Q1 : Alternating tensor $\varphi : V^k \to \mathbb{R}$:

1) linear coordin.-wise $\varphi(v_1, \dots \lambda_i v_i + w_i \dots v_k) =$

$= \lambda_i \varphi(v_1, \dots v_i \dots v_k) + \varphi(v_1, \dots w_i, \dots v_k)$

2) $\varphi(v_1, \dots \overset{\substack{i\text{-th} \\ \downarrow}}{v}, \dots \overset{\substack{j\text{-th} \\ \downarrow}}{v}, \dots v_k) = 0$ equal!

3) $\varphi(v_1, \dots v_j, \dots v_i, \dots v_k) = - \varphi(v_1, \dots v_i, \dots v_j, \dots v_k)$

swap

(take $x = y$)

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}\right) := x_1 y_2 - x_2 y_1 + x_1 y_1$$

h ← is even linear?

V – n-dim vector space

$e_1, \dots e_n \leftarrow$ basis

$f_1, \dots f_n \leftarrow$ dual basis of V^*,

$f_i(e_j) = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases}$

Rmk: ω_I look like determinants! (not a coincidence!)

Idea: $\psi(v_1, v_2) := f_1(v_1) \cdot f_2(v_2) \leftarrow$ tensor!

$I = \{1 \leq i_1 < i_2 < i_3 < \dots < i_k \leq n\} \leftarrow$ directed k-tuples

Ex: $n = 4$, $k=2$: $\{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

$$\omega_I = \omega_{i_1 i_2 \dots i_k}(v_1, \dots v_k) \overset{\text{def}}{=} \frac{1}{k!}^{?} \sum_{\sigma \in S_k} (-1)^\sigma f_{i_1}(v_{\sigma(1)}) f_{i_2}(v_{\sigma(2)}) \dots f_{i_k}(v_{\sigma(k)})$$

ω_I is an alternating tensor.

Rk: ω_I does depend on the choice of e_i.

Ex: $V = \mathbb{R}^2$, with the stand. basis. $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$f_1(v) \to$ first coord

$f_2(v) \to$ second coord

$f_1\begin{pmatrix}3\\2\end{pmatrix} = 3$

$\underbrace{I = (1,2)}_{\text{works for } n=2!}$

$$\omega_{12}\left(\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}, \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}\right) = \frac{1}{2}\left(f_1\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix} f_2\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix} - f_1\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix} f_2\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}\right) =$$

$$= \frac{1}{2}\left(a_{11} a_{22} - a_{12} a_{21}\right)$$

$L: \mathbb{R}^m \to \mathbb{R}^n$ - linear op.

L can be repr. by a matrix A.
(in the stand. basis)

$\omega_I \leftarrow$ altern. k-tensor on $\mathbb{R}^n$ (codomain!)

$L^*: \left\{ \text{alt tensors on } \mathbb{R}^n \text{ (codom)} \right\} \to \left\{ \text{alt tensors on } \mathbb{R}^m \text{ (dom)} \right\}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{f \circ g} & \downarrow g \\ & & Z \end{array}$$

ω_I on codom

φ_J - basis of k-tensors on dom

find!

$$(L^*\omega_I)\underbrace{(v_1, \dots v_k)}_{\in \mathbb{R}^m} = \omega_I(Lv_1, \dots Lv_k) = \sum_{J \in m_a^k} c_J \varphi_J .$$

$\mathbb{R}^m$ admits the stand. basis

$e_1, \dots e_m$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Recall: $\varphi_J(e_{J'}) = \delta_{JJ'} = \begin{cases} 1, & J = J' \\ 0, & J \neq J' \end{cases}$

$\Downarrow$

Lemma: $\underbrace{(L^*\omega_I)(e_{j_1}, \dots e_{j_k})}_{\text{apply the def-n!}} = c_{j_1 j_2 \dots j_k}$ (prove the lemma, and you're good to go!)

Hints:

$\Lambda^k(V) = \Omega^k(V)$ ← notations for altern. tensors

1) $S^k(V)$ should be a vector space! (should be able to + · const)

2) you can "alternate" a tensor ($\varphi(v_1, \dots v_k) \rightsquigarrow \sum_{\sigma \in S^k} (-1)^\sigma \varphi(v_{\sigma(1)} \dots v_{\sigma(k)})$)

any

standard notation

alternating

it turns out that you can "symmetrize" a tensor in a similar fashion!

3) for $k=2$ $S^2(V) \oplus \Lambda^2(V)$ has the dim $(\dim V)^2$!

alternating

(not mandatory)

(doesn't work for $k>2$)

4) $\{1 \leq i_1 < i_2 < \dots i_k \leq n\}$ ← altern. case

think about this: can $i_1 = i_2$ in the symmetric case? (problems?)