$V = \mathbb{R}^3$,

$\Lambda^1(\mathbb{R}^3) = \{a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3\}$

φ_i - dual to e_i

$\dim \Lambda^1(\mathbb{R}^3) = \dim \Lambda^2(\mathbb{R}^3) = 3$.

(include this in sol-n)

$$\underbrace{\sum a_i \varphi_i \mapsto (a_1, a_2, a_3)}$$

$$(a_1\varphi_1 + \dots + a_3\varphi_3) \wedge (b_1\varphi_1 + \dots + b_3\varphi_3) = a_1 b_2\, \varphi_1 \wedge \varphi_2 + \underbrace{b_1 a_2\, \varphi_2 \wedge \varphi_1}_{-a_2 b_1\, \varphi_1 \wedge \varphi_2} + \dots$$

$$\underbrace{(a_1 b_2 - a_2 b_1)}\, \underbrace{\varphi_1 \wedge \varphi_2}$$

Choose the correct order of $\varphi_1 \wedge \varphi_2$, $\varphi_1 \wedge \varphi_3$, $\varphi_2 \wedge \varphi_3$.

Q2. $L : \mathbb{R}^n \to \mathbb{R}^n$ (invertible), w.r.t. stand. orient.

L is orient.-preserv. $\Leftrightarrow \det L > 0$.

a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \det = -1$

b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \det = -1$

$\vdots$

(Hint: answer might depend on whether n is odd or even)

Q3. $\chi : \Lambda^n(V) \to \mathbb{R}$. given, not expl.

$0 \le k \le n$ $\quad \binom{n}{k} = \binom{n}{n-k} = \binom{n}{k}$

$\psi_k : \Lambda^{n-k}(V) \xrightarrow{\sim} (\Lambda^k(V))^*$

(forbidden to fix a basis, ...)

$\Lambda^0 \oplus \Lambda^1 \oplus \dots \oplus \Lambda^n$

Hint:

$\varphi \in \Lambda^{n-k}(V) \rightsquigarrow \mathbb{R}$

$\theta \in \Lambda^k(V)$

$\rightsquigarrow \underset{\Lambda^n(V)}{\varphi \wedge \theta} \xrightarrow{\chi} \mathbb{R}$

$\psi_k(\varphi)(\theta) \in \mathbb{R}$, $\quad \varphi \in \Lambda^{n-k}(V)$, $\theta \in \Lambda^k(V)$

Answer: $\underline{\varphi \longmapsto \chi(\varphi \wedge \cdot)}$

What is left to do:
show that is, indeed, bijective.

$V = \langle v_1, \dots v_n \rangle$

$\langle \varphi_1 \dots \varphi_n \rangle \leftarrow$ dual

ω_I, $\langle \omega_I, \omega_J \rangle \overset{\text{def}}{=} \delta_{IJ}$ (ω_I form (by def) an orthon. basis w.r.t. $\langle \cdot , \cdot \rangle$)

Ex: $(\varphi_1 \wedge \varphi_3) \wedge (\varphi_2 \wedge \varphi_4) = \omega_4 \cdot (-1) = -\omega_4$.

a) $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$, $\forall \lambda, \eta \in \Lambda^k$ $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$

Define $*$ on basis elements.

Choose

$\lambda = \omega_K$. $\eta = \sum_I c_I \omega_I$

$\uparrow$ multi-index K-index

$*\eta = \sum_J d_J \omega_J$

$\uparrow$ $(n-k)$-index.

$$\omega_K \wedge \left(\sum_J d_J \omega_J\right) = \langle \omega_K, \sum_I c_I \omega_I \rangle \omega_n.$$

$$\omega_K \wedge \left(\sum_J d_J \omega_J\right) = \sum_J d_J \underbrace{\omega_K \wedge \omega_J}_{\substack{\text{zero for} \\ \text{almost all} \\ J}} = d_{K^c} \omega_K \wedge \omega_{K^c}$$

$$\langle \omega_K, \sum_I c_I \omega_I \rangle \omega_n = c_K \omega_n.$$

$K = \{1, 2, 3, \dots k\}$. if $K \cap J \neq \emptyset \Leftarrow$ $\omega_K \wedge \omega_J = 0$.

Ex: $K = \{1\ 2\ 3 \dots k\}$ $K^c = \{k+1\ k+2 \dots n\}$

$n = 4$ $K = \{1\ 3\}$ $K^c = \{2\ 4\}$

To sum it up. $c_K \omega_n = d_{K^c} \omega_K \wedge \omega_{K^c} = (-1)^? d_{K^c} \omega_n$

b) do it yourselves!

c) check the basis elements.

$*$ is adjoint w.r.t. $\wedge$, $\langle \cdot , \cdot \rangle$

An attempt to come up with a coordinate-free proof

$\lambda \in \Lambda^k$

$\eta \in \Lambda^{n-k}$

$$\lambda \wedge **\eta = \langle \lambda, *\eta \rangle \omega_n = \langle *\eta, \lambda \rangle \omega_n = *\eta \wedge *\lambda$$

$$\eta \wedge **\lambda = \langle \eta, *\lambda \rangle \omega_n = \langle *\lambda, \eta \rangle \omega_n = *\lambda \wedge *\eta = (-1)^{k(n-k)} *\eta \wedge *\lambda .$$

does not immed. yield a proof!

Conjecture: $\langle *\lambda, *\eta \rangle = (-1)^{k(n-k)} \langle \lambda, \eta \rangle$

$\forall \lambda, \eta \in \Lambda^k$