Q1a) $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = \omega^1_{\text{grad} f}$ $\quad dx_1 \wedge dx_2 = - dx_2 \wedge dx_1$

$$d(\underset{\substack{\text{diff.}\\ \text{w.r.t.}\\ x_2, x_3}}{F_1} dx_1 + \dots + \underset{\substack{\text{diff}\\ \text{w.r.t.}\\ x_1, x_2}}{F_3} dx_3) = \frac{\partial F_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial F_1}{\partial x_3} dx_3 \wedge dx_1 +$$

correct signs!

$$+ \frac{\partial F_2}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial F_2}{\partial x_3} dx_3 \wedge dx_2 +$$

$$+ \frac{\partial F_3}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial F_3}{\partial x_2} dx_2 \wedge dx_3$$

double-check the signs!

$$\text{curl } F = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1} \right)$$

Q1b) is, essentially, $d^2 = 0$.

$F = (F_1, F_2, F_3)$

$\mathbb{R}^3 \to \mathbb{R}$ fn-ns.

$F : \mathbb{R}^3 \to \mathbb{R}^3$.

$\omega^1_F \in \Omega^1(\mathbb{R}^3)$

$\omega^2_F \in \Omega^2(\mathbb{R}^3)$

Q2: $g : U \to \underset{\sim}{V}$

$$\boxed{\forall \omega \in \Omega^k U : \underbrace{d\omega = 0}_{\text{closed}} \iff \exists \eta \in \Omega^{k-1} U : \underbrace{d\eta = \omega}_{\text{exact}} .}$$

d commutes with the pull back!

$g^* : \Omega^k V \to \Omega^k U$, $\boxed{d(g^*\omega) = g^*(d\omega) \quad \forall \omega \in \Omega^k}$

Q3: $\underbrace{d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy}_{\text{Chain rule}}$

Q4: $d^2 \omega = 0$

$$\boxed{|x| = \sqrt{x_1^2 + \ldots + x_n^2}}$$

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge (d\eta)$$

Compute $d\omega \overset{\text{prob}}{\neq} 0$ $\qquad d(d\omega) = 0$

$d\eta \neq 0$ $\qquad d(d\eta) = 0$

$\omega \wedge \eta \neq 0$ $\qquad d(\omega \wedge \eta) \neq 0$

The reason why we expect $d^2 = 0$ is because partial deriv-s commute!

Q5. $$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial}{\partial x_i} \left(\frac{x_i}{|x|^p} \right) dx_i \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n =$$

$$= \left(\sum_{i=1}^{n} (-1)^{2i-2} \frac{\partial}{\partial x_i} \left(\frac{x_i}{|x|^p} \right) \right) \underbrace{dx_1 \wedge \ldots \wedge dx_n}_{\text{no indic. omitted}}$$

$\leftarrow$ a volume form (if the fn-n is not zero)

$\mathbb{R}^2$

$$\omega = \frac{x_1}{\left(\sqrt{x_1^2+x_2^2}\right)^p} dx_1 - \frac{x_2}{\left(\sqrt{x_1^2+x_2^2}\right)^p} dx_2$$

For $\mathbb{R}^2$ $p=2$

$$dx \wedge dy \wedge dx = -\underbrace{dx \wedge dx}_{0} \wedge dy = 0.$$

If $p=2$ then $\omega = d\theta$

$d\omega = d^2\theta = 0$

$dx_j \wedge (dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n) = 0$

for $i \neq j$

For $\mathbb{R}^n$ we can do a direct computation:

$$\frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|^p}\right)$$

Use product rule + chain rule

Smth to think about: try spherical:

$$\begin{pmatrix} r\cos\varphi\cos\phi \\ r\cos\varphi\sin\phi \\ r\sin\varphi \end{pmatrix}$$