to the very last <u>sentence</u>

$$\int_{\partial M} z n_3 \, dA = \int_{\partial M} \left\langle \underset{\substack{\| \\ F}}{\begin{pmatrix}0\\0\\z\end{pmatrix}}, \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} \right\rangle dA \overset{?}{=} \int_M \operatorname{div} F \, dx\,dy\,dz = \mathrm{Vol}(M).$$